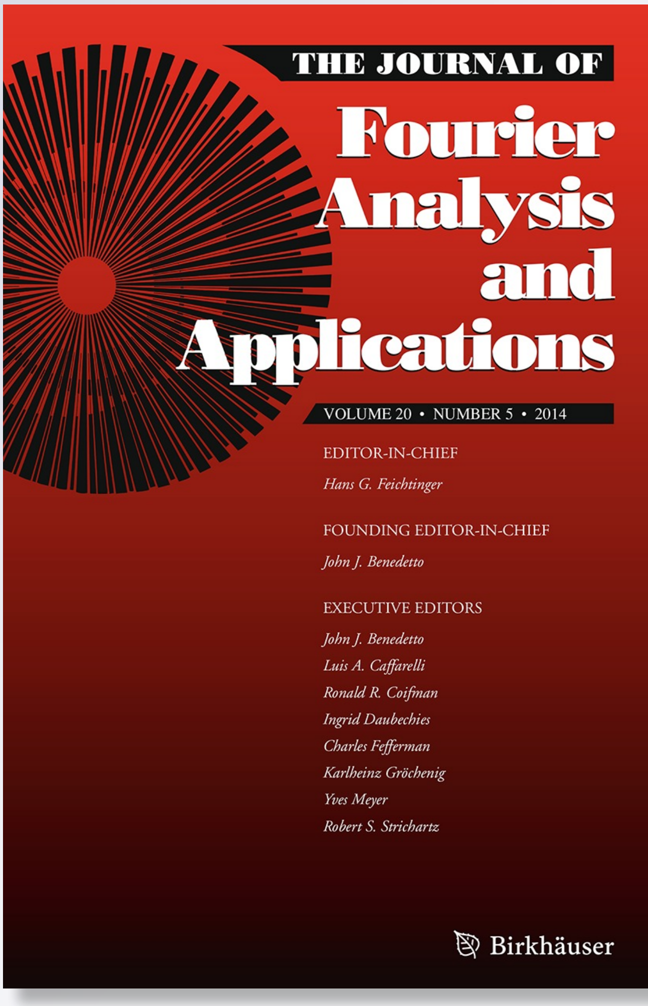
Fourier Duality in Integral Geometry and Reconstruction from Ray Integrals

Victor Palamodov

Journal of Fourier Analysis and Applications

ISSN 1069-5869
Volume 20
Number 5

J Fourier Anal Appl (2014) 20:947-960
DOI 10.1007/s00041-014-9340-7



Springer

J Fourier Anal Appl (2014) 20:947–960
DOI 10.1007/s00041-014-9340-7

Fourier Duality in Integral Geometry and Reconstruction from Ray Integrals

Victor Palamodov

Received: 16 October 2012 / Revised: 27 April 2014 / Published online: 4 July 2014

Abstract Analytic reconstruction of a function defined in an affine space from data of its integrals along lines or rays is in focus of the paper. Basic tools are the Fourier transform of homogeneous distributions and a self-duality equation in integral geometry. Three dimensional case is of special interest.

Keywords Ray transform · Completeness condition · Homogeneous distribution · Duality · Support theorem

Mathematics Subject Classification Primary 53C65 · 65R10 · Secondary 94A08

1 Introduction

We address the problem of reconstruction of a function in real affine space from data of line or ray integrals. In Sect. 2 we give a formula for the Fourier transform of homogeneous distributions including singular degrees. The self-duality relation is applied for reconstruction from line integrals in Sects. 3, 4. We focus on inversion of the ray transform in Sect. 5 in an Euclidean space. In Sect. 6 a quantitative version of Helgason's support theorem is stated.

Given a function f with compact support in a real vector space $V = V^n$ of dimension n, a *ray* integral of f is defined by

Communicated by Eric Todd Quinto.

