Some recent developments in the Radon transform theory

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November 6, 2017

1 Introduction

In this survey several papers related to the Radon transform appeared after 2013 are discussed. For the short survey of the first 100 years of the history of the theory, I refer to historical notes in the books of Helgason [7] Natterer [5] and in [50]. Pioneering papers of A. Cormack 1963-64 on "Representation of a function by its line integrals" were devoted to medical applications [3]. Few years later he wrote that "his first mathematical problem had been already solved by Radon's theory". These papers triggered numerous applications of the Radon transform to inverse problems. A fragment of the prehistory of these applications was described by A. Cormack [4]: "In 1906 H. B. A. Bockwinkel used a reconstruction formula of H. A. Lorentz in a paper on propagation of light in biaxial crystals. Lorentz's result was generalized by G. Uhlenbeck in 1925. In 1936 H. Cramér and H. Wold proved their theorem on marginal distribution (which is widely used in probability theory). Also in 1936 V. A. Ambartsumyan found reconstruction procedures and used these to calculate the distribution of velocities of stars from their radial velocities in various directions. This is the first numerical inversion of the Radon transform and it gives the lie to the often made statement that computed tomography would be impossible without computers." In 1947 J. Szarski and T. Ważewski presented a reconstruction procedure by starting from an elementary method which the Polish physician M. S. Majerek 1932 used for the reconstruction of the human head from X-ray pictures of it taken from various directions. R. N. Bracewell ran into Radon's problem investigating the sun about 1956."

2 The classical Radon transform

Helgason-Ludwig conditions

Ruhlandt *et al* [8] describe an iterative algorithm of reconstruction of parameters of 3D object illuminated by a monochromatic high frequency plane wave whose phaseless near-field is detected. The back propagation provides a rude approximation at each step of the algorithm. Helgason-Ludwig consistency conditions are used to improve reconstruction of low frequencies of images taken from different angles. This helps to retriev the phase and improve evaluation of the attenuation and refraction coefficients of the object.

Photoacoustic reconstructions

Elbau and Scherzer [9] have considered a model for photoacoustic plane sectional imaging with integrating half-cylinder shaped acoustic detector serving as focusing detector. ?Haltmeier et al [11] analyzed reconstruction in thermoacoustic tomography with a non standard acquisition geometry. The integral data is registered on many planes tangent to boundary of spherical cavity as a short time series. They obtain approximate reconstruction formulas for the case of constant acoustic speed.

Reconstruction formulae for photoacoustic transform are known for the center sets which are spheres, ellipsoids or more complicated algebraic compact sets Z; a survey is given in [50]. Fawcett [23] and Anderssen [24] considered the problem for the central set $Z = \{x_1 = 0\}$ and functions f supported in the half-space. This acquisition geometry is significant for the SAR technic. Fawcett's method was based on the backprojection of the function $g = \mathrm{R} f$ where $g\left(y, r\right)$ is the spherical integral a function f with the center $y \in Z$ of radius r. The backprojection is given by the improper integral

$$\mathrm{R}^* g\left(x\right) = \int_Z g\left(y, \left|x - y\right|\right) \mathrm{d}y.$$

This integral diverges for any x such that $f\left(x\right) > 0$ if f is continuous and non-negative since $g\left(y, \left|x - y\right|\right)$ has the positive limit as $y \to \infty$. Therefore the backprojection does not exists on an open set. Andersson's method is based on the Fourier transform of the function g. This method is also not complete since the Fourier transfrom $F\left(g\right)$ is not a point function and further steps in [24] have to be approved.

Haltmeier and Perversyev [13], [14] stated reconstruction for non compact second order hypersurfaces Z that can be approximated by ellipsoids. The authors apply the reconstruction formula for ellipsoids that was known since([6] and [46]). The problem of convergence of the improper integrals is not in the focus in this papers. The case of elliptic cylinders Z was addressed again by Haltmeier and Moon in [15]. The authors apply the reconstruction method of BPF type (backprojection-filtration) which does not work since the backprojection is not well defined. We show in the Addendum that the FBP method is applicable for Fawcett-Andersson's case $Z = \{x_1 = 0\}$ as well as for any quadratic Z.

Ray transform of tensor fields

Evaluation of residual elastic strain is used for study of structure of complicated materials. The strain tensor has six components in three dimensions.

Tomographic approach is applied for the reconstruction of small residual strain fields in a body from data of diffraction patterns under penetrated x-ray or neutron radiation. The mathematical model is the longitudinal (axial) line transform $X\varepsilon$ of a strain tensor ε. This data is gauge invariant since all the integrals vanish if ε is a potential tensor; that is, $\varepsilon = Du$ for small deformations u.

The polarization tomography is another method of reconstruction of a strain field in a transparent solid. It is based on measurements of transformation of the polarization ellipse of the penetrating light. The mathematical model is the line integral transform $T\varepsilon$ of the traceless normal part of the stress field ε. An analytic algorithm of complete reconstruction of an arbitrary strain tensor ε from non-redundant data of ray integrals $X\varepsilon$ and $T\varepsilon$ will be described.

