Non-Euclidean Geometry (spring 2011)

Partial Solutions to Exercise No. 7 - Hyperbolic Geometry

1. Show that the map $z \mapsto \frac{z-a}{\bar{a}z-1}$ is an hyperbolic isometry of the Poincaré disk model $\triangle$ that sends the point a to the origin, and moreover, it is its own inverse.

2. (*) Show that the group $\text{Iso}^+(\mathbb{H})$ (resp. $\text{Iso}^+(\mathbb{D})$) acts transitively on equidistance pairs of points in $\mathbb{H}$ (resp. $\mathbb{D}$), as well as on ordered triples in $\partial\mathbb{H}$ (resp. $\partial\mathbb{D}$).

   Solution: We shall proof this for $\mathbb{H}$ (in order to prove the result in $\mathbb{D}$ you can either make direct calculations or apply the Cayley transformation) (i) We have to show that if $P, P', Q, Q' \in \mathbb{H}$ with $d_{\mathbb{H}}(P, P') = d_{\mathbb{H}}(Q, Q')$, then there exists an isometry $T \in \text{Isom}^+(\mathbb{H})$ such that $T(P) = P'$ and $T(Q) = Q'$. It will be sufficient to prove this for the case in which $Q = i$ and $Q' = iexp(d_{\mathbb{H}}(P, P'))$ (Why?). Let S_1 be the map which maps the semicircle C with center on $\partial\mathbb{H}$ to the imaginary axis. Next let S_2 be the map $z \mapsto z/a$ which maps the imaginary axis to itself sending ai to i, so in particular $S_2S_1(P) = i$. Since S_2S_1 is an isometry we have $d_{\mathbb{H}}(S_2S_1(P), S_2S_1(P')) = d_{\mathbb{H}}(P, P')$, so that $d_{\mathbb{H}}(i, S_2S_1(P')) = d_{\mathbb{H}}(P, P')$ and hence $S_2S_1(P') = iexp(\pm d_{\mathbb{H}})(P, P'))$. If $S_2S_1(P') = iexp(d_{\mathbb{H}}(P, P')) = Q'$ we are done taking $T = S_2S_1$. Otherwise, apply $S_3 : z \mapsto -1/z$ which fixes$Q = i$ and takes it to i/t for any $t > 0$ (note that $S_3 \in SL(2, \mathbb{R})$). Then $S_3S_2S_1(P) = Q$ and $S_3S_2S_1(P') = Q'$ so we are done.

   (ii) Say $\xi_1, \xi_2, \xi_3 \in \mathbb{R} \cup \infty$. The map $T : z \mapsto [z, \xi_1, \xi_2, \xi_3] = \frac{z-\xi_1}{z-\xi_2} \cdot \frac{\xi_3-\xi_2}{\xi_3-\xi_1}$ carries $\xi_1 \mapsto 0$, $\xi_2 \mapsto \infty$, $\xi_3 \mapsto 1$. Now $\det(T) = \frac{(\xi_3-\xi_2)(\xi_1-\xi_2)}{\xi_3-\xi_1}$, so $\det(T) > 0$ if and only if the two triples $(0, \infty, 1)$ and (ξ_1, ξ_2, ξ_3) have the same cyclic order on $\widehat{\mathbb{R}}$. Then we can normalize to get $T \in SL(2, \mathbb{R})$.

3. Given two hyperbolic triangles T_1, T_2 in $\triangle$ with interior angles α, β, γ. Show that there is a hyperbolic isometry taking T_1 to T_2.

   Solution: Consider two triangles of angles α, β, γ at vertices A, B, C and A', B', C'. Applying an isometry, we may arrange the triangles so that $A = A'$ and so that the sides AB, AB' are contained in a common line through A, as are AC, AC'. Suppose without loss of generality that $|AB| \leq |AB'|$. If $|AC| \leq |AC'|$ then either triangle ABC is strictly contained inside triangle $AB'C'$, which is impossible because they have the same area, or $B = B', C = C'$. If $|AC| > |AC'|$ then the lines BC and $B'C'$ cross at a point X say. Then the angle sum of triangle BXB' is at least $\beta + (\pi - \beta)$, which is impossible.

4. Show that any two triply asymptotic triangles (triangles with all 3 vertices on the boundary of the disc) are congruent.

5. Show that a hyperbolic circle with hyperbolic radius r has length $2\pi \sinh r$ and encloses a disc of hyperbolic area $4\pi \sinh^2 \frac{1}{2}r$. Sketch these as functions of r.