V. Palamodov (✉)
Tel Aviv University, Tel Aviv, Israel
e-mail: palamodo@post.tau.ac.il

Birkhäuser

$$\mathrm{X}f(x,v)=\int_0^\infty f(x+tv)\,\mathrm{d}t,$$

where $x\in V$ (*source* point) and $v\in V\backslash 0$ (direction vector). The ray integral is a homogeneous function of the variable v of degree -1. If V is supplied with an Euclidean structure, then

$$\mathrm{X}f(x,v)=\frac{1}{|v|}\int_{R(x,v)} f\,\mathrm{d}s.$$

where $R(x,v)$ is the corresponding ray. The *line* transform of a function f is defined in a similar way:

$$\mathrm{L}f(x,v)=\int_{-\infty}^{\infty} f(x+tv)\,\mathrm{d}t=\mathrm{X}f(x,v)+\mathrm{X}f(x,-v).$$

1.1 Reconstruction Problem

Let $\mathrm{L}(V)$ be the manifold of all lines (or all rays) in V and $\Sigma\subset\mathrm{L}(V)$ be a submanifold, to find a reconstruction formula $\mathrm{L}f|\Sigma\mapsto f$ for smooth functions f supported in K. A subvariety Σ is called *redundant* if $\dim\Sigma>n$. In particular the manifold $\mathrm{L}(V)$ has dimension $2n-2$ and is redundant for $n>2$ and not redundant for $n=2$.

Definition We say that a family of lines Σ is *complete* in a compact set $K\subset V$, if for any point $x\in K$ and any hyperplane H through x, there exists a line $L\in\Sigma$ such that $x\in L\subset H$. This definition can be applied to a family Σ of rays, provided source points belong to $V\backslash K$. It is easy to show that for a closed family Σ lack of completeness causes non-uniqueness or severe instability of a reconstruction.

Let Γ be a curve in V and $\Sigma(\Gamma)$ be the set of rays with vertices in Γ or set of lines that meet Γ. The condition of completeness of $\Sigma(\Gamma)$ in a compact set K means that any hyperplane H that has a common point with K must meet Γ.

1.2 Short History

After Radon's paper of 1917 on reconstruction from data of line integrals in a plane there was no progress in the problem until seventies of the last century. In sixties the problem of reconstruction of a function f in $\mathbb{C}^3$ from data of integrals over complex lines was studied by Gelfand and Graev [3] for special line complexes. The case of complex lines meeting a surface Γ in $\mathbb{C}^n$ a reconstruction was proposed by Kirillov [12]. These results were not immediately rearranged for complexes of real lines complexes since there is no obvious similarity between odd and even dimensions in integral geometry. Later Orlov [14,15] found a reconstruction method for functions in

$\mathbb{R}^3$ from its line integrals over the family $\Sigma(\Gamma)$ where Γ is a closed improper curve. In Haltmeier et al [5] a relation between the line and the Radon transforms was stated. For functions with compact support in $\mathbb{R}^3$, Tuy [20] found a reconstruction formula for any proper curve Γ such that the family $\Sigma(\Gamma)$ is complete in some compact set, Finch [2] studied reconstruction for curves Γ which do not satisfy Tuy's condition. Grangeat's method [4] was to recover first normal derivatives of the Radon transform in $\mathbb{R}^3$. It is similar to that of [5] but does not involve the Hilbert transform. Katsevich [9] proposed another method for the case Γ is a helix. His method was generalized and developed in for more general curves Γ in Katsevich [10], Pack–Noo [16], Zhuang–Chen [21], Katsevich–Kapralov [11], see also references therein.

2 Fourier Transform of Homogeneous Distributions

Let V be a vector space of dimension n. A distribution or a generalized function u in V is said *homogeneous* of degree $\lambda \in \mathbb{C}$, if it satisfies the equation $L(\mathbf{e})\, u = \lambda u$ where $\mathbf{e} = \sum_0^n x_i \partial/\partial x_i$ is the Euler field and $L(\mathbf{e})$ is the corresponding Lie derivative. This condition is equivalent to the equation

$$u(L(\mathbf{e})\,\varphi) = -\lambda u(\varphi)$$

that holds for any test function or any test density φ, respectively.

