Non-Euclidean Geometry (spring 2011)

Partial Solutions to Exercise No. 9

1. Show that the radius R of the inscribed circle in a hyperbolic triangle $T = \triangle ABC$ is given by:

$$\tanh^2 R = \frac{\cos^2\alpha + \cos^2\beta + \cos^2\gamma + 2\cos\alpha\cos\beta\cos\gamma - 1}{2(1+\cos\alpha)(1+\cos\beta)(1+\cos\gamma)}$$

   Solution: We proved in Exercise no. 7 that the angle bisectors of T meet at a point ξ at T. Denote by W_C the point on the edge c for which $\angle AW_C\xi = BW_c\xi = \pi/2$. In a similar manner we definte the points W_a and W_b. Check that the inscribed circle is the circle centered at ξ passing through W_a, W_b and W_c. Next, denote $x = \rho_{\mathbb{H}}(A, W_c)$, and $y = \rho_{\mathbb{H}}(W_c, B)$. Then

$$\frac{\cos\alpha\cos\beta + \cos\gamma}{\sin\alpha\sin\beta} = \cosh x\cosh y + \sinh x\sinh y$$

   From this we obtain that

$$[(\cos\alpha\cos\beta + \cos\gamma) - (\sin\alpha\sinh x)(\sin\beta\sinh y)]^2 =$$

$$= [(1-\cos^2\alpha) + \sin^2\alpha\sinh^2 x][(1-\cos^2\beta) + \sin^2\beta\sinh^2 y]$$

   The identity $\sin\theta = (1+\cos\theta)\tan(\theta/2)$ together with the relation $\tanh R = \sinh x\tan(\alpha/2)$ yields $\sin\alpha\sinh x = (1+\cos\alpha)\tanh R$. A similar relation holds for β, γ and R and substituting yields (after some simplifications) the desired result.

2. Let α be the angle of parallelism (i.e., the angle at one vertex of a right hyperbolic triangle that has two asymptotic parallel sides), and let b be the segment length between the right angle and the vertex of the angle of parallelism. In class we proved that $\cosh(b)\sin\alpha = 1$. Show that this condition is equivalnet to the following:

$$(i)\ \sinh(b)\tan\alpha = 1, \quad (ii)\ \tanh(b)\sec\alpha = 1, \quad (iii)\ e^{-b} = \tan(\alpha/2)$$

3. Prove that $\Gamma = PSL(2,\mathbb{Z})$ is a discrete subgroup of $PSL(2,\mathbb{R})$.

   Solution: There are many ways to prove this. For example, you can use the following criterion: The subgroup $G \in SL(2,\mathbb{C})$ is discrete if and only if for every positive k one has that $\{A \in G \mid \|A\| \le k\}$ is finite. Here $\|A\|^2 = (a^2+b^2+c^2+d^2)$. If this set is finite for each k, then G clearly cannot have any limit points (the norm function is continuous) and so G is discrete. On the other hand, if this set is infinite then there are distinct elements A_n in G with $\|A_n\| \le k$, for $n = 1, 2, \ldots$. If A_n

has coefficients (a_n, b_n, c_n, d_n) then $|a_n| \leq k$ and so the sequence a_n has a convergent subsequence. The same is true for the other coefficients, and using the well known "diagonal process" we get a subseqence on which each of the coefficient converge. On this subsequence, $A_n \to B$ say, for some B and as the determinante is continuous, $B \in SL(2, \mathbb{C})$, and thus G is not discrete. Using this criterion, it follows immediatly that $G \in SL(2, \mathbb{C})$ is discrete.

4. Find a fundamental domain in $\mathbb{H}^2$ for the group $\Gamma = \{\gamma_n | \gamma_n(z) = 2^n z\}$

5. Show that the following conditions on a subgroup $G < SL(2, \mathbb{R})$ are equivalent: (i) There are no accumulation points in G; (ii) G has no accumulation points in $SL(2, \mathbb{R})$; (iii) The identity is an isolated point of G.

   Solution:

   Remark: Note that condition (ii) is on the face of it somewhat stronger than (i). In fact for general metric spaces $X \subset Y$, the conditions (i) and (ii) are not equivalent. For example, suppose $X = \{1/n\} \subset [0, 1] = Y$. Then X has no accumulation points in itself, so (i) holds, but it does have an accumulation point $0 \in Y$, so (ii) fails. The proof which follows shows that if X, Y are topological groups, then the two conditions are the same.

