Non-Euclidean Geometry (spring 2011)

Exercise No. 6 - Partial Solutions

1. Use what we did in class to prove that for any open ball $B \subset \overline{\mathbb{C}}$ and $z_1, z_2 \in B$, there is a unique circle $D \subset \overline{\mathbb{C}}$ passing through z_1 and z_2 such that $D \perp \partial B$.

2. Let $\rho_\triangle, \rho_\mathbb{H}$ be the hyperbolic distances in the unit disc model $\triangle = \{|z| < 1\}$, and in the upper-half plane model $\mathbb{H} = \{\mathrm{Im} z > 0\}$ respectively. Show that

$$\cosh \rho_\triangle(z_1, z_2) = \frac{2|z_1 - z_2|^2}{(1 - |z_1|^2)(1 - |z_2|^2)} + 1, \quad \text{for all } z_1, z_2 \in \triangle,$$

$$\sinh\Big(\frac{1}{2}\rho_\mathbb{H}(z_1, z_2)\Big) = \frac{|z_1 - z_2|}{2\sqrt{\mathrm{Im} z_1 \cdot \mathrm{Im} z_2}}, \quad \text{for all } z_1, z_2 \in \mathbb{H}$$

Solution (sketch): In $(\mathbb{H}, \rho_\mathbb{H})$, for z and w on the imaginary axis we can check that the result holds by a straightforward computation. If $z, w \in \mathbb{H}$ we can always find a $g \in \mathrm{Isom}(\mathbb{H})$ such that $g(z)$ and $g(w)$ are on the imaginary axis. We can also check by a straightforward computation that the quantity $\frac{2|z-w|^2}{(1-|z|^2)(1-|w|^2)}$ is invariant under elements of $\mathrm{Isom}(\mathbb{H})$. (Note: it is actually enough to check this for $z \to z + a, a \in \mathbb{R}$, for $z \to cz$, where $c > 0$ and for $z \to 1/z$.)

A similar argument would work for $(\triangle, \rho_\triangle)$ where instead of the imaginary axis, we will"move" the points to the x-axis by an isometry and check that the righthand side of the formula about is invariant under the isometries of this model.

3. Find the set of points in $\triangle$ which are equidistant from points 0 and 1/3 w.r.t. $\rho_\triangle$.

4. In the upper-half plane model, a hyperbolic circle of radius r with center z is the set $S(z, r) = \{w \in \mathbb{H} : \rho_\mathbb{H}(z, w) = r\}$. Show that $S(z, r)$ is necessarily a Euclidean circle.

Hint: You can use question 2 above to show that in the Poincaré disk model $\triangle$, a ball of radius r centered at 0 is a Euclidean ball centered at 0 and radius $\tanh(r/2) < 1$. Next, use isometries of this model to show that this results for any ball. Finally, "translate" these results to the upper-half plane model.

5. Complete the details (if necessary) in the proofs we gave in class of the following: Let $\triangle$ be a hyperbolic triangle with angles α, β, γ and corresponding side lengths a, b, c.

   (i) The hyperbolic law of sines:

$$\frac{\sinh a}{\sin \alpha} = \frac{\sinh b}{\sin \beta} = \frac{\sinh c}{\sin \gamma}$$

(ii) The second hyperbolic law of cosines:

$$\cos(\gamma) = -\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)\cosh(c)$$

Note that (ii) implies that the angles of a triangle determines its side lengths.

Solution: First, in order to prove the hyperbolic sines law, we consider the quantity $\frac{\sinh^2 C}{\sin^2 \gamma} > 0$. Writing $\cosh A = a, \cosh B = b$ and $\cosh C = c$, we have from the 1st law of cosines that:

$$\begin{aligned}
\sin^2 \gamma &= 1 - \cos^2 \gamma = 1 - (\frac{ab - c}{\sinh A \sinh B})^2 \\
&= \frac{\sinh^2 A \sinh^2 B - (ab - c)^2}{\sinh^2 A \sinh^2 B} \\
&= \frac{(a^2 - 1)(b^2 - 1) - a^2 b^2 - c^2 + 2abc}{\sinh^2 A \sinh^2 B} \\
&= \frac{1 - a^2 - b^2 - c^2 + 2abc}{\sinh^2 A \sinh^2 B} \\
&= \frac{1 - \cosh^2 A \cosh^2 B - \cosh^2 C + 2\cosh A \cosh B \cosh C}{\sinh^2 A \sinh^2 B}
\end{aligned}$$

Hence we conclude that

$$\frac{\sin^2 \gamma}{\sinh^2 C} = \frac{1 - \cosh^2 A \cosh^2 B - \cosh^2 C + 2\cosh A \cosh B \cosh C}{\sinh^2 A \sinh^2 B \sinh^2 C}$$

The right hand side remains unchanged after permuting A, B, C and α, β, γ respectively, and so the left hand side also remains unchanged. This gives us our result.