Definition Let $x_1, \ldots, x_n$ be a coordinate system in V. A *singular* differential form a in V is a tensor $a = \sum a_{i_1,\ldots,i_k} \mathrm{d}x_{i_1} \wedge, \cdots, \wedge \mathrm{d}x_{i_k}$ whose coefficients $a_{i_1,\ldots,i_k}$ are generalized functions. Linear operations, exterior differential and inner product with a smooth field are well defined in a class of singular forms. In particular, any generalized function is a singular 0 form, any distribution u is a singular n form. An arbitrary distribution u in V is a product of a generalized function f and of a volume form $\mathrm{d}x = \mathrm{d}x_1 \wedge, \cdots, \wedge \mathrm{d}x_n$, and vice versa. The inner product $\mathbf{e} \lrcorner u = f\omega$ is a singular $n-1$ form where $\omega \doteqdot \mathbf{e} \lrcorner \mathrm{d}x$. A k form v is called (positively) *homogeneous* of degree $\lambda \in \mathbb{C}$ if $L(\mathbf{e})\, v \doteqdot \mathrm{d}(\mathbf{e} \lrcorner v) + \mathbf{e} \lrcorner \mathrm{d}v = \lambda v$. If f is a homogeneous generalized function of degree λ in V, then the distribution $u = f\mathrm{d}x$ is homogeneous of degree $\lambda + n$ in E since

$$\begin{aligned} L(\mathbf{e})\, u &= L(\mathbf{e})\,(f\,\mathrm{d}x) \\ &= \mathrm{d}f \wedge \mathbf{e} \lrcorner \mathrm{d}x + f\mathrm{d}(\mathbf{e} \lrcorner \mathrm{d}x) = L(\mathbf{e})\,(f)\,\mathrm{d}x + nf\mathrm{d}x = (\lambda + n)\, u. \end{aligned}$$

For example, the Dirac function $\delta_0(\phi\,\mathrm{d}x) = \phi(0)$ is a homogeneous generalized function of degree $-n$, the Dirac distribution $\delta_0\mathrm{d}x$ is homogeneous of degree 0. For any homogeneous distribution u of degree 0, the $n-1$ form $\mathbf{e} \llcorner u$ is closed. We have $\mathrm{d}(\mathbf{e} \llcorner u) = L(\mathbf{e})\,(u) = 0$. Any homogeneous distribution is tempered. The Fourier transform of a tempered distribution u in an Euclidean space E^n is the generalized function defined by

$$\widehat{u}(\xi) = F(u) \doteqdot \lim_{\varepsilon \to 0+} u\left(\exp\left(-\mathrm{j}\,\langle \xi, x\rangle - \varepsilon\,|x|^2\right)\right),$$

where we denote $\mathrm{j} = 2\pi i$.

Proposition 1 *Let u be a homogeneous distribution in E of degree* $\alpha \neq 0, -1, -2, \ldots$. *The Fourier transform* $\widehat{u}$ *is a homogeneous generalized function in E of degree* $-\alpha$ *given by*

$$\widehat{u}(\xi) = \Gamma(\alpha) \int\limits_{\mathrm{S}(E)} (\mathrm{j}\langle \xi, x\rangle + 0)^{-\alpha} \mathbf{e} \lrcorner u, \quad \xi \neq 0. \tag{1}$$

A special case of (1) is Herglotz–Petrovskiĭ's for fundamental solutions of hyperbolic equations see [8]. In singular cases, we have

Proposition 2 *Any homogeneous distribution u in* $E \backslash 0$ *of degree* $\alpha = -k$, $k = 0, 1, 2, \ldots$ *such that*

$$\int\limits_{\mathrm{S}(E)} q\, \mathbf{e} \lrcorner u = 0 \tag{2}$$

for arbitrary homogeneous polynomial q of degree k, can be extended to E as a homogeneous distribution. Its Fourier transform is equal to

$$\widehat{u}(\xi) = -\frac{(-\mathrm{j})^k}{k!} \int\limits_{\mathrm{S}(E)} \langle \xi, x\rangle^k \log(\mathrm{j}\langle \xi, x\rangle + 0)\, \mathbf{e} \lrcorner u(x) + p(\xi), \quad \xi \neq 0, \tag{3}$$

where the first term is a homogeneous generalized function, and p is a homogeneous polynomial of degree k.

