Non-Euclidean Geometry (spring 2011)

Exercise No. 8 - Hyperbolic Geometry

1. Verify that the hyperbolic disk $D_{\triangle}(a, r)$ is the Euclidean disk with center c and radius R, where
$$c = \frac{a(1-\tanh^2(r/2))}{1-|a|^2\tanh^2(r/2)}, \text{ and } R = \frac{(1-|a|^2)\tanh(r/2)}{1-|a|^2\tanh^2(r/2)}$$

2. "Recall" from differential geometry that if $ds^2 = g_1(x,y)dx^2 + g_2(x,y)dy^2$ is a Riemannian metric on a surface, then the Gaussian curvature K is given by
$$K = \frac{-1}{\sqrt{g_1 g_2}}\Big[\frac{\partial}{\partial x}\big(\frac{1}{\sqrt{g_1}}\frac{\partial}{\partial x}\sqrt{g_2}\big) + \frac{\partial}{\partial y}\big(\frac{1}{\sqrt{g_2}}\frac{\partial}{\partial y}\sqrt{g_1}\big)\Big]$$
Show that the different models of Hyperbolic geometry discussed in class $(\mathbb{H}, \mathbb{I}, \mathbb{J}, \mathbb{K}, \mathbb{L})$ are spaces of constant negative curvature -1.

3. Show that the map $\gamma : \mathbb{J} \to \mathbb{K}$ that was defined in class is an isometry between the Hemi-sphere and the Klein models.

4. Let $g \in PSL(2, \mathbb{R})$ be an isometry of the hyperbolic plane $\mathbb{H}$. Denote the translation length of g by $l(g) = \inf_{z\in\mathbb{H}} \rho_{\mathbb{H}}(z, gz)$. Show that $l(g) > 0$ if and only if g is hyperbolic (as a Möbius transformation). Next, assume that g is hyperbolic. Find l(g) in terms of $tr^2(g)$. Show that the infimum in the definition of the translation length is attained at any point of the unique hyperbolic line invariant under g. (**Hints:** 1) the distance l is invariant under conjugation, 2) use canonical forms.)

5. A non-identity Möbius transformation T that maps a disc $\triangle$ onto itself is one of the following:
   - (a) Elliptic with two fixed points, one in $\triangle$ and one in the complementary disc;
   - (b) Hyperbolic with two fixed points, both on the boundary $\partial\triangle$ of $\triangle$;
   - (c) Parabolic with one fixed point, which lies on $\partial\triangle$.

6. Let $g \in PSL(2, \mathbb{R})$ given by $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ Show that in the upper-half model
$$2\cosh(d_{\mathbb{H}}(i, gi)) = a^2 + b^2 + c^2 + d^2$$

7. (*) Use the above exercise to show that any discrete subgroup of $PSL(2, \mathbb{R})$ acts properly on the upper-half plane. ("Recall" that an action of a discrete group G on a locally compact space X is said to be properly discontinuous if for every compact K in X, one has $K \cap gK = \emptyset$ except for a finite number of $g \in G$). Hint: start with the fact that G is countable.