Reconstructions from integrals over non-analytic manifolds

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Abstract The known integral transforms of Funk-Radon type are applied to manifolds which have algebraic structure (planes, spheres, ellipsoids, hyperboloids, paraboloids etc.). A variety of new exact reconstructions is described in this paper for the integral transforms on arbitrary smooth manifolds X^n embedded in an affine space E^{n+1} with an additional structure.

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1 Introduction

Any even function defined on 2-sphere is reconstructed from its integrals over big circles by means of the classical Minkowski-Funk formula [1]. It was generalized by Helgason [4] and Semyanistiy [3] for the geodesic transform on spheres of arbitrary dimension. For the non-geodesic Funk transform, there are inversion formulas similar to that for the Funk transform. These formulas [6] and used in [9] for inversion of the singular cone integral transform. Here we obtain generalizations for arbitrary open smooth hypersurfaces X in an affine space. A function f defined on X is integrated over intersections of X with hyperplanes tangent to an ellipsoid Σ (called *katod*). The exact reconstruction of the function is possible if there are no collinear points x, y, σ such that $x, y \in \operatorname{supp} f$, $\sigma \in \Sigma$. Note that no reconstruction is known so far for functions on non-analytic submanifolds X of an affine space.

2 Preliminaries

Let X and Σ be manifolds of dimension $n > 1$ and Φ be a real smooth function defined on $X \times \Sigma$ such that $\mathrm{d}\Phi(x, \sigma) \neq 0$ as $\Phi(x, \sigma) = 0$. The (generalized) Funk-Radon transform M_Φ generated by Φ is defined by

$$\mathrm{M}_\Phi f(\sigma) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_{|\Phi| \leq \varepsilon} f \mathrm{d}X = \int_{Z(x)} f(x) \frac{\mathrm{d}_{\mathrm{g}} X}{\mathrm{d}\Phi}, \ \sigma \in \Sigma \tag{1}$$

where $\mathrm{d}_{\mathrm{g}} X$ is a volume form on X of a Riemannian metric g and $Z(x) = \{\sigma : \Phi(x, \sigma) = 0\}$. Condition **I:** $\mathrm{D}_\Phi : Z \to S^*(X)$ is a diffeomorphism, where

$Z=\{(x,\sigma):\Phi(x,\sigma)=0\}$, $S^{*}(X)$ is the bundle of unit cotangent vectors on X and

$$\mathrm{D}(x,\sigma)=\|\mathrm{d}_x\Phi\|_{\mathrm{g}}^{-1}\,\mathrm{d}_x\Phi(x,\sigma).$$

where $\|t\|_{\mathrm{g}}$ means the g norm of a covector t at a point $x\in X$. This implies that for any $x\in X$, the set $Z(x)=\{\sigma:\Phi(x,\sigma)=0\}$ is diffeomorphic to the sphere S^{n-1}. Points $x,y\in X,\ x\neq y$ are called *conjugate* if $\Phi(x,\sigma)=\Phi(y,\sigma)=0$ and $\mathrm{d}_\sigma\Phi(x,\sigma)\parallel\mathrm{d}_\sigma\Phi(y,\sigma)$ for some $\sigma\in\Sigma$. Condition **II:** there are no conjugate points. For any smooth volume form $\mathrm{d}\Sigma$ on Σ, the integral

$$Q_n(x,y)=\int_{Z(y)}(\Phi(x,\sigma)-i0)^{-n}\,\frac{\mathrm{d}\Sigma}{\mathrm{d}_\sigma\Phi(y,\sigma)},\ Z(y)=\{\sigma:\Phi(y,\sigma)=0\}$$

is well defined for any $x,y\in X,\ y\neq x$ since of conditions **I** and **II.**