A series of paper was published on reconstruction of vector plane fields from ray or line integrals. An exact reconstruction was done in [1] from data of all line integrals. Denisjuk [17] considered reconstruction of the solenoidal part of a tensor field of arbitrary degree from ray integrals i9n a flat space. Sharafutdinov [18] applied exact reconstruction from integral data for lines parallel to one of three plane in general position. According to [44] data of first derivatives of the line integrals are sufficient for stable reconstruction if the lines meet the source curve Γ satisfying Tuy's condition. Similar results were obtained later in [20] and [21]. The authors impose a different geometric condition: any plane P in R^3 that meet the support of the unknown function must have at least 3 common points with Γ. According to [44] only one point in $P \cap \Gamma$ is necessary where 3 first order derivatives of the ray data are known. Note that a line integral of vector field depends on the euclidean structure of the space whereas the integral of one form does not. The number 3 is minimal anyway since the first order differential form has 3 components. Some improvements to the method are done in [22].

Paternain, Salo and Uhlmann [16] have shown that on simple Riemannian surface the geodesic axial transform acting on solenoidal symmetric tensor fields of arbitrary order is injective. It is shown in [48] that in the Euclidean 3-space the strain 2-tensor can be reconstructed from data of the longitudinal and of the traceless normal integrals. Sharafutdinov [19] develops the theory of magneto-potoelasticity for determination of dielectric tensor ε of a medium by application of the exterior magnetic field. In the linear approximation, the traceless transversal part of ε is evaluated in terms of the integrals of the gyration field along rays corresponding to the refraction coefficient of the medium. This tensor ε may be related to the tensor of mechanical stress.

Cone Radon transform

The cone of rotation in an Euclidean space E^n can be written in the form

$$C(\lambda) = \{x \in E^n : \lambda x_1 = r\}, \ \lambda > 0, \ r^2 = x_2^2 + ... + x_n^2$$

in a coordinate system. The line $r = 0$ is the axis and $\theta \doteqdot \arctan \lambda$ is the half-opening of the cone. The integral operator

$$\mathrm{D}_{\lambda,k}f\left(y\right)=\int_{x\in C(\lambda)}f\left(y+x\right)\left|x\right|^{-k}\mathrm{d}S,\ y\in E^{n}$$

is discussed in recent publications under the name of *cone Radon* or *cone* transform. Here $\mathrm{d}S$ is the Euclidean hypersurface element. This transform is called regular in the case $k<n-1$ and singular in the case $k=n-1$. The realistic model of point spread function for single-scattering optical 3D tomography is based on the photometric law of scattered radiation modelled by the singular cone transform.

This method provides multiple views of the object which can be considered as a Radon-type cone transform of the photon source distribution. Cree and Bones [28] proposed reconstruction from data of the regular cone transform on R^3 with apices restricted to a plane orthogonal to the axis and all openings. Analytic reconstructions from the regular cone transform with restricted apex were obtained by Nguyen and Truong [30], Smith [31], Grangeat *et al* [32], Maxim *et al* [33], Maxim [34], Haltmeier [25], Terzioglu [37], Kuchment and Terziogly [38], Moon [40]. Papers [33], Maxim [34] contains representative numerical reconstructions. These reconstructions are based on the reduction of redundant data of cone integrals for all openings. These data is converted to the Radon transform in several ways e. g.

$$\int_{0}^{\pi}\mathrm{D}_{\lambda,0}f\mathrm{d}\theta=c_{n}\mathrm{R}^{*}\mathrm{R}f$$

where R^* is the back projection operator for the Radon transform R. Jung and Moon [39] obtained inversion formulae for the regular cone transform on R^n using non redundant data from a line of detectors and rotating axis. In [26] and [27] other reconstruction methods were proposed based on the integral data from all cones with apices on a sphere and all openings.

Basko *et al* [29] proposed a numerical method based on developing the unknown function into spherical harmonics from cone integrals with swinging axis and only one opening. Gouia-Zarrad and Ambartsoumian [36], [35] found the reconstruction formula for the regular cone transform on a half-space with free apex and one opening. This paper contains new reconstruction formulas for reconstruction of an unknown density function on the half-space from its weighted integrals over cones with constant axis and apex running the half-space. In the regular case (the weight density is integrable over the cone) the cone transform is the convolution with a distribution supported by the cone. In [51] a reconstruction from a regular cone transform $\mathrm{D}_{\lambda,k}$ with constant axis and opening on R^3 is given in terms of the operator $\mathrm{D}_{\lambda,1-k}$. This approach does not work when the weight density is not integrable. Then the cone integral data can not be collected for all positions of the apex. A more complicated reconstruction method is proposed in [51] from data of cone integrals with apices running a 1D set. The inversion formula for the nongeodesic Funk transform is applied (see below).

3 The general Funk-Radon transform

There is a variety problems of integral geometry that have similar features with the classical Funk and Radon transforms. A systemization of this variety can be done using the term geometric integral transforms. Let X be a smooth manifold of dimension $n > 1$ and $\mathrm{d}X$ be a volume form on X; we call $(X, \mathrm{d}X)$ the physical space. Another smooth manifold Σ of the same dimension n plays tho role of the registration space. Take a smooth real function Φ on $X \times \Sigma$ (called generating function) satisfying the condition:

i) $\mathrm{d}_x\Phi \neq 0$ and $\mathrm{d}_\sigma\Phi \neq 0$ on the set $Z \doteqdot \{\Phi = 0\}$ where $\mathrm{d}_x\Phi + \mathrm{d}_\sigma\Phi = \mathrm{d}\Phi$. We have $\mathrm{d}\Phi = 0$ on Z hence these inequalities are equivalent.

