Reconstruction of the refractive index by near-field phaseless imaging

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Abstract. An explicit method is described for reconstruction of the complex refractive index by the method of contrast transfer function

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1 Introduction

New lensless diffractive x-ray technic for micro-scale imaging of biological tissue is based on quantitative phase retrieval schemes. A sample is illuminated by a parallel beam of coherent X-rays; the intensity of the diffracted pattern (hologram) is registered at a plane detector. The distribution of attenuation and refractivity in the sample are to be found. By incorporating refraction, this method yields improved contrast compared to purely absorption-based radiography but involves a phase retrieval problem. The linearized version of this problem is known as the contrast transfer function model (CTF). This model is applied to thin optically weak objects and the Helmholtz equation is replaced by the paraxial approximation. Maretzke and Hohage [1] stated Lipschitz stability of reconstruction of objects with compact support. Uniqueness of this problem was shown earlier by Maretzke [4]. The Newton's method was applied for reconstruction in [2]. See more references on the subject in [1], [2]. The method of phase contrast imaging can be applied for resolving the refractive index of an unknown object in three dimensions [3]. In the paper [6] the near-field phase retrieval problem has been proven to be uniquely solvable for general compactly supported objects, given at least two independent intensity patterns recorded at different detector distances or incident wavelengths.

Here a theoretically exact analytic method is proposed for reconstruction of the ray integral complex refraction index of the object from the linearized intensity obtained from one intensity pattern.

2 CTF-model

The ray integral

$$\mu + i\varphi = k\int (\beta + i\delta)\,\mathrm{d}z$$

satisfies equation

$$T(\mu + i\varphi) = -2F^{-1}\left[\sin\left(\frac{|\xi|^2}{2\mathfrak{f}}\right)F(\varphi) - \cos\left(\frac{|\xi|^2}{2\mathfrak{f}}\right)F(\mu)\right], \tag{1}$$

where $\mathbf{n} = 1 - \delta + i\beta$ is the dimensionless refractive index of the object, β is the attenuation function; z is the coordinate along the central ray, k is the space frequency. The Fresnel number $\mathfrak{f}$ is calculated by

$$\mathfrak{f} = \frac{kb^2}{d},$$

where $z = d$ is distance to the detector, b is the diameter of the disc Ω containing the support of $\psi \doteqdot \mu + i\varphi$. The observable intensity I is related to the image ψ by the nonlinear forward operator

$$I = |\exp(\mathcal{D}\psi)|\,,$$

where $\mathcal{D}$ is the Fresnel propagator generating the near field hologram. According to Paganin [5]

$$\mathcal{D}(\psi) \doteqdot \exp(ikd)\,F^{-1}(m_{\mathfrak{f}}F(\psi));\;\; m_{\mathfrak{f}}(\xi) \doteqdot \exp\left(\frac{i\,|\xi|^2}{2\mathfrak{f}}\right)$$

and $T = 2\,\mathrm{Re}\,\mathcal{D}(\psi)$ is the linearization of $I = |\exp(\mathcal{D}(\psi))|$. The function ϕ and μ are uniquely determined from (1) [4] and the norm of the inverse operator T^{-1} defined in $L_2(\Omega)$ is estimated by an exponential function of the Fresnel number according to [1].

3 The reconstruction method

Theorem 3.1 *Functions φ and μ with compact support can be found analytically from*

$$\frac{1}{2}\hat{T}(\psi) = -\sin\left(\frac{|\xi|^2}{2\mathfrak{f}}\right)\hat{\varphi} + \cos\left(\frac{|\xi|^2}{2\mathfrak{f}}\right)\hat{\mu} \tag{2}$$

Proof. Here $\hat{\varphi} = F(\varphi)$ denote the Fourier transform with the kernel $\exp(-i\langle x,\xi\rangle)$ of a function φ on $\mathbb{R}^2$. Suppose that φ and μ are supported in the central disc Ω of radius b. To portray Ω as the central disc of radius π, we introduce the coordinates $y_i = bx_i/2\pi,\ i = 1,2$. We have $|\xi|^2/2\mathfrak{f} = \eta^2/g$, where $\eta_i = 2\pi\xi_i/b,\ i = 1,2$ are the corresponding coordinates on the frequency plane and $g = 8\pi^2 b^{-2}\mathfrak{f}$.

Lemma 3.2 *For any $k > \pi g$, there exists a root $t_k > 0$ of $\sin(\lambda^2/g)$ such that $|t_k - k| \le c < 1/4$ for some c.*

Proof. The points $\sigma_j = \sqrt{\pi g j},\ j = 0, \pm1, \pm2, ...$ are roots of this function. It is easy to see that for $j \ge \pi g$,

$$\sigma_{j+1} - \sigma_j \le \rho < 1/2,\ \ \rho = 1/(2 + 1/\pi g).$$

Therefore for any $k > \pi g$, there exists a root $\sigma_{j(k)} \in (k - \rho/2, k + \rho/2)$. We set $\lambda_k = \sigma_{j(k)}$. ▶

Set $k(\mathfrak{f}) = [\pi g]$ and define function

$$S_{\mathfrak{f}}(t^2) = \frac{\sin(\varepsilon t)}{t}\prod\nolimits_{j=1}^{k(\mathfrak{f})}\frac{t^2 - j^2}{t^2 - (j\pi/\varepsilon)^2}. \tag{3}$$

The right hand side is a even function of t; it is bounded on the real axis and has holomorphic continuation to the whole space since $\sin(\varepsilon t)$ vanishes on roots of denominator. Therefore it has holomorphic continuation $S_{\mathfrak{f}}(\lambda^2)$ to the complex plane which satisfies

$$\left|S_{\mathfrak{f}}(\lambda^2)\right| \le C\exp(\varepsilon\,|\mathrm{Im}\,\lambda|). \tag{4}$$