Theorem 1 *Let Φ be a smooth function on $X\times\Sigma$ satisfying **I**, **II** and condition **III**:*

$$\operatorname{Re} i^n Q_n(x,y)=0 \text{ for all } x,y\in X,\ \ x\neq y.$$

An arbitrary function $f\in L_2(X)$ with compact support can be reconstructed from the Funk-Radon transform by

$$f(x)=\frac{1}{2\mathrm{j}^{n-1}D_n(x)}\int_\Sigma\delta^{(n-1)}(\Phi(x,\sigma))\,\mathrm{M}_\Phi f(\sigma)\,\mathrm{d}\Sigma \tag{2}$$

for any odd n, and by

$$f(x)=\frac{(n-1)!}{\mathrm{j}^n D_n(x)}\int_\Sigma\frac{\mathrm{M}_\Phi f(\sigma)\,\mathrm{d}\Sigma}{\Phi(x,\sigma)^n} \tag{3}$$

for even n, where for any n

$$D_n(x)=\frac{1}{|\mathrm{S}^{n-1}|}\int_{Z(x)}\|\mathrm{d}_x\Phi(x,\sigma)\|_{\mathrm{g}}^{-n}\,\frac{\mathrm{d}\Sigma}{\mathrm{d}_\sigma\Phi(x,\sigma)}. \tag{4}$$

In both cases $\mathrm{M}_\Phi f\in W_2^{(n-1)/2}(\Sigma)$ and integrals (2), (3) converge in quadratic mean on any compact set in X. If $f\in\mathcal{C}_0^{n-1}(X)$, respectively $f\in\mathcal{C}_0^{n-1+\varepsilon}(X)$, then the integrals converges uniformly on each compact set in X.

See [6] for the proof. Singular integrals like (3) and (2) are defined as follows

$$\begin{aligned}\int\frac{\omega}{\Phi^n} &= \frac{1}{2}\left(\int\frac{\omega}{(\Phi-i0)^n}+\int\frac{\omega}{(\Phi+i0)^n}\right),\\ \int_\Sigma\delta^{(n-1)}(\Phi)\,\omega &= (-1)^{n-1}\frac{(n-1)!}{2\pi i}\left(\int\frac{\omega}{(\Phi-i0)^n}-\int\frac{\omega}{(\Phi+i0)^n}\right)\end{aligned}$$

for any smooth volume form ω.

3 Reconstructions on smooth manifolds

Theorem 2 *Let X be a smooth hypersurface in an affine space E^{n+1} with a volume form $\mathrm{d_g}X$. Let Σ be an ellipsoid or a point in E^{n+1} (called katod) such that the condition **E** is satisfied: any line that meets X at least twice does not touch Σ. Then for any $n \geq 2$, any function $f \in L_2(X)$ with compact support can be recovered from data of integrals $M_\Phi f(\sigma)$ generated by function*

$$\Phi(x, \sigma) = \langle x - \sigma, \nabla q(\sigma) \rangle .$$

The reconstruction is given in the form (2) for odd n and in the form (3) for even n where $\mathrm{d}\Sigma = \mathrm{d}V/\mathrm{d}q$, $\mathrm{d}V$ is the invariant volume form in E^{n+1} and q is the second order polynomial such that $q(\sigma) = 1$ on Σ. The integrals converge in the sense of Theorem 2.

The hyperplanes tangent to Σ are shown in Fig.1 by light red. The set

$$Z(\sigma) = \{x \in X : \Phi(x, \sigma) = 0\}$$

is the intersection of X with the hyperplane tangent to the katod at the point σ :

