Non-Euclidean Geometry (spring 2011)

Exercise No. 8 - Hyperbolic Geometry

1. Verify that the hyperbolic disk $D_\triangle(a, r)$ is the Euclidean disk with center c and radius R, where
$$c = \frac{a(1 - \tanh^2(r/2))}{1 - |a|^2 \tanh^2(r/2)}, \text{ and } R = \frac{(1 - |a|^2)\tanh(r/2)}{1 - |a|^2 \tanh^2(r/2)}$$

2. "Recall" from differential geometry that if $ds^2 = g_1(x, y)dx^2 + g_2(x, y)dy^2$ is a Riemannian metric on a surface, then the Gaussian curvature K is given by
$$K = \frac{-1}{\sqrt{g_1 g_2}}\Big[\frac{\partial}{\partial x}\big(\frac{1}{\sqrt{g_1}}\frac{\partial}{\partial x}\sqrt{g_2}\big) + \frac{\partial}{\partial y}\big(\frac{1}{\sqrt{g_2}}\frac{\partial}{\partial y}\sqrt{g_1}\big)\Big]$$
Show that the different models of Hyperbolic geometry discussed in class ($\mathbb{H}, \mathbb{I}, \mathbb{J}, \mathbb{K}, \mathbb{L}$) are spaces of constant negative curvature -1.

   Solution: This follows from a direct computation. For example, in the case of $\mathbb{H}$ one has $g_1 = g_2 = 1/y^2$ and hence
$$\frac{\partial}{\partial x}\sqrt{g_2} = 0, \ \frac{\partial}{\partial y}\sqrt{g_1} = -1/y^2, \text{ and } \frac{\partial}{\partial y}\Big(\frac{1}{\sqrt{g_2}}\frac{\partial}{\partial y}\sqrt{g_1}\Big) = \frac{1}{y^2},$$
from which the result follows. A similar computation shows the other cases.

3. Show that the map $\gamma : \mathbb{J} \to \mathbb{K}$ that was defined in class is an isometry between the Hemi-sphere and the Klein models.

   Solution: Set $y_1 = x_1, \ldots, y_n = x_n$, and $y_{n+1}^2 = 1 - y_1^2 - \cdots - y_n^2 = 1 - x_1^2 - \cdots - x_n^2$. Then, $dy_i = dx_i$ for $i = 1, \ldots, n$ and $y_{n+1}dy_{n+1} = -(x_1 dx_1 + \ldots x_n dx_n)$. Thus,
$$\gamma^*(ds_J^2) = \frac{1}{y_{n+1}^2}(dy_1^2 + \ldots + dy_n^2) + \frac{1}{y_{n+1}^2}dy_{n+1}^2 =$$
$$= \frac{1}{(1 - x_1^2 - \ldots - x_n^2)}(dx_1^2 + \ldots + dx_n^2) + \frac{(x_1 dx_1 + \ldots + x_n dx_n)^2}{(1 - x_1^2 - \ldots - x_n^2)^2} = ds_K^2$$

4. Let $g \in PSL(2, \mathbb{R})$ be an isometry of the hyperbolic plane $\mathbb{H}$. Denote the translation length of g by $l(g) = \inf_{z \in \mathbb{H}} \rho_{\mathbb{H}}(z, gz)$. Show that $l(g) > 0$ if and only if g is hyperbolic (as a Möbius transformation). Next, assume that g is hyperbolic. Find l(g) in terms of $tr^2(g)$. Show that the infimum in the definition of the translation length is attained at any point of the unique hyperbolic line invariant under g. (**Hints:** 1) the distance l is invariant under conjugation, 2) use canonical forms.)

Solution: In the hyperbolic case, using the fact that the distance is invariant under conjugation, we can consider the case where $g(z) = kz$ for $k > 0$. Using the formula proven in "Daf" 6 exercise 2, we conclude that

$$\sinh \frac{1}{2}\rho_{\mathbb{H}}(z, gz) = \frac{|z||1-k|}{2Im(z)\sqrt{k}}$$

Thus, it follows that this function attains its minimum (over $z \in \mathbb{H}$) at and only at each z on the axis $x = 0$. Moreover , in this case $l(g) > 0$, and (by invariance under conjugation)

$$\cosh^2(\frac{1}{2}l) = 1 + \frac{(1-k)^2}{4k} = \text{trace}^2(g)/4$$

In the non-hyperbolic case, using canonical forms of the transformation, we can easily show that $l(g) = 0$. Indeed if g is parabolic or elliptic then they would have fixed points (at "infinity" in the case of g parabolic) and hence $l(g) = 0$.

5. A non-identity Möbius transformation T that maps a disc $\triangle$ onto itself is one of the following:

   - (a) Elliptic with two fixed points, one in $\triangle$ and one in the complementary disc;
   - (b) Hyperbolic with two fixed points, both on the boundary $\partial\triangle$ of $\triangle$;
   - (c) Parabolic with one fixed point, which lies on $\partial\triangle$.

   Solution: We showed in class (it's easy) that T has either one or two fixed points. Suppose that T has two fixed points. Then T is conjugate to $M_k : z \mapsto kz$ for some k$\neq 0, 1$. When $|k| = 1$, the only discs M_k maps onto themselves are $\{|z| < r\}$ and $\{|z| > r\}$. Hence we are in case (a). When $k > 0 (k \neq 1)$, the only discs mapped onto themselves are the half planes$\{Re(e^{i\theta})z > 0\}$. Hence we are in case (b). (All other values for k give loxodromic transformations and these map no disc onto itself.) Suppose that T has only one fixed point. Then T is conjugate to $z \mapsto z + 1$. The only discs mapped onto themselves in this case are the half planes $\{Im(z) > c\}$. Hence we are in case (c).

6. Let $g \in PSL(2, \mathbb{R})$ given by $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ Show that in the upper-half model

   $$2\cosh(d_{\mathbb{H}}(i, gi)) = a^2 + b^2 + c^2 + d^2$$

   Solution: From the formula we proved in class one has

   $$\cosh(d(i, g \cdot i)) = 1 + \frac{|i - g \cdot i|^2}{2\text{Im}(g \cdot i)} = 1 + \frac{c^2 + d^2}{2}\Big[\frac{(bd + ac)^2}{(c^2 + d^2)^2} - \frac{(1 - c^2 + d^2)^2}{(c^2 + d^2)^2}\Big]$$

   $$= 1 + \frac{1}{2(c^2 + d^2)}\Big[(bd + ac)^2 + 1 + (c^2 + d^2)^2 - 2(c^2 + d^2)\Big]$$

   $$= 1 + \frac{1}{2(c^2 + d^2)}\Big[(bd + ac)^2 + (ad - bc)^2 + (c^2 + d^2)^2 - 2(c^2 + d^2)\Big]$$

$$= 1 + \frac{1}{2(c^2+d^2)}\Big[(a^2+b^2)(c^2+d^2) + (c^2+d^2)^2 - 2(c^2+d^2)\Big]$$

$$= \frac{1}{2}(a^2+b^2+c^2+d^2)$$

7. (*) Use the above exercise to show that any discrete subgroup of $PSL(2,\mathbb{R})$ acts properly on the upper-half plane. ("Recall" that an action of a discrete group G on a locally compact space X is said to be properly discontinuous if for every compact K in X, one has $K \cap gK = \emptyset$ except for a finite number of $g \in G$). Hint: start with the fact that G is countable.