It follows that the set Z is a smooth manifold of dimension $2n - 1$ and $Z_\sigma \doteqdot \{\Phi(\cdot, \sigma) = 0\}$, $Z_x \doteqdot \{\Phi(x, \cdot) = 0\}$ are smooth hypersurfaces in X, respectively in Σ. The projections $p_X : Z \to X$ and $p_\Sigma : Z \to \Sigma$ have rank n. The *geometric integral transform* is defined by

$$\mathrm{M}_\Phi f(\sigma) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_{|\Phi(x,\sigma)| \leq \varepsilon} f \mathrm{d}X$$

for a integrable function f on X with compact support. It is well defined for almost any registration point $\sigma \in \Sigma$.

The classical Radon transform fulfils (**i**) for $X = R^n$, $\Sigma = R \times S^{n-1}$, $\Phi(x; p, \omega) = p - \langle x, \omega \rangle$. The Funk transform ([1] (n=2), Helgason 1959 (even n), Semjanistiy 1961 (odd n)) is obtained for $X = S^n$, $\Sigma = S^n$, $\Phi(x, \sigma) = \langle x, \sigma \rangle$. The relation between Radon transform M_R ([2]) and the Funk transform M_F can be written in the form $\mathrm{M}_R(f)(p, \omega) = \left(1 + p^2\right)^{-1/2} \mathrm{M}_F(g)(\sigma)$, where

$$g\left(\left(1 + |x|^2\right)^{-1/2} x\right) = \left(1 + |x|^2\right)^{n/2} f(x).$$

This relation is extended for the totally geodesic transformation of on hyperbolic space of constant curvature by means of the central projection in E^{n+1} (gnomonic projection).

Let w be a smooth function on $X \times \Sigma$. The Funk-Radon transform with the weight w is defined by the integral

$$\mathrm{M}_{\Phi,\mathrm{w}} f(\sigma) \doteqdot \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_{|\Phi(x,\sigma)| \leq \varepsilon} \mathrm{w}(x, \sigma) f(x) \mathrm{d}X = \int_{Z(\sigma)} \mathrm{w}\delta(\Phi) f \mathrm{d}X.$$

If a function f is supported by a compact K, then $\mathrm{M}_{\Phi,\mathrm{w}} f$ is supported by the set $\Lambda = p_\Sigma p_X^{-1}(K)$ that is also compact, since p_X is a proper map. The propriety of M_Φ holds also for the operator $\mathrm{M}_{\Phi,\mathrm{w}}$ for any smooth weight w.

Properties of the general Funk-Radon transform

Let $\mathcal{E}(X)$ be the space of smooth functions on X and $\mathcal{D}(X)$ be the subspace of smooth functions on X with compact support.

Proposition 1 *If a generating function* Φ *satisfies* (***i***)*, then the backprojection transform* $\mathrm{M}^*:\mathcal{D}(\Sigma)\to\mathcal{E}(X)$ *is well defined by*

$$\mathrm{M}^*_\Phi(\varphi)(x)=\lim\frac{1}{2\varepsilon}\int_{|\Phi|\leq\varepsilon}\varphi\mathrm{d}\Sigma$$

and M_Φ *defines a continuous operator* $\mathcal{E}'(X)\to\mathcal{D}'(\Sigma)$ *by*

$$\mathrm{M}_\Phi(u)(\varphi)=u\left(\mathrm{M}^*_\Phi(\varphi)\right),\ \varphi\in\mathcal{D}(\Sigma).$$

Proposition 2 *For an arbitrary distribution* f *with compact support on* X,

$$\mathrm{WF}(\mathrm{M}_\Phi f)\subset\cup(\sigma,\eta)\in T^*_0(\Sigma):\exists x\in X:\Phi(x,\sigma)=0,\ \eta\|\mathrm{d}_\sigma\Phi(x,\sigma),$$

$$(x,\mathrm{d}_x\Phi(x,\sigma))\in\mathrm{WF}(f).$$

This result is known in several particular cases. Suppose that the only set $\mathrm{WF}(\mathrm{M}_\Phi f)$ is known. Then a point $(x,\xi)\in T^*(X)$ can be recognized as a point in $\mathrm{WF}(f)$ if there exists a point σ such that $\Phi(x,\sigma)=0$, $(\sigma,\mathrm{d}_\sigma\Phi(x,\sigma))\in\mathrm{WF}(\mathrm{M}_\Phi f)$ and $\mathrm{d}_x\Phi(x,\sigma)\parallel\xi$. A point $(x,\xi)\in\mathrm{WF}(f)$ may not be recognized in this way, if there is a point $y\neq x$ such that $\mathrm{d}_x\Phi(y,\sigma)\parallel\xi$.

The following condition is more strong than (**i**)

(**i+**) the map $\mathrm{D}:Z\times(\mathbb{R}\setminus 0)\to T^*(X)\setminus 0$ is a diffeomorphism, where $\mathrm{D}(x,\sigma,t)=(x,t\mathrm{d}_x\Phi(x,\sigma))$. This condition is equivalent to the following. the map

$$\mathrm{D}_S:Z\to S^*(X),\ \mathrm{D}_S(x,\sigma)=\left(x,\frac{\mathrm{d}_x\Phi(x,\sigma)}{|\mathrm{d}_x\Phi(x,\sigma)|_{\mathrm{g}}}\right)$$

is a diffeomorphism where g is an arbitrary Riemannian metric on X and $S^*(X)$ means the bundle of unit spheres in the cotangent bundle. This implies that p_X is a proper and for any $x\in X$, $Z(x)$ is diffeomorphic to a $n-1$-sphere.