   Solution: The solution was given in class.

   For distinct points A, B, X on an hyperbolic line, define their "hyperbolic ratio" by

$$h(A, X, B) := \begin{cases} \sinh(d(A,X))/\sinh(d(X,B)) & \text{if } X \text{ is between } A \text{ and } B, \\ \sinh(d(A,X))/\sinh(d(X,B)) & \text{otherwise.} \end{cases}$$

   Before we prove the next claims, note (check this!) the following basic Properties of the Hyperbolic Ratio:

   (1) $h(A, X, B) = -h(B, X, A)$,

   (2) if X is between A and B, then $h(A, X, B) \in (0, 1)$,

   (3) if X is on AB, beyond B, then $h(A, X, B) \in (-\infty, -1)$,

   (4) if X is on AB, beyond A, then $h(A, X, B) \in (-1, 0)$.

6. (Menelaus's theorem for hyperbolic triangles) If L is an hyperbolic line not through any vertex of an hyperbolic triangle ABC such that L meets BC in Q, AC in R, and AB in P, then $h(A, P, B)h(B, Q, C)h(C, R, A) = -1$. Moreover, (the converse of Menelaus's theorem for hyperbolic triangles): If P lies on the hyperbolic line AB, Q on BC, and R on CA such that $h(A, P, B)h(B, Q, C)h(C, R, A) = -1$, then P, Q and R are hyperbolic collinear.

   Solution: We start with the first part: a little consideration shows that, if we change the labels of the vertices, then either the factors are simply permuted, or are inverted and permuted. Since our aim is to show that the product is -1, the labelling of the vertices of the hyperbolic-triangle is immaterial. It follows that there are two cases, depending on the position of the hyperbolic line $L = PQR$ relative to the Hyperbolic triangle. Observe that either one cut is external, or all three are external. It follows that the product of the ratios is negative. It is therefore enough to show that the absolute value is $+1$.

   Case I: (one cut is external - the point Q) Applying the Hyperbolic Sine Rule to the hyperbolic triangle APR, we obtain

$$\sinh(d(A,R))/\sin(\angle APR)) = \sinh(d(A,P))/\sin(\angle ARP).$$

   Similarly, from the hyperbolic triangles BPQ and CRQ, one has

$$\sinh(d(B,Q))/\sin(\angle BPQ)) = \sinh(d(B,P))/\sin(\angle BQP),$$

   and

$$\sinh(d(C,R))/\ sin(\angle CQR)) = \sinh(d(C,Q))/\sin(\angle CRQ).$$

   Observe that: $\angle APR = \angle BPQ$, so $\sin(\angle APR) = \sin(\angle BPQ)$, , $\angle BQP = \angle CQR$, so $\sin(\angle BQP) = \sin(\angle CQR)$, and $\angle ARP = \pi - \angle CRQ$, so $\sin(\angle ARP) = \sin(\angle CRQ)$. Some elementary algebra now yields $|h(A, P, B)h(B, Q, C)h(C, R, A)| = 1$.

Case II: (the points P, Q, R are all external): This follows in a similar way, with minor changes in the argument showing that the sines of the relevant angles are equal.

Before we continut to prove the other direction we shall need the following observation:

Lemma(the Hyperbolic Ratio Theorem): if X and Y are points on the hyperbolic line AB such that $h(A, X, B) = h(A, Y, B)$, then $X = Y$.

Proof of the Lemma: There are three cases, depending on whether the common hyperbolic ratio is in $(0, 1), (-\infty, -1)$ or $(-1, 0)$. Case I (ratio in $(0, 1)$): Then X and Y lie between A and B. Interchanging the labels X and Y if necessary, we may assume that Y is between X and B. From the triangle inequality, one has, if $X \neq Y$, then $d(A, Y) = d(A, X) + d(X, Y) > d(A, X)$, and $d(Y, B) = d(X, B) - d(X, Y) < d(X, B)$. Since sinh is increasing, this gives $h(A, Y, B) > h(A, X, B)$. Thus, if $h(A, X, B) = h(A, Y, B)$, then $X = Y$.