   (i) implies (ii): Suppose that (ii) fails so that $g_n \to h$ for some $h \in SL(2, \mathbb{R})$. Then $g_n g_{n+1}^{-1} \to hh^1 = \mathrm{Id}$. Since G is a subgroup, $\mathrm{Id} \in G$ and hence (i) fails.

   (ii) implies (iii) since if the identity is not an isolated point, then clearly (ii) fails.

   (iii) implies (i): Choose a neighbourhood $\mathrm{Id} \in U$ such that $G \cap U = \{\mathrm{id}\}$. Then if $g \in G$, we have $G \cap gU = \{g\}$ which implies (i).

6. (**) Let $T \in \mathrm{Isom}^+(\mathbb{H})$ be parabolic. If $S \in \mathrm{Isom}(\mathbb{H})$ commutes with T, what can we say about the group generated by S and T?

7. How many points are there in the projective plane $\mathbb{P}(\mathbb{Z}_2^3)$? How many lines? How many points does each line contain? How many lines pass through each point?

   Solution:

   (a) The same argument as that given in lectures establishes that, for any field $\mathbb{F}$, one has $\mathbb{P}(\mathbb{F}^2) \simeq \mathbb{F} \sqcup \{\infty\}$ so that $|\mathbb{P}(\mathbb{F}^2)| = |\mathbb{F}| + 1$. When $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$, this yields$|\mathbb{P}(\mathbb{F}^2)| = 2+1 = 3$. Alternatively, we view $\mathbb{P}(\mathbb{F}^2)$ as $\mathbb{F}^2 \setminus \{0\}/\sim$ and observe that all equivalence classes have the same size which is $|\mathbb{F} \setminus \{0\}|$ because, for any $v \in \mathbb{F}^2 \setminus \{0\}$, the map $\lambda \mapsto \lambda v$ is a bijection from $\mathbb{F} \setminus \{0\}$ to the equivalence class of v. Since equivalence classes partition, we have $|\mathbb{P}(\mathbb{F}^2)| = |\mathbb{F}^2 \setminus \{0\}|/|\mathbb{F} \setminus \{0\}| = (2^2 - 1)/(2 - 1) = 3$

   (b) Either argue that $\mathbb{P}(\mathbb{F}^3) \simeq \mathbb{F}^2 \sqcup \mathbb{F} \sqcup \{\infty\}$ so that $|\mathbb{P}(\mathbb{F}^3)| = 2^2 + 2 + 1 = 7$ or that $|\mathbb{P}(\mathbb{F}^3)| = |\mathbb{F}^3 \setminus \{0\}|/|F \setminus \{0\}| = (2^3 - 1)/(2 - 1) = 7$. As for the lines in $\mathbb{P}(\mathbb{F}^3)$, a line is determined by any pair of distinct points lying on it. There are $7 \times 6/2$ such pairs and any particular line is equally well determined by any of the 3 pairs of the 3 points on it. Thus there are $7 = 7 \times 6/2 \times 3$ lines in the plane $\mathbb{P}(\mathbb{F}^2)$.

(c) Fix a point, then there are 6 pairs that include that point and each determines a line which is counted twice over: once for each of the other two points on the line. Thus there are $6/2 = 3$ lines through each point.

(d) The same arguments apply for the field with 3 elements: $|\mathbb{P}(\mathbb{F}^2)| = 3 + 1 = 4 = (3^2 - 1)/(3 - 1)$. Similarly, $|\mathbb{P}(\mathbb{F}^3)| = 13$. Meanwhile, each line has 4 points and is determined by any of the 6 pairs of point that lie thereon. In all, there are $13 \times 12/2 = 13 \times 6$ pairs of points in $\mathbb{P}(\mathbb{F}^3)$ and thus 13 lines. Finally, there are 12 pairs that include a given point and each determines a line through that point which is counted 3 times over, giving 4 lines through a given point.

8. Let $\mathbb{P}(V)$ be a projective space of dimension at least 2 (thus $dim V \geq 3$). Prove that:

   (a) Distinct projective lines in $\mathbb{P}(V)$ intersect in at most one point. (Very easy!)