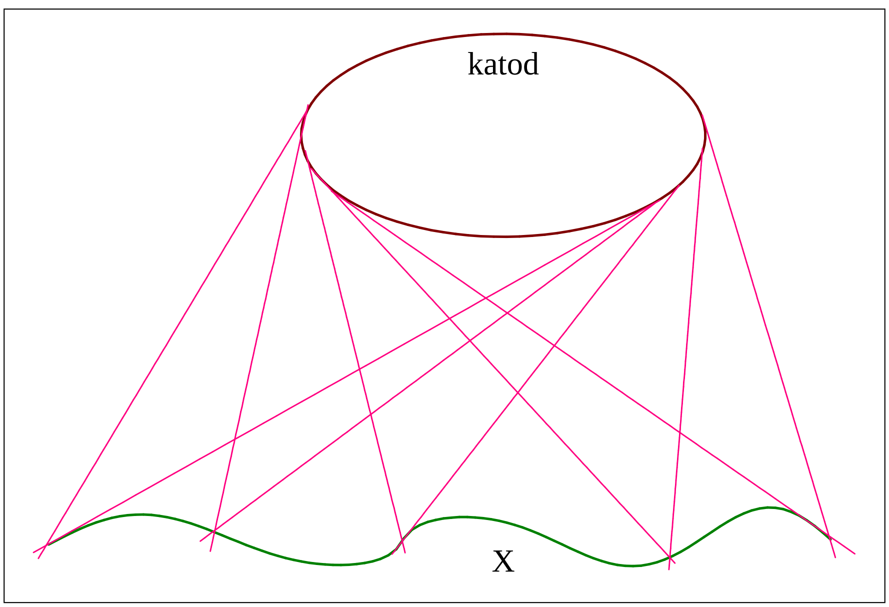

Fig. 1 Geometry of the generalized Funk-Radon transform

Remark 1. Theorem 2 was obtained in [6] for the case $X \subset \mathrm{S}^n$ and the katod is a sphere or a point inside S^n. A reconstruction for the case of one-point katod was also done by Salmon [7] ($n = 2$), [8] (arbitrary n) by a different method.

Remark 2. The generating function $\langle x - \sigma, \nabla q(\sigma) \rangle$ of this geometry does not depend on a specific affine coordinate system on E^{n+1}. Therefore we may assume that the katod is a sphere. The ellipsoid can be replaced by an arbitrary hyperboloid H in E^{n+1}; the volume form $\mathrm{d}V$ is to be replaced by the form $T^{*}(\mathrm{d}V)$ where T is the projective transform such that $T(H)$ is an ellipsoid.

Remark 3. If X is a hyperplane and the katod is a point in E^{n+1} or at infinity Theorem 2 is equivalent to the classical Radon's inversion theorem [2]. In this case **I** is fulfilled if we take each hyperplane through the katod point two times with the opposite conormal vectors. The hyperbolic space of constant curvature can be realized as the hyperboloid

$$\mathrm{H} = \left\{ (x_0, x) \in E^{n+1} : x_0^2 = |x|^2 + 1,\ x_0 > 0 \right\}$$

with the metric induced from the euclidean metric of E^{n+1}. Any totally geodesic hypersurface is the intersection of H with a subspace P in E^{n+1} of dimension n. The inversion of the totally geodesic transform for functions on H was obtained by Helgason 1959-1961 (even n) [4] and Semyanistiy in 1960-1961 (odd n) [3]. The alternative approach was applied by Gelfand and Graev [5]. In the case $X = \mathrm{H}$, $\Sigma = \{0\}$ Theorem 2 gives Semyanistyi's reconstruction and Helgason's inversion in the equivalent form. Theorem 2 applied to any elliptic katod $\Sigma \subset \left\{ x_0 < |x|^2 + 1 \right\}$ provides inversions for the family of non-equivalent non-geodesic integral transforms on any open subset $X \subset \mathrm{H}$ that fulfils **E.** If $\Sigma \subset \left\{ x_0 \leq -|x|^2 \right\}$ the inversion holds for $X = \mathrm{H}$. These reconstructions were not previously known.

4 Proof

The function $\Phi(x, \sigma) = \langle x - \sigma, \nabla q(\sigma) \rangle$ generates the family of hyperplane sections $Z(\sigma) = \{x \in X : \Phi(x, \sigma) = 0\}$ with hyperplanes tangent to Σ. Now we check that Φ satisfies conditions **I**, **II**, **III** as in Sect. 2.

Lemma 3 *Φ fulfils **I**.*

Proof. We have $\mathrm{d}_x \Phi \neq 0$ on Z since of **I.** For any point $x \in X$ and any covector $v \in T_x^{*}(X)$, $v \neq 0$, there exists one and only one hyperplane $Z(\sigma)$ such that $x \in Z(\sigma)$ and $v = t\mathrm{d}_x \Phi(x, \sigma)$ on $T_x^{*}(X)$ for some $t > 0$. It follows that the map D_Φ is bijective. We prove that D_Φ is a local diffeomorphism. This condition can be written in the form

$$\det J_{\xi,\tau}(x, \sigma) \neq 0,\ (x, \sigma) \in Z, \tag{5}$$

where

$$J_{\xi,\tau} = \begin{pmatrix} 0 & \nabla_\tau \Phi \\ {}^t\nabla_\xi \Phi & \nabla_\xi \nabla_\tau \Phi \end{pmatrix}$$

