Non-Euclidean Geometry (spring 2011)

Exercise No. 9

1. Show that the radius R of the inscribed circle in a hyperbolic triangle $T = \triangle ABC$ is given by:
$$\tanh^2 R = \frac{\cos^2\alpha + \cos^2\beta + \cos^2\gamma + 2\cos\alpha\cos\beta\cos\gamma - 1}{2(1+\cos\alpha)(1+\cos\beta)(1+\cos\gamma)}$$

2. Let α be the angle of parallelism (i.e., the angle at one vertex of a right hyperbolic triangle that has two asymptotic parallel sides), and let b be the segment length between the right angle and the vertex of the angle of parallelism. In class we proved that $\cosh(b)\sin\alpha = 1$. Show that this condition is equivalnet to the following:
$$(i)\ \sinh(b)\tan\alpha = 1, \quad (ii)\ \tanh(b)\sec\alpha = 1, \quad (iii)\ e^{-b} = \tan(\alpha/2)$$

3. Prove that $\Gamma = PSL(2,\mathbb{Z})$ is a discrete subgroup of $PSL(2,\mathbb{R})$.

4. Find a fundamental domain in $\mathbb{H}^2$ for the group $\Gamma = \{\gamma_n | \gamma_n(z) = 2^n z\}$

5. Show that the following conditions on a subgroup $G < SL(2,\mathbb{R})$ are equivalent: (i) There are no accumulation points in G; (ii) G has no accumulation points in $SL(2,\mathbb{R})$; (iii) The identity is an isolated point of G.

6. (**) Let $T \in \mathrm{Isom}^+(\mathbb{H})$ be parabolic. If $S \in \mathrm{Isom}(\mathbb{H})$ commutes with T, what can we say about the group generated by S and T?

7. How many points are there in the projective plane $\mathbb{P}(\mathbb{Z}_2^3)$? How many lines? How many points does each line contain? How many lines pass through each point?

8. Let $\mathbb{P}(V)$ be a projective space of dimension at least 2 (thus $dimV \geq 3$). Prove that:

   (a) Distinct projective lines in $\mathbb{P}(V)$ intersect in at most one point. (Very easy!)

   (b) Distinct projective lines in $\mathbb{P}(V)$ intersect if and only if they lie in some (unique) projective plane.

   (c) Give an example of two non-intersecting projective lines in $\mathbb{P}(\mathbb{R}^4)$.

9. Let $\mathbb{P}(V)$ be a projective space and $\mathbb{P}(W_1), \mathbb{P}(W_2) \subset \mathbb{P}(V)$ distinct projective subspaces of complementary dimension: $dim\mathbb{P}(W_1) + dim\mathbb{P}(W_2) = dim\mathbb{P}(V)$. Prove that the $\mathbb{P}(W_i)$ intersect. What does this tell us when $dim\mathbb{P}(V)) = 3$?

10. Suppose that T and T' define the same projective transformation, then $T = \lambda T'$ for some $\lambda \neq 0$.