Next we turn to prove the 2nd Hyperbolic Law of Cosines. From the 1st law of cosines we have

$$cos\alpha = \frac{\cosh B \cosh C - \cosh A}{\sinh B \sinh C} = \frac{\cosh B \cosh C - \cosh A}{\sqrt{(\cosh^2 B - 1)(\cosh^2 C - 1)}}$$

Since $\sin\alpha = \sqrt{1 - \cos^2 \alpha}$, one has

$$\begin{aligned}
\sin\alpha &= \sqrt{1 - \frac{(\cosh B \cosh C - \cosh A)^2}{(\cosh^2 B - 1)(\cosh^2 C - 1)}} \\
&= \sqrt{\frac{(\cosh^2 B - 1)(\cosh^2 C - 1) - (\cosh B \cosh C - \cosh A)^2}{(\cosh^2 B - 1)(\cosh^2 C - 1)}} \\
&= \sqrt{\frac{1 + 2\cosh A \cosh B \cosh C - \cosh^2 A - \cosh^2 B - \cosh^2 C}{(\cosh^2 B - 1)(\cosh^2 C - 1)}}
\end{aligned}$$

Similarly we have

$$\cos\beta = \frac{\cosh A \cosh C - \cosh B}{\sqrt{(\cosh^2 A - 1)(\cosh^2 C - 1)}},$$

$$\sin\beta = \frac{\sqrt{1+2\cosh A\cosh B\cosh C - \cosh^2 A - \cosh^2 B - \cosh^2 C}}{\sqrt{(\cosh^2 A - 1)(\cosh^2 C - 1)}},$$

$$\cos\gamma = \frac{\cosh A\cosh B - \cosh C}{\sqrt{(\cosh^2 A - 1)(\cosh^2 B - 1)}}.$$

From this we see that:

$$\frac{-\cos\gamma + \cos\alpha\cos\beta}{\sin\alpha\sin\beta} = \frac{(bc-a)(ac-b)+(ab-c)(c^2-1)}{1+2abc-a^2-b^2-c^2} = c,$$

where $\cosh A = a, \cosh B = b$ and $\cosh C = c$. This completes the proof.

6. Show that there is no local isometry between the Euclidean and the hyperbolic planes.

   Solution: Proof by contradiction: Let f be an isometry between a domain in the hyperbolic plane $U \in \mathbb{H}$, and a domain in the Euclidean plane V. Let $\triangle ABC$ be an Equilateral triangle in U with side length a, and let $\triangle f(A)f(B)f(C)$ be its image, an Equilateral triangle with sidelength a in V. Let E be the midpoint of the edge AC. Since f is an isometry, $f(E)$ is the mid-point of $f(A)f(C)$. Hence, the distance between A and E must be $\frac{\sqrt{3}}{2}a$ (since f is an isometry). However, from the hyperbolic cosines theorem in the triangle ABE we conclude that

   $$\cosh a = \cosh(a/2)\cosh(\frac{\sqrt{3}}{2}a) - \sinh(a/2)\sinh(\frac{\sqrt{3}}{2}a)\cos(\pi/2) = \cosh(a/2)\cosh(\frac{\sqrt{3}}{2}a)$$

   Hence, for every $a > \epsilon > 0$ one has $\cosh(\epsilon) = \cosh(\epsilon/2)\cosh(\frac{\sqrt{3}}{2}\epsilon)$. Now we can use Taylor Expansion to obtain a contradiction (as we did in the spherical case).

7. Show that the bisectors of a hyperbolic triangle meet at a common point.

   Solution: see the solution to Daf Targil 7.

8. Show that the angle sum of a hyperbolic triangle is strictly less than π.

   Solution: Put the hyperbolic triangle $\triangle$ in $\mathbb{D}$ (Poincaré unit-disk model) with one vertex at the center O. The sides through O are radii. If P, P' are the other two vertices, it is easy to see that the Euclidean triangle with vertices OPP' strictly contains $\triangle$, and in particular, that the $\angle OPP'$, $\angle OP'P$ in $\triangle$ are less than the corresponding Euclidean angles, from which the result follows.

9. Show that in hyperbolic space there exists a unique perpendicular from a point P to a line L. This perpendicular minimizes the distance from P to L.

   Solution: Place P at $O \in \mathbb{D}$ (Poincaré disk model). It is clear from Euclidean geometry that there is a unique radial line PQ perpendicular to the hyperbolic line L. The relation $d_{\mathbb{D}}(P,Q) < d_{\mathbb{D}}(P,Q')$ follows since from Euclidean, any other point $Q' \in L$ lies outside the hyperbolic circle center 0 and radius $|PQ|$.

10. (*) Let $L, L' \subset \mathbb{H}$ be disjoint lines which do not meet at ∞. Show that in contrast to Euclidean geometry, L, L' have a **unique** common perpendicular which minimizes the distance between them. Moreover, (also in contrasts with Euclidean space), perpendicular projection onto a line L strictly decrease distances.

    Solution: *Existence:* We arrange the line L to be the imaginary axis in $\mathbb{H}$. We may assume (explain why??) that L' is a Euclidean semicircle with centered on the real axis (and let P and Q be the intersection of L' with the real axis). It follows from Euclidean geometry that the circle with center O, and radius s, where $s^2 = |OP||OQ|$, cuts L' orthogonally.

    Uniqueness: Any other common perpendicular would produce either a quadrilateral whose angle sum is 2π or a triangle of angle sum larger then π. Both of these cases are impossible by what we proved above.

    *The common perpendicular minimizes distance.*let PP' be the common perpendicular from L to L', from the previous question it follows that $|PP'| < |PQ|$ unless $\angle Q = \pi/2$, in which case $Q = P'$.