is a $n+1 \times n+1$ matrix and ξ, τ are arbitrary local systems of coordinates on X and Σ, respectively. We have

$$J_{\xi,\tau} = \begin{pmatrix} 0 & \left\langle (x - \sigma) \times \frac{\partial \sigma}{\partial \tau}, \nabla^2 q(\sigma) \right\rangle \\ \left\langle \frac{\partial x}{\partial \xi}, \nabla q \right\rangle & \left\langle \frac{\partial x}{\partial \xi} \times \frac{\partial \sigma}{\partial \tau}, \nabla^2 q(\sigma) \right\rangle \end{pmatrix},$$

where $\langle \partial\sigma/\partial\tau, \nabla q(\sigma)\rangle = \partial q(\sigma)/\partial\tau = 0$ since q is constant on Σ. If $T = \left(t_0, t^1, ..., t^n\right)$ is a vector such that $TJ = 0$ then

$$\left\langle t^i \frac{\partial x_i}{\partial \xi}, \nabla q \right\rangle = 0, \tag{6}$$

$$\left\langle \left(t^i \frac{\partial x_i}{\partial \xi} + t_0 (x - \sigma) \right) \times \frac{\partial \sigma}{\partial \tau_j}, \nabla^2 q(\sigma) \right\rangle = 0, \; j = 1, ..., n \tag{7}$$

where summation over i is assumed. Vector $t^i \partial x_i / \partial \xi$ is tangent to X and (6) means that it is tangent to Σ at σ. Vector $x - \sigma$ is also tangent to Σ since of $\Phi(x, \sigma) = 0$. Therefore there exist constants $c_1, .., c_n$ such that

$$\theta \doteqdot c_j \frac{\partial \sigma}{\partial \tau_j} = t_i \frac{\partial x_i}{\partial \xi} + t_0 (x - \sigma).$$

Taking the linear combination of equations (7) with coefficients $c_i c_j$ we get

$$\left\langle \theta \times \theta, \nabla^2 q(\sigma) \right\rangle = 0$$

which implies $\theta = 0$ since the form $\nabla^2 q$ is strictly positive. This yields $t^i \partial x_i / \partial \xi + t_0 (x - \sigma) = 0$ where the first term is tangent to X and the second one is transversal to X. It follows that $t^1 = ... = t^n = 0$, $t_0 = 0$ and $T = 0$ which completes the proof of (5) and of the Lemma. ▶

Condition **II**. Check that generating function Φ coincides with

$$\tilde{\Phi}(x, \sigma) = \langle x - e, \nabla q(\sigma) \rangle - r, \; r = 2 - 2q(e). \tag{8}$$

This follows from

$$\tilde{\Phi}(x, \sigma) - \Phi(x, \sigma) = \langle \sigma - e, \nabla q(\sigma) \rangle - r = 2(q(\sigma) - q(e)) - r = 0 \tag{9}$$

since $q(\sigma) - q(e)$ is a quadratic form of $\sigma - e$, $\sigma \in \Sigma$. Suppose that **II** violates for $\tilde{\Phi}$ and some points $x, y \in X$. We have then $a \left\langle x - e, \nabla^2 q(\sigma) \right\rangle = b \left\langle y - e, \nabla^2 q(\sigma) \right\rangle$ for a vector $(a, b) \neq (0, 0)$ and a point $\sigma \in \Sigma$. This implies that $a(x - e) = b(y - e)$ since the matrix $\nabla^2 q$ is nonsingular. It follows that x, y, and e belong to one line. This line crosses the ellipsoid which is impossible since of **E**.