Choose ε is so small that the supports of φ and μ are contained in the disc of radius $\pi - \varepsilon\pi$. For an arbitrary unit vector $\theta \in \mathbb{R}^2$, we evaluate both sides of (2) at $\eta = t_k\theta$ for $k > k(\mathfrak{f})$. The first term vanishes and the cosine factor is equal ± 1 hence

$$\hat{\mu}(\pm t_k\theta) = (-1)^{k+1}\frac{1}{2}\hat{T}(\psi)(\pm t_k\theta), \; k > k(\mathfrak{f}). \tag{5}$$

The function

$$M(\eta) = S_{\mathfrak{f}}\left(\eta^2\right)\hat{\mu}(\eta)$$

belongs to $L_2(\mathbb{R}^2)$ since $S_{\mathfrak{f}}$ is bounded on the real line. On the other hand by the Paley-Wiener theorem $\hat{\mu}$ has holomorphic continuation which fulfils

$$|\hat{\mu}(\zeta)| \leq C\exp((1-\varepsilon)|\mathrm{Im}\,\zeta|), \zeta \in \mathbb{C}$$

and by (4)

$$|M(\zeta)| = \left|S_{\mathfrak{f}}\left(\zeta^2\right)\hat{\mu}(\zeta)\right| \leq C\exp(|\mathrm{Im}\,\zeta|).$$

For any unit vector θ, function $f(t) = M(t\theta)$ fulfils $|f(t)| = O(\exp|\mathrm{Im}\,t|)$ and belongs to $L_2(\mathbb{R})$ since the factor $S_{\mathfrak{f}}(\zeta^2)$ is bounded on the real axis. Therefore $\hat{f}$ is supported in $[-\pi, \pi]$ and Theorem 5.1 (Sect.5) can be applied to f and to the sampling of knots $t_j = j$ for $j \leq k(\mathfrak{f})$ and t_j as in Lemma 3.2 for $j > k(\mathfrak{f})$. The values of $f(t_j) = M(t_j\theta)$ are calculated by (5) for $j > k(\mathfrak{f})$ and for $j = 0$; we set $f(t_j) = 0$ for $j = 1, ..., k(\mathfrak{f})$. Function $M(t\theta)$ is reconstructed by means of the series (7) that converges in $L_2(\mathbb{R})$ for any θ. Finally we get

$$\mu(x) = F^{-1}_{\eta\mapsto y}\left(\frac{M(\eta)}{S_{\mathfrak{f}}(\eta^2)}\right)\Bigg| y = \frac{b}{2\pi}x \tag{6}$$

The function φ is recovered in the same way by interpolation with knots at roots of the cosine factor. ►

4 Conclusion

The above arguments can be formulated as an algorithm:

Precomputations: choose a natural N that show the range of radial frequencies of φ and μ to be reconstructed. For given Fresnel number $\mathfrak{f}$, to choose the sequence of real numbers λ_k, $k = k(\mathfrak{f}) + 1, ..., N$ by the method of to Lemma 3.2 completed by $t_k = 0, 1, ..., k(\mathfrak{f})$ for $k \leq k(\mathfrak{f})$. Values $S_{\mathfrak{f}}(\lambda_k)$ are to be calculated for $k = k(\mathfrak{f}) + 1, ..., N$. Next the function G is to be calculated taking a finite product as in (8).

Algorithm: first step: for a unit vector θ, formula (7) is applied for values

$$f(\pm t_k) = (-1)^{k+1} \frac{1}{2} S_{\mathfrak{f}}\left(t_k^2\right) \hat{T}(\psi)(\pm t_k \theta), \ k = 0, k(\mathfrak{f}), ..., N$$

and for values $f(t_k) = 0$ for $0 < k \le k(\mathfrak{f})$. The construction is repeated for θ in a sampling in the unit circle sufficient for the chosen range of angular frequency

Second step: the function $M(\lambda\theta)$ obtained on the previous step is substituted in (6).

Third step: a similar procedure is applied for determination of φ by means of interpolation with knots t_k satisfying $|t_k - k - 1/2| \le c < 1/4$.

5 Appendix

Theorem 5.1 *If* $|t_k - k| \le c < 1/4, \ k \in \mathbb{Z}$ *for some* c *then for any function* $f \in L_2(\mathbb{R})$ *such that* $\hat{f}$ *is supported in* $[-\pi, \pi]$ *can be interpolated by*

$$f(t) = \sum_{-\infty}^{\infty} f(t_k) \frac{G(t)}{(t - t_k) G'(t_k)}, \tag{7}$$

where

$$G(t) = (t - t_0) \prod_{k=1}^{\infty} \left(1 - \frac{t}{t_k}\right)\left(1 - \frac{t}{t_{-k}}\right). \tag{8}$$

Series (7) converges in $L_2(\mathbb{R})$ *and the functions*

$$G_k(t) \div \frac{G(t)}{(t - t_k) G'(t_k)}, \quad k \in \mathbb{Z}$$

forms a Riesz basis in the space $F(L_2[-\pi, \pi])$. *It follows the series (7) converges in* $L_2(\mathbb{R})$ *for any sequence* $\{f(t_k)\} \in L_2$. *The constant* $1/4$ *is maximal possible.*

This result follows from famous "1/4-theorem" of Mikhail Kadec' (Kadets) see [8], [7]. For any constant $c < 1/\pi^2$ this property was stated by Paley and Wiener of 1934.

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