Case II (Ratio in $(-\infty, -1)$): Then X and Y lie beyond B. Interchanging the labels X and Y if necessary, we may assume that Y lies beyond X. By the triangle inequality, we have, if $X \neq Y$, $d(A, Y) = d(A, X) + d(X, Y) = \alpha + \delta$, say, and $d(B, Y) = d(B, X) + d(X, Y) = \beta + \delta$, say. Thus, if $h(A, X, B) = h(A, Y, B)$, then, $-\sinh(d(A, X)))/\sinh(d(X, B)) = -\sinh(d(A, Y))/\sinh(d(Y, B))$. So that

$$0 = \sinh(\alpha + \delta) \sinh(\beta) - \sinh(\beta + \delta) \sinh(\alpha) = \sinh(\delta) \sinh(\beta - \alpha)$$

Thus $\delta = 0$, or $\alpha = \beta$, either of which implies that $X = Y$.

Case III (Ratio in $(-1, 0)$): This follows from the previous case as $h(A, Z, B) = 1/h(B, Z, A)$ for all Z.

Next we turn to prove the converse of Menelaus's theorem for hyperbolic triangles:

A little consideration shows that, if we change the labels of the vertices, then either the factors are simply permuted, or are inverted and permuted. Since our hypothesis is that the product is -1, the labelling of the vertices of the hyperbolic triangle is immaterial. The key is to show that an hyperbolic line through two of P, Q, R cuts the third side of the hyperbolic triangle. We apply Menelaus's Theorem to this line. Since the product is negative, at least one of the cuts is external. Relabelling if necessary, we may assume that Q lies beyond C on BC.

If P lies on the hyperbolic ray AB, then the hyperbolic line QP cuts AC, at R^* say. Applying Menelaus's theorem to the hyperbolic triangle ABC, cut by PQR^*, we get $h(A, P, B)h(B, Q, C)h(C, R^*, A) = -1$. Comparing this with the hypothesis, $h(C, R^*, A) = h(C, R, A)$. Then, by the Hyperbolic Ratio Theorem, $R = R^*$, so P, Q, R are collinear.

Note that there are two other possibilities: If R lies on the hyperbolic ray CA, then the hyperbolic line QR cuts AB, at P^* say. The only other possibility is that P lies beyond A (e.g. at P'), and that R lies beyond C (e.g. at R'). Then $P'R'$ cuts BC, at Q^* say. In each case a similar application of Menelaus gives the result.

7. (Ceva's Theorem for Hyperbolic Triangles) If X is a point not on any side of an hyperbolic triangle ABC such that AX and BC meet in Q, BX and AC in R, and

CX and AB in P, then $h(A,P,B)h(B,Q,C)h(C,R,A) = 1$. Moreover, (the converse of Ceva's theorem for hyperbolic triangles): If P lies on the hyperbolic line AB, Q on BC, and R on CA such that $h(A,P,B)h(B,Q,C)h(C,R,A) = 1$, and two of the hyperbolic lines CP, BR and AQ meet, then all three are concurrent.

Solution: A little consideration shows that, if we change the labels of the vertices, then either the factors are simply permuted, or are inverted and permuted. Since our aim is to show that the product is 1, the labelling of the vertices of the hyperbolic triangle is immaterial. It follows that there are essentially three cases, depending upon the position of X relative to A:

Case I (X is in between A and Q inside the triangle): Applying Menelaus's Theorem to the hyperbolic triangle AQC, cut by BXR, gives $|h(A,X,Q)h(Q,B,C)h(C,R,A)| = 1$,so that

$$(\sinh(d(A,X)/\sinh(d(X,Q))\ (\sinh(d(Q,B)/\sinh(d(B,C))\ (\sinh(d(C,R)/\sinh(d(R,A)) = 1$$

Similarly, applying Menelaus's Theorem to the hyperbolic triangle AQB, cut by CXP gives

$$(\sinh(d(A,X)/\sinh(d(X,Q))\ (\sinh(d(Q,C)/\sinh(d(C,B))(\sinh(d(B,P)/\sinh(d(P,A)) = 1$$

Dividing these two expressions and cancelling common factors, we get

$$(\sinh(d(A,P)/\sinh(d(P,B))\ (\sinh(d(B,Q)/\sinh(d(Q,C))\ (sinh(d(C,R)/sinh(d(R,A)) = 1$$

Since, in this case, the sides are cut internally, $h(A,P,B)h(B,Q,C)h(C,R,A) = 1$, as required.

The other two cases are almost identical and are left as an exercisc.