Lemma 4 *Function Φ fulfils **III**.*

Proof. We are going to show that integral

$$Q_n(x, y) = \operatorname{Re} i^n \int_{Z(y)} (\Phi(x, \sigma) - i0)^{-n} \, \mathrm{d}Z(y),$$

$$\mathrm{d}Z(y) \doteqdot \frac{\mathrm{d}\Sigma}{\mathrm{d}\langle y - \sigma, \nabla q(\sigma) \rangle}$$

vanishes for all $x, y \in X, \; y \neq x$. We have

$$\Phi(x, \sigma) = \Phi(x, \sigma) - \Phi(y, \sigma) = \langle x - y, \nabla q(\sigma) \rangle$$

on $Z(y)$. The right hand side does not change its sign if and only if the point x is contained in the convex closed cone bounded by the lines through points y that are tangent to $Z(y)$. It is not the case since of **E**. Therefore $\Phi(x,\sigma)$ does change its sign on $Z(y)$. By (9)

$$\langle y-e, \mathrm{d}\nabla q(\sigma)\rangle - \mathrm{d}_\sigma \langle y-\sigma, \nabla q(\sigma)\rangle = \mathrm{d}(\sigma - e, \nabla q(\sigma)) = \mathrm{d}(q(\sigma) - q(e)) = 0$$

on Σ since $q(\sigma) = 1$. Therefore

$$\mathrm{d}Z(y) = \frac{\mathrm{d}V}{\mathrm{d}q \wedge \langle y-e, \mathrm{d}\nabla q(\sigma)\rangle}.$$

The reconstruction formulas to be proved are invariant with respect to affine transformations. Therefore we can introduce affine coordinates $\xi_0, ..., \xi_n$ in E^{n+1} such that $q(\sigma) = |\xi|^2/2$ .We have then $\mathrm{d}V = C\mathrm{d}\xi_0 \wedge ... \wedge \mathrm{d}\xi_n$ for some constant C. This yields $\mathrm{d}q = \sum \xi_i \mathrm{d}\xi_i$and $\langle y-e, \mathrm{d}\nabla q(\sigma)\rangle = \langle s, \mathrm{d}\xi\rangle$ for some vector $s \in E^{n+1}$. It follows that

$$\mathrm{d}Z(y) = \frac{\mathrm{d}V}{\xi \mathrm{d}\xi \wedge \langle s, \mathrm{d}\xi\rangle} = C\frac{\Omega_n}{\langle s, \mathrm{d}\xi\rangle} = C'\Omega_{n-1}$$

for a constant C' where Ω_k denotes the volume form of the euclidean k-sphere S^k. Finally we apply [6] Theorem A.20 to Φ and to the sphere $Z(y) \cong \mathrm{S}^{n-1}$. This implies $Q_n(x,y) = 0$ which proves the Lemma. ►

Application of Theorem 1 completes the proof of Theorem 2 for any elliptic katod. In the case of one-point katod $\{e\}$ one can take the generating function $\tilde{\Phi}(x,\sigma) = \langle x-e, \sigma\rangle$, $\sigma \in \mathrm{S}^n$ and follow the above arguments.

5 Spheres instead of hyperplanes

An analog of Theorem 2 for spheres reads

Theorem 5 *Let X be a smooth manifold of dimension n embedded in the unit ball $B\backslash\{0\}$ in an Euclidean space E^{n+1} and $\Sigma \doteqdot \partial B$ is the katod. Suppose that there are no four points $\{0\}, x_1, x_2, \sigma$ on one circle where $x_1, x_2 \in X$, $\sigma \in \Sigma$. The function $\Psi(x,\sigma) = \langle x, \sigma\rangle - |x|^2$ generates the integral transform*

$$\mathrm{M}_\Psi f(\sigma) = \int_{Z(\sigma)} f(x) \frac{\mathrm{d}X}{\langle \mathrm{d}x, \sigma - 2x\rangle}, \ \sigma \in \Sigma$$

where for any σ, $Z(\sigma) = \left\{x : \langle x, \sigma\rangle = |x|^2\right\}$ is a sphere in B tangent to Σ and containing the origin. This transform can be inverted by (2) for odd n and by (3) for even n.

Proof. The inversion map $I : x \mapsto x(y) = y/|y|^2$ is identical on Σ and we have

$$\Psi(x(y), \sigma) = |y|^{-2}(\langle y, \sigma\rangle - 1) = |y|^{-2}\tilde{\Phi}(y,\sigma)$$

where $\tilde{\Phi}$ is defined in (8) for the katod $\Sigma = \mathrm{S}^n$. The function Ψ fulfils **I**,**II**,**III** since so does $\tilde{\Phi}$ and the factor $|y|^{-2}$ does not vanish on X. Finally we apply Theorem 1 to Ψ. ►

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