   (b) Distinct projective lines in $\mathbb{P}(V)$ intersect if and only if they lie in some (unique) projective plane.

   (c) Give an example of two non-intersecting projective lines in $\mathbb{P}(\mathbb{R}^4)$.

   Solution: (a) In class we proved that there is a unique line through two distinct points. Thus distinct lines can intersect in at most one point.

   (b) Let $L_i = \mathbb{P}(U_i), i = 1, 2$, be distinct lines in $\mathbb{P}(V)$. Thus each U_i is a 2-dimensional subspace of V and the lines meet if and only if $U_1 \cap U_2 \neq \emptyset$. In this case, we have $1 \leq \dim(U_1 \cap U_2) < \dim U_i = 2$ where the latter inequality is strict since $U_1 \neq U_2$. Thus the lines intersect if and only if $\dim(U_1 \cap U_2) = 1$. In this case,

$$\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2) = 2 + 2 - 1 = 3$$

   Thus, L_1 and L_2 lie in the projective plane $\mathbb{P}(U_1 + U_2)$. The converse, which uses a similar argument is left as an exercise. For uniqueness, any projective plane $\mathbb{P}(W)$ that contains L_1 and L_2 must have $U_i \subseteq W$ and so $U_1 + U_2 \subseteq W$ whence $W = U_1 + U_2$ since both have dimension 3.

   (c) We just need linear subspaces $U_i \subseteq \mathbb{R}^4$ with $U_1 \cap U_2 = \{0\}$. For example, take $U_1 = \{(\lambda_0, \lambda_1, 0, 0),\ \lambda_0, \lambda_1 \in \mathbb{R}\}$ and $U_2 = \{(0, 0, \lambda_2, \lambda_3),\ \lambda_2, \lambda_3 \in \mathbb{R}\}$.

9. Let $\mathbb{P}(V)$ be a projective space and $\mathbb{P}(W_1), \mathbb{P}(W_2) \subset \mathbb{P}(V)$ distinct projective subspaces of complementary dimension: $dim\mathbb{P}(W_1) + dim\mathbb{P}(W_2) = dim\mathbb{P}(V)$. Prove that the $\mathbb{P}(W_i)$ intersect. What does this tell us when $dim\mathbb{P}(V)) = 3$?

   Solution: let $\dim\mathbb{P}(V) = n$ and $\dim\mathbb{P}(W_1) = k$ so that $\dim V = n+1$, $\dim W_1 = k+1$, and $\dim W_2 = nk + 1$. We want to show that $\dim(W_1 \cap W_2) \geq 1$. Now $W_1 + W_2 \subseteq V$ so that

$$n + 1 = \dim V \geq \dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$$

$$= (k + 1) + (n - k + 1) - \dim(W_1 \cap W_2)$$

   and rearranging this gives what we want. When $n = 3$, the only interesting possibility is $k = 1, 2$ and the result asserts that any projective line must intersect any projective plane. It is easy to see that this intersection point is unique $\dim(W_1 \cap W_2) = 1$ or the line lies in the plane $\dim(W_1 \cap W_2) = 2$.

10. Suppose that T and T' define the same projective transformation, then $T = \lambda T'$ for some $\lambda \neq 0$.

    Solution: Suppose $[T'(v)] = [T(v)]$ for all $v \in V \setminus \{0\}$. This implies $T'(v) = \lambda(v)T(v)$ for some non-zero scalar $\lambda(v)$ which may a priori depend on v. We have to show that it does not. So suppose $v, w \in V \setminus \{0\}$. If v, w are linearly dependent, then it is obvious from the definition of $\lambda(v)$ that $\lambda(v) = \lambda(w)$. So assume v, w are linearly independent. Now

    $$T'(v + w) = T'(v) + T'(w) = \lambda(v)T(v) + \lambda(w)T(w)$$

    but also

    $$T'(v + w) = \lambda(v + w)T(v + w) = \lambda(v + w)(T(v) + T(w))$$

    Since $T(v)$ and $T(w)$ are also linearly independent this implies $\lambda(v) = \lambda(v+w) = \lambda(w)$, which completes the proof.