Next we prove the Converse of Ceva's Theorem: a little consideration shows that, if we change the labels of the vertices, then either the factors are simply permuted, or are inverted and permuted. Since our hypothesis is that the product is 1, the labelling of the vertices of the hyperbolic triangle is immaterial. The key is to show that the hyperbolic line through the intersection X of two of CP, AQ, BR and the third vertex cuts the third side of the hyperbolic triangle. We apply Ceva's Theorem to this point X. If P lies between A and B, then CP cuts the hyperbolic segment AQ, at X say. Also, BX cuts the hyperbolic segment AC, at R^*,say. Applying Ceva's Theorem to the hyperbolic triangle ABC, and the point X, we get $h(A,P,B)h(B,Q,C)h(C,R,A) = 1$. Comparing this with the hypothesis, $h(C,R,A) = h(C,R^*,A)$. By the Hyperbolic Ratio Theorem (the lemma above), $R = R^*$, so AQ, BR, CP concur at X. A similar argument applies if R lies between A and C.

Now suppose that P lies beyond B, and R beyond C. Then BR and CP meet at X, which lies within angle BAC. Now AX cuts the hyperbolic segment BC at Q^*, say. It now follows as before that $Q = Q^*$, so the hyperbolic lines are concurrent.

Next suppose that P lies beyond B, and R beyond A. Then the hyperbolic ray RB enters angle PBC at B, and so cuts CP at X, say. Since X lies within the angleBAC,AX cuts the segment BC, at Q^*, say. It now follows as before that $Q = Q^*$, so the lines are concurrent. The case where P is beyond A, and R beyond C is similar. **Note that, in all of the above cases, the lines AQ,BR,CP must meet**. We are left with cases where both P and R lie beyond A. The hypotheses include the condition that at least two of AQ, BR, CP meet. Suppose first that BR and CP meet, at X, say. Then X lies within angle RAP, so AX meets the segmentBC, atQ^*, say. It now follows as before that $Q = Q*$, so the lines are concurrent. Suppose next that BR and AQ meet, at X, say. Now X and C lie on opposite sides of AB, so CX meets AB, at P^*, say. Much as before, we deduce that $P = P^*$, so the lines are concurrent. The case where CP and AQ meet is similar to the previous case.

8. Use the above to show that: 1) The hyperbolic medians of a hyperbolic triangle are concurrent. 2) The internal angle bisectors of a hyperbolic triangle are concurrent. 3) If any two hyperbolic altitudes of a hyperbolic triangle meet, then the hyperbolic altitudes are concurrent.

   Solution: (i) The Medians Theorem for Hyperbolic Triangles: i.e., The hyperbolic medians of a hyperbolic triangle are concurrent:

   Let ABC be an hyperbolic triangle, and let the hyperbolic medians be AQ, BR and CP. Since P is the hyperbolic midpoint of BC, one has $h(A, P, B) = \sinh(d(A, P))/\sinh(d(P, B)) = 1$, and similarly $h(B, Q, C) = h(C, R, A) = 1$. Hence $h(A, P, B)h(B, Q, C)h(C, R, A) = 1$. The h-medians all lie within the hyperbolic triangle, so any two must meet. Thus, by the Converse of Ceva's Theorem, the hyperbolic medians AQ, BR and CP are collinear.

   (ii) The Angle Bisectors Theorem for Hyperbolic Triangles: The internal angle bisectors of a hyperbolic triangle are concurrent:

   Let ABC be an hyperbolic triangle, and let the angle bisectors be AQ, BR and CP. By the Hyperbolic Sine Rule applied to the hyperbolic triangles AQC and AQB,

   $$\sinh(d(Q, C))/\sin(\angle QAC) = \sinh(d(A, Q))/\sin(\angle ACQ),$$

   and

   $$\sinh(d(B, Q))/\sin(\angle QAB) = \sinh(d(A, Q))/\sin(\angle ABQ).$$

   As AQ bisects $\angle BAC, \angle QAC = \angle QAB$, so these equations yield $h(B, C, Q) = \sin(\angle ACB)/\sin(\angle ABC)$. SImilarly, using the bisectors BR and CP, one get $h(A, B, P) = \sin(\angle CBA)/\sin(\angle CAB)$, and $h(C, A, R) = \sin(\angle BAC)/\sin(\angle BCA)$. Multiplying the ratios, $h(A, B, P)h(B, C, Q)h(C, A, R) = 1$. The angle bisectors all lie within the triangle, so any two must meet. Thus, by the Converse of Ceva's Theorem, the bisectors AQ, BR and CP are collinear.