Non-Euclidean Geometry (spring 2011)

Exercise No. 7 - Hyperbolic Geometry

1. Show that the map $z \mapsto \frac{z-a}{\bar{a}z-1}$ is an hyperbolic isometry of the Poincaré disk model $\triangle$ that sends the point a to the origin, and moreover, it is its own inverse.

2. (*) Show that the group $\mathrm{Iso}^+(\mathbb{H})$ (resp. $\mathrm{Iso}^+(\mathbb{D})$) acts transitively on equidistance pairs of points in $\mathbb{H}$ (resp. $\mathbb{D}$), as well as on ordered triples in $\partial\mathbb{H}$ (resp. $\partial\mathbb{D}$).

3. Given two hyperbolic triangles T_1, T_2 in $\triangle$ with interior angles α, β, γ. Show that there is a hyperbolic isometry taking T_1 to T_2.

4. Show that any two triply asymptotic triangles (triangles with all 3 vertices on the boundary of the disc) are congruent.

5. Show that a hyperbolic circle with hyperbolic radius r has length $2\pi \sinh r$ and encloses a disc of hyperbolic area $4\pi \sinh^2 \frac{1}{2}r$. Sketch these as functions of r.

   For distinct points A, B, X on an hyperbolic line, define their "hyperbolic ratio" by

$$h(A,X,B) := \begin{cases} \sinh(d(A,X))/\sinh(d(X,B)) & \text{if } X \text{ is between } A \text{ and } B, \\ \sinh(d(A,X))/\sinh(d(X,B)) & \text{otherwise.} \end{cases}$$

6. (Menelaus's theorem for hyperbolic triangles) If L is an hyperbolic line not through any vertex of an hyperbolic triangle ABC such that L meets BC in Q, AC in R, and AB in P, then $h(A,P,B)h(B,Q,C)h(C,R,A) = -1$. Moreover, (the converse of Menelaus's theorem for hyperbolic triangles): If P lies on the hyperbolic line AB, Q on BC, and R on CA such that $h(A,P,B)h(B,Q,C)h(C,R,A) = -1$, then P, Q and R are hyperbolic collinear.

7. (Ceva's Theorem for Hyperbolic Triangles) If X is a point not on any side of an hyperbolic triangle ABC such that AX and BC meet in Q, BX and AC in R, and CX and AB in P, then $h(A,P,B)h(B,Q,C)h(C,R,A) = 1$. Moreover, (the converse of Ceva's theorem for hyperbolic triangles): If P lies on the hyperbolic line AB, Q on BC, and R on CA such that $h(A,P,B)h(B,Q,C)h(C,R,A) = 1$, and two of the hyperbolic lines CP, BR and AQ meet, then all three are concurrent.

8. Use the above to show that: 1) The hyperbolic medians of a hyperbolic triangle are concurrent. 2) The internal angle bisectors of a hyperbolic triangle are concurrent. 3) If any two hyperbolic altitudes of a hyperbolic triangle meet, then the hyperbolic altitudes are concurrent.