Proposition 3 *[45]If* Φ *fulfils* (***i+***)*, then* M_Φ *can be extended to a bounded operator* $\mathrm{M}_\Phi:H^\alpha_K(X)\to H^{\alpha+(n-1)/2}_\Lambda(\Sigma)$ *for any* $\alpha\in\mathbb{R}$*, an arbitrary compact set* $K\subset X$ *and* $\Lambda=p_\Sigma\left(p_X^{-1}(K)\right)$.

Points $x,y\in X$ are called conjugate for a generating function Φ, if $x\neq y$, $\Phi(x,\sigma)=\Phi(y,\sigma)$ and $\mathrm{d}_\sigma\Phi(x,\sigma)\parallel\mathrm{d}_\sigma\Phi(y,\sigma)$ for some $\sigma\in\Sigma$.

We call the generating function Φ *regular,* if it satisfies condition (**i+**) and (**ii**) there are no conjugate points.

This condition under the name "Bolker condition" was introduced by E Quinto [56] in a more general situation.

Homan and Zhou [57] define "Generalized Radon transforms" in a slightly different way by means of the "defining" function $\varphi=\varphi(x,\theta)$ that is positive homogeneous of degree 1 in $\theta\in R^n$. The generating function $\Phi(x,\sigma)=s-\varphi(x,\theta)$ defines the same acquisition geometry for $\sigma=(s,\theta)$. The global Bolker conditions formulated by the authors plays role of "absence of conjugate points". The author focus on injectivity of the weight integral transform that essentially coincides with the operator $\mathrm{M}_{\Phi,\mathrm{w}}$. They show injectivity and stability for an open, dense subset of generalized Radon transforms with analytic defining functions satisfying the Bolker condition. Homan and Zhou [57] define "Generalized Radon transforms" in a slightly different way. To the "defining" function φ corresponds the generating function $\Phi(x,s,\theta) = s - \varphi(x,\theta)$ that is positive homogeneous of degree 1 in θ and $\sigma = (s,\theta)$. The global Bolker conditions formulated by the authors plays role of "absence of conjugate points". The author focus on injectivity of the weight integral transform that essentially coincides with the operator $\mathrm{M}_{\Phi,\mathrm{w}}$. They show injectivity and stability for an open, dense subset of generalized Radon transforms satisfying the Bolker condition.

Parametrix on Sobolev spaces

Let X and Y be compact manifolds with boundaries of class C^{κ}, where κ is a natural number. The Sobolev spaces $H^{\alpha}(X)$ and $H^{\alpha}(Y)$ are well defined for any real $\alpha, |\alpha| < \kappa$.

Definition. We say that a linear operator $A : L_2(X) \to L_2(Y)$ with a dense domain is a Sobolev operator of order $d \in \mathbb{R}$ (or $-d$-smoothing operator if $d < 0$) if it defines a bounded operator $A : H^{\alpha}(X) \to H^{\alpha-d}(Y)$ for any $\alpha,\ |\alpha| < \kappa,\ |\alpha - d| < \kappa$ which is a restriction of A for positive α and a closure of A for negative α.

Definition. For a number $s > 0$, an operator $P : L_2(Y) \to L_2(X)$ is said a s-parametrix for A if it is an Sobolev operator of order $-d$ and

$$R = \mathrm{Id} - PA$$

is a s-smoothing operator (called the remainder). If P_1 is a 1-parametrix and R_1 is a remainder, then for any natural k, a k-parametrix P_k can be found for any natural k recursively by $P_k = P_{k-1} - R_{k-1}P_1,\ R_k = -R_{k-1}R_1$ for $k = 2, ..., \kappa$. Any 1-smoothing operator is compact on $L_2(X)$ hence P_1A is a Fredholm operator and the image of A is closed. A s-parametrix P_s recovers the singularity of an arbitrary function $f \in H^{\alpha}(X)$ from Af up to a function $h = R_s f \in H^{\alpha+s}(X)$. In particular, if $f = \delta_y$ is the delta-function at a point $y \in X$ and $s > n$ then the function $h = R_s\delta_y$ is continuous. In fact, we have $\delta_y \in H^{\alpha}(X)$ for any $\alpha < -n/2$ which implies $h \in H^{\alpha+s}(X)$. The space $H^{\alpha+s}(X)$ is contained in $C(X)$ if we take $\alpha > n/2 - s$ hence h is continuous. The equation $P_sA\delta_y = \delta_y + h$ shows that any delta function can be recognized from data of $A\delta_y$ by means of a s-parametrix P_s.

A parametrix operator recovers not only the wave front of a function f but also the profile of its singularity.

Singular integral operators

Let E^n be a Euclidean space of dimension $n \geq 1$, and let $a(x,s)$ be a locally bounded function on $E^n \times E^n \backslash 0$. Consider an integral transform A defined by

$$Af(x) = \lim_{\varepsilon \to 0} \int_{|s|>\varepsilon} a(x,s)\, f(x+s)\, \mathrm{d}s \tag{1}$$

for functions $f \in L_{2\text{comp}}(E^n)$. Let S be the unit sphere in E^n.

Theorem 4 *Let a_0 be a locally bounded on $E^n \times \mathrm{S}$ and positively homogeneous function of degree $-n$ in variables s fulfilling the condition*

$$\int_{\mathrm{S}} a_0(x,s)\,\mathrm{d}\Omega(s) = 0,\ \ x \in E^n, \tag{2}$$

where $\mathrm{d}\Omega$ is the Euclidean measure on the unit sphere. Then (1) defines a continuous operator $A : L_{2\text{comp}}(E^n) \to L_{2\text{loc}}(E^n)$.

This is a simplified version of the Calderon-Zygmund theorem [43].

Theorem 5 *Let*

$$a(x,s) = a_0(x,s) + r_1(x,s)$$

be a kernel supported on $X \times E^n \backslash 0$ of class C^κ for some natural κ, a_0 is a homogeneous function of s of degree $-n$ satisfying (2) and with remainder r_1 that fulfils

$$\max_{i+j\le\kappa}\max_{x\in X}|s|^{i+n}\left|\nabla_s^i\nabla_x^j r_1(x,s)\right| \le C|s| \tag{3}$$

where C is a constant. Then for any compact set $X \subset E^n$ with a boundary of class C^κ, operator A defined in (1) is a Sobolev operator on $L_2(X)$ of order 0.

Construction of the parametrix

A parametrix for a class of integral operators $\mathrm{M}_{\Phi,\mathrm{w}}$ was constructed by Beylkin [42] in terms of the Fourier integral operators.

An exact inversion of the transform $\mathrm{M}_{\Phi,\mathrm{w}}$ is only known for special types of acquisition geometries Φ see a survey in [50] while a parametrix can be constructed for a wide class of geometries. Pestov and Uhlmann [52] gave a construction of an approximate inversion for the geodesic integral transform on simple Riemannian surfaces. The reconstruction formula of Natterer [6] provides a parametrix for photoacoustic acquisition geometry. The construction of a parametrix for general geometry is described below following [49] with some modifications.

Theorem 6 *Let Φ be a regular generating function and $\mathrm{w} \neq 0$ is a smooth function on $X \times \Sigma$ and $\mathrm{d}X = \mathrm{d}x_1 \wedge ... \wedge \mathrm{d}x_n$ for some coordinate functions $x_1, ..., x_n$ on X. Then the operator*

$$\mathrm{P}_{\Phi,\mathrm{w}}g(x) = \frac{(n-1)!}{\mathrm{j}^n D_n(x)}\int_\Sigma \frac{g(\sigma)}{\mathrm{w}(x,\sigma)\,\Phi(x,\sigma)^n}\Omega_\Phi \wedge \mathrm{d}_\sigma\Phi(x,\sigma) \tag{4}$$

defined for even n and $g \in C^n(\Sigma)$ and

$$\mathrm{P}_{\Phi,\mathrm{w}}g(x) = \frac{1}{2\mathrm{j}^{n-1}D_n(x)}\int_\Sigma \frac{\delta^{(n-1)}(\Phi(x,\sigma))}{\mathrm{w}(x,\sigma)}g(\sigma)\,\Omega_\Phi \wedge \mathrm{d}_\sigma\Phi(x,\sigma) \tag{5}$$

defined for odd n and $g \in C^{n-1}(\Sigma)$ is a parametrix for $M_{\Phi,\mathrm{w}}$, where

$$\begin{aligned} \Omega_\Phi &= \frac{1}{(n-1)!} \nabla \Phi \wedge (\mathrm{d}_\sigma \nabla_x \Phi)^{\wedge(n-1)}, \\ D_n(x) &= \frac{1}{|\mathrm{S}^{n-1}|} \int_{Z(x)} \frac{\Omega_\Phi}{|\nabla \Phi(x,\sigma)|^n}, \end{aligned}$$

and

$$\mathrm{R}_{\Phi,\mathrm{w}} = \mathrm{Id} - \mathrm{P}_{\Phi,\mathrm{w}} \mathrm{M}_{\Phi,\mathrm{w}}$$

is an operator of Sobolev order -1.

The wedge product $(\mathrm{d}_\sigma \nabla_x \Phi)^{\wedge(n-1)}$ is defined by means of the exterior product of n vectors and the wedge product of one-forms. Singular integrals (4),(5) are defined as in [50], A.5.1.

The differential form in (4-5) can be written in the form looking similar to classical inversion integral

$$\Omega_\Phi \wedge \mathrm{d}_\sigma \Phi(x,\sigma) = |\nabla \Phi(x,\sigma)|^n \varphi^*(\Omega_{n-1})$$

where

$$\varphi : \Sigma \to \mathrm{S}^{n-1},\ \varphi(x,\sigma) = \frac{\mathrm{d}_x \Phi(x,\sigma)}{|\mathrm{d}_x \Phi(x,\sigma)|}.$$

In fact, the forms $\varphi^*(\Omega_{n-1})$ and $\mathrm{d}_\sigma \Phi$ are analogous of the angular form $\mathrm{d}\omega$ and the differential $\mathrm{d}p$ in the inverse Radon transform formula, respectively. The appearing factor $|\nabla \Phi|^n$ balances the denominator Φ^n making the quotient calibration independent.

Remark. An explicit inversion of the operator $\mathrm{M}_{\Phi,\mathrm{w}}$ is well known in the case of attenuated Radon transform on a plane that is $\mathrm{w}(x,\sigma)$ is the integral along the line σ of a known function (attenuation coefficient) from infinity to the point x. In the general case existence of a Sobolev parametrix does not imply injectivity of $\mathrm{M}_{\Phi,\mathrm{w}}$.

Stefanov and Uhlmann [54] constructed an approximate time reversal operator for medium with variable sound speed in a compact domain (non trapping). The time reversal is a Fredholm operator with convergent Neumann series. It is not shown that the remainder is a smoothing operator. Ilmavirta [55] studies the reconstruction problem in R^3 for the weighted line transform of unknown field of Hamiltonian 3-matrices. He considers this as the model for tomography of the earth from data of neutrino oscillations emanated by the artificial sources. The author stated uniqueness of reconstruction under simplifying assumptions.

4 Addendum

Photoacoustic reconstruction with non compact centers set

Theorem 7 *Let* Z *be the zero set of an elliptic second order polynomial* p *in* E^n. *Any function* $f \in C^{n-1}(E^n)$ *with compact sup in* $\mathrm{H} = \{p < 0\}$ *can be reconstructed by the formula*

$$f(x) = \mathrm{j}^{1-n} p(x) \int_{\mathrm{Z}} \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{n-1} \frac{\mathrm{R}f(r,\xi)}{r}\Bigg|_{r=|x-\xi|} \frac{\mathrm{d}\xi}{\mathrm{d}p} \tag{6}$$

for odd n, *and by*

$$f(x) = 2\mathrm{j}^{-n} p(x) \int_{\mathrm{Z}} \frac{\mathrm{d}\xi}{\mathrm{d}p} \int_0^\infty \frac{\mathrm{d}r^2}{|x-\xi|^2 - r^2} \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{n-1} \frac{\mathrm{R}f(r,\xi)}{r} \tag{7}$$

for even n *from data of spherical integrals*

$$\mathrm{R}f(r,\xi) = \int_{|x-\xi|=r} f(x)\,\mathrm{d}S, \ \ \xi \in \mathrm{Z}. \tag{8}$$

A proof is given in [50], p.102.

Corollary 8 *Reconstructions (7) and (6) hold for any second order polynomial* p *on* E^n *that is non negative on* E^n_+.

Proof. For simplicity we assume that $p = x_1$ and take the sequence of polynomials

$$p_\varepsilon = \varepsilon |x|^2 - 2x_1$$

The cavity $\mathrm{H}_\varepsilon = \{p_\varepsilon < 0\}$ is the ball of radius $\varepsilon^{-1/2}$ in the upper half-space E^n_+ with the center $x_1 = \varepsilon^{-1/2}$... The sphere $\mathrm{Z}_\varepsilon = \{p_\varepsilon = 0\}$ is contaned in E_+ and contains the origin. We have $p_\varepsilon \to p_0 = -2x_1$ and $\mathrm{Z}_\varepsilon \to \mathrm{Z}_0 = \{p_0 = 0\}$ uniformly on any compact set in E^n_+ as $\varepsilon \to 0$, and Z_0 is boundary of $\mathrm{H}_0 = E_+$. We apply Theorem 7 to p_ε and show that the integral has the limit as $\varepsilon \to 0$.We have

$$\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{n-1} \frac{\mathrm{R}f(r,\xi)}{r} = \frac{\partial_r^n \mathrm{R}f(r,\xi)}{r^n} + (n-1)\frac{\partial_r^{n-1}\mathrm{R}f(r,\xi)}{r^{n+1}} + ... + c_n \frac{\mathrm{R}f(r,\xi)}{r^{2n-1}}$$

For any function $f \in C^n$ with compact support, we have by (8)

$$\left|\partial_r^k \mathrm{R}f(r,\xi)\right| \le C_1 \min(r,1)^{n-1-k}, \ \ k \le n-1$$

where C_1 does not depend on r and $\xi \in \mathrm{Z}$. Therefore for $r = |x-\xi|$,

$$\left|\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{n-1} \frac{\mathrm{R}f(r,\xi)}{r}\right| \le \frac{C_2}{(r+1)^n} = \frac{C_2}{(|x-\xi|+1)^n} \tag{9}$$

for some constant C_2. This shows that the integrand in (6) for $p = p_0$ converges absolutely uniformly for $\xi \in E_+$, $r > 0$ and $x \in \mathrm{supp} f$. The spheres Z_ε are convex, contained in H_0. We have $\mathrm{Z}_\varepsilon \to \mathrm{Z}_0$ in the obvious geometric sense. This together with (9) guarantee that the limit in (6) as $\varepsilon \to 0$ can be done under integral and tends to the volume form $\mathrm{d}\xi/\mathrm{d}p_0$ uniformly. We can do the same in the reconstruction (7). ►

Exact inversion of the geometric integral transform

Let X and Σ be manifolds of dimension n with volume forms $\mathrm{d}X$ and $\mathrm{d}\Sigma$ and $\Phi = \Phi(x,\sigma)$ be a regular generating function defined on $X \times \Sigma$. The singular integral

$$Q_n(x,y) = \int_{Z(y)} (\Phi(x,\sigma) - i0)^{-n} \frac{\mathrm{d}\Sigma}{\mathrm{d}_\sigma \Phi(y,\sigma)} \tag{10}$$

is well defined for any $x, y \in X$, $y \neq x$ and is a continuous function. We have

$$\begin{aligned} \operatorname{Re} Q_n(x,y) &= \int_{Z(y)} \Phi(x,\sigma)^{-n} \frac{\mathrm{d}\Sigma}{\mathrm{d}_\sigma \Phi}, \\ \operatorname{Im} Q_n(x,y) &= \frac{\pi}{(n-1)!} \int_{Z(y)} \delta^{(n-1)}(\Phi(x,\sigma)) \frac{\mathrm{d}\Sigma}{\mathrm{d}_\sigma \Phi}. \end{aligned}$$

Theorem 9 *Suppose that $\mathrm{d}X = \mathrm{d}_{\mathrm{g}} X$ is the volume form of a Riemannian metric g on X and Φ is a regular generating function Φ satisfying condition **(iii)**:*

$$\operatorname{Re} i^n Q_n(x,y) = 0 \text{ for all } x, y \in X \text{ such that } x \neq y. \tag{11}$$

Then an arbitrary function $f \in C^n(X)$ with compact support can be reconstructed from data of $\mathrm{M}_\Phi f$ by the formula:

$$f(x) = \frac{(n-1)!}{\mathrm{j}^n D_n(x)} \int_\Sigma \frac{\mathrm{M}_\Phi f(\sigma)\,\mathrm{d}\Sigma}{\Phi(x,\sigma)^n} \tag{12}$$

for even n, and by

$$f(x) = \frac{1}{2\mathrm{j}^{n-1} D_n(x)} \int_\Sigma \delta^{(n-1)}(\Phi(x,\sigma))\, \mathrm{M}_\Phi f(\sigma)\,\mathrm{d}\Sigma \tag{13}$$

for odd n where

$$D_n(x) = \frac{1}{|\mathrm{S}^{n-1}|} \int_{Z(x)} \frac{1}{|\mathrm{d}_x \Phi(x,\sigma)|_{\mathrm{g}}^n} \frac{\mathrm{d}\Sigma}{\mathrm{d}_\sigma \Phi(x,\sigma)}. \tag{14}$$

The integral (12) or (13) converges to f uniformly on any compact set $K \subset X$.

Hyperplane sections of an ovaloid

Let X be an ovaloid in the Euclidean space E^{n+1} that is a compact smooth convex hypersurface. Let Σ be a ellipsoid contained in the interior of X. It can be defined by the equation $q(\sigma) = 1,\ \sigma \in E^{n+1}$ where q is a second order polynomial on E^{n+1} with positive principal part q_2. Any hyperplane H tangent to Σ can be written in the form $H = \{x; \langle x - \sigma, \nabla q(\sigma)\rangle = 0\}$ for some $\sigma = H \cap \Sigma$. The intersection $X \cap H$ is a smooth $n-1$ manifold. It can be defined by the generating function

$$\Phi(x, \sigma) = \langle x - e, \nabla q(\sigma)\rangle - r,\ x \in X,\ \sigma \in \Sigma$$

where e is the center of Σ and $r = 2 - 2q(e)$. For $\sigma = \tau + e$, wed have $q(\sigma) = q_2(\tau) + q(e)$ since the linear part $q_1(\tau)$ of q vanishes. Therefore $q_2(\tau) = q(\sigma) - q(e)$ and by the Euler identity we have

$$\langle \sigma - e, \nabla q(\sigma)\rangle = \langle \tau, \nabla q_2(\tau)\rangle = 2q_2(\tau) = 2(q(\sigma) - q(e)) = 2 - 2q(e) = r$$

hence $\langle x - e, \nabla q(\sigma)\rangle - r = \langle x - \sigma, \nabla q(\sigma)\rangle$ for any $\sigma \in \Sigma$.

Let g be a Riemannian metric on X. The integral transform M_Φ generated by Φ and by the Riemannian volume form $\mathrm{d}_\mathrm{g} X$ is

$$\mathrm{M}_\Phi f(\sigma) = \int_{Z(\sigma)} \frac{f \mathrm{d}_\mathrm{g} X}{\mathrm{d}_x \Phi}.$$

where $\mathrm{d}_x\Phi = \langle \nabla q(\sigma), \mathrm{d}x\rangle$. We take the quotient $\mathrm{d}\Sigma = \mathrm{d}V/\mathrm{d}q$ as the volume form on Σ, where $\mathrm{d}V$ is the volume form on E^{n+1}.

Theorem 10 *Let X, Σ and Φ be as above. Let $\sigma_0 \in \Sigma$ and $Y = \{x : \Phi(x, \sigma_0) > 0\}$. For any odd $n \geq 3$, an arbitrary function $f \in \mathcal{C}_0^{n-1}(X)$ supported by Y can be recovered from $\mathrm{M}_\Phi f$ by*

$$f(x) = \frac{1}{2\mathrm{j}^{n-1} D_n(x)} \int_\Sigma \delta^{(n-1)}(\Phi(x, \sigma)) \mathrm{M}_\Phi f(\sigma) \mathrm{d}\Sigma. \tag{15}$$

For any even n, any $f \in \mathcal{C}_0^n(X)$ supported by Y can be reconstructed by

$$f(x) = \frac{(n-1)!}{\mathrm{j}^n D_n(x)} \int_{\mathrm{S}^n} \frac{\mathrm{M}_\Phi f(\sigma)}{\Phi(x, \sigma)^n} \mathrm{d}\Sigma, \tag{16}$$

where for any n,

$$D_n(x) = \frac{1}{|\mathrm{S}^{n-1}|} \int_{Z(x)} \frac{1}{|\mathrm{d}_x \Phi|_\mathrm{g}^n} \frac{\mathrm{d}\Sigma}{\mathrm{d}_\sigma \Phi} \tag{17}$$

where $Z(x) = \{\sigma;\ \Phi(x, \sigma) = 0\}$.

Proof. We check that Φ satisfies conditions (**i+**), (**ii**) and (**iii**) of Theorem 9. We have

$$\partial_x \Phi = \nabla q(\sigma),\ \partial_\sigma \Phi = \left\langle x - e, \nabla^2 q(\sigma)\right\rangle,\ \partial_x \partial_\sigma \Phi = \nabla^2 q(\sigma).$$

Let $t = (t_x, t_0)$ be a $n+1$-vector such that $tJ = 0$. We have

$$\langle t_x, \nabla q(\sigma)\rangle = 0,\ \left\langle t_x + t_0(x - e), \nabla^2 q(\sigma) \mathrm{d}\sigma\right\rangle = 0.$$

The first equation means that t_x is tangent to Σ. The second one yields $t_x + t_0(x - e) = 0$ since the quadratic form $\nabla^2 q(\sigma)$ is not singular. This equation is implies $t_x = 0$ and $t_0 = 0$ since t_x is tangent to Σ and the vector $x - e$ is not. This proves (**i+**). Condition(**ii**) is easy to check. To verify (**iii**)**,** we have to show that the integral

$$Q_n(x, y) = \operatorname{Re} i^n \int_{Z(y)} \frac{1}{(\Phi(x, \sigma) - i0)^n} \frac{\mathrm{d}\Sigma}{\langle y - e, \mathrm{d}\sigma \rangle} \tag{18}$$

vanishes for arbitrary $x, y \in X_t,\ y \neq x$. Note that this integral does not depend on X. The set $Z(y)$ is an ellipsoid of dimension $n - 1 > 0$. Check that the linear function $\Phi(x, \sigma)$ has a zero on $Z(y)$. Let P be the plane through x, y and e. The line L through y and x does not touch Σ since $x, y \in X_\tau$ see the picture below Let $L_1, L_2 \subset P$ be the rays started from y tangent to the ellipse $P \cap \Sigma$ at σ_1 and σ_2, respectively. The vectors $n_1 = \nabla q(\sigma_1),\ n_2 = \nabla q(\sigma_2)$ are the exterior normals. Function $\langle x - y, \nabla q(\sigma) \rangle$ is continuous on this set $Z(y)$ and its values $\langle x - y, \nabla q(\sigma_1) \rangle > 0$ and $\langle x - y, \nabla q(\sigma_2) \rangle < 0$ have different signs see pict.1. Therefore this function has a zero σ_0 on $Z(y)$. It follows $\Phi(x, \sigma_0) = \Phi(y, \sigma_0) + \langle x - y, \sigma_0 \rangle = 0$, hence $\Phi(x, \sigma_0)$ also has a zero. The integral (18) is taken over the $n-1$-dimensional ellipsoid $Z(y) = \{\sigma \in \Sigma;\ \Phi(y, \sigma) = 0\}$ against volume form

$$\frac{\mathrm{d}\Sigma}{\langle y - e, \mathrm{d}\sigma \rangle} = \frac{\mathrm{d}V}{\mathrm{d}q \wedge \langle y - e, \mathrm{d}\sigma \rangle}.$$

Change coordinates $\sigma = A\xi + e$ on E^{n+1} where A is a constant matrix such that $q(A\xi + e) = |\xi|^2$. Then $\mathrm{d}V(A\xi + e) = \mathrm{const}\ \mathrm{d}\xi_1 \wedge ... \wedge \mathrm{d}\xi_{n+1},\ \mathrm{d}q = 2\xi \mathrm{d}\xi$ and $\langle y - e, \mathrm{d}\sigma \rangle = \langle s, \mathrm{d}\xi \rangle$ for some constant $s \in E^{n+1}$ hence the volume form in the right side of (18) is equal to

$$\mathrm{const} \frac{\mathrm{d}V}{\xi \mathrm{d}\xi \wedge \langle s, \mathrm{d}\xi \rangle} = \mathrm{const} \frac{\Omega_n}{\langle s, \mathrm{d}\xi \rangle} = \mathrm{const}\ \Omega_{n-1}$$

where Ω_k denotes the volume form of the euclidean k-sphere. The function $\Phi(x, \sigma)$ is a linear function on the sphere S^{n-1} that changes its sign. By [50],Theorem A.20 $Q_n(x, y) = 0$ for $x \neq y$ Q.E.,D. ▶

Remarks. Convexity property of X can be be replaced by the condition $\mathrm{d}_x \Phi(x, \sigma) \mid T(X) \neq 0$ for any point (x, σ) such that $\Phi(x, \sigma) = 0$.

The analogon of this theorem holds for any ellipsoid Σ contained in the exterior part of a strictly convex ovaloid X. Let X_+ be set of points $x \in X$ such that the tangent plane T_x separates X and Σ except occasionally tangency points. The reconstructions (15) and (16) are true for any function f supported by X_+.

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Proposition 11 *Map* D *is a local diffeomorphism if and only if* $\det J_{x,\sigma}(\Phi) \neq 0$ *on* Z, *where*

$$\mathrm{J}_{x,\sigma}(\Phi) = \begin{pmatrix} \frac{\partial\Phi}{\partial x_1} & \frac{\partial^2\Phi}{\partial x_1\partial\sigma_1} & \cdots & \frac{\partial^2\Phi}{\partial x_1\partial\sigma_n} \\ \cdots & \cdots & & \cdots \\ \frac{\partial\Phi}{\partial x_n} & \frac{\partial^2\Phi}{\partial x_n\partial\sigma_1} & \cdots & \frac{\partial^2\Phi}{\partial x_n\partial\sigma_n} \\ 0 & \frac{\partial\Phi}{\partial\sigma_1} & \cdots & \frac{\partial\Phi}{\partial\sigma_n} \end{pmatrix} \tag{19}$$

and $x_1, ..., x_n$; $\sigma_1, ..., \sigma_n$ *are arbitrary local coordinates on* X *and* Σ, *respectively. Vice versa, if* D *is a local diffeomorphism then* $\det \mathrm{J}_{x,\sigma}(\Phi) \neq 0$.

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