Proof Consider the integral

$$v(\xi) \doteqdot \int\limits_{\mathrm{S}(E)} \langle \xi, x\rangle^k \log(\mathrm{j}\langle \xi, x\rangle + 0)\, \mathbf{e} \lrcorner u(x).$$

We have for any $t > 0$

$$\begin{aligned} v(t\xi) &= \int \langle t\xi, x\rangle^k \log(\mathrm{j}\langle t\xi, x\rangle + 0)\, \mathbf{e} \lrcorner u \\ &= t^k \int \langle \xi, x\rangle^k (\log(\mathrm{j}\langle \xi, x\rangle + 0) + \log t)\, \mathbf{e} \lrcorner u \\ &= t^k v(\xi) + t^k \log t \int \langle \xi, x\rangle^k\, \mathbf{e} \lrcorner u. \end{aligned}$$

The last integral vanishes by (2), hence v is a homogeneous function of degree k. For an arbitrary $\delta \in \mathbb{R}$, the product $u_\delta \doteqdot |x|^\delta u$ is a homogeneous distribution in $E \backslash 0$ of degree $\delta - k$. It admits the unique extension to a homogeneous distribution in E, if $k - \delta$ is not an integer ≥ 0. According to Proposition 1, the Fourier transform of u_δ is equal to

$$\widehat{u}_\delta(\xi) = \int\limits_{\mathrm{S}(E)} \Gamma(\delta - k)(\mathrm{j}\langle \xi, x\rangle + 0)^{k-\delta}\, \mathbf{e} \lrcorner u$$

since $u_\delta = u$ on $\mathrm{S}(E)$. By (2) we can write

$$\widehat{u}_\delta(\xi) = \int\limits_{\mathrm{S}(E)} \Gamma(\delta - k)\left[(\mathrm{j}\langle\xi, x\rangle + 0)^{k-\delta} - (\mathrm{j}\langle\xi, x\rangle)^k\right]\mathbf{e} \lrcorner u.$$

To resolve indeterminacy $\infty \cdot 0$ as $\delta \to 0$, we apply the equation $\Gamma(\delta - k) = (-1)^k (k!\delta)^{-1} + O(1)$ for the first factor and the Lagrange formula for the second:

$$\left[(\mathrm{j}\langle\xi, x\rangle + 0)^{k-\delta} - (\mathrm{j}\langle\xi, x\rangle)^k\right] = -(\mathrm{j}\langle\xi, x\rangle)^k \log(\mathrm{j}\langle\xi, x\rangle + 0)^k \delta + O\left(\delta^2\right).$$

This yields

$$\lim_{\delta\to 0}\widehat{u}_\delta(\xi) = w(\xi) \doteqdot -\frac{(-\mathrm{j})^k}{k!}v(\xi), \tag{4}$$

which coincides with the first term of (3). The distribution $w\mathrm{d}\xi$ is homogeneous of degree $k+n$ and by Proposition 1 $F^{-1}(w\mathrm{d}\xi)$ is a homogeneous generalized function of degree $-k-n$. The distribution $U_\delta \doteqdot F^{-1}(\widehat{u}_\delta \mathrm{d}\xi)\,\mathrm{d}x$ is equal to u_δ and tends to $W \doteqdot F^{-1}(w\mathrm{d}\xi)\,\mathrm{d}x$ as $\delta \to 0$ since of (4). On the other hand $u_\delta \to u$ in the space of distributions in $E\backslash 0$, hence $u = W$ in $E\backslash 0$ that is W is an extension of u. We have $\hat{u} = \hat{W} = w$, which gives the first term in (3). Any other extension of u to E is equal to $W + h$, where h is a homogeneous distribution supported by the origin. Therefore the polynomial term $p = F(h)$ arises in (3). □

Remark The existence of a homogeneous extension under condition (2) and necessity of this condition were stated in [7] Theorem 3.2.4.

Corollary 3 *For any odd natural k and any C^{k-1} smooth even distribution $u = f\mathrm{d}x$ of degree k (that is f is an even homogeneous function of degree $k-n$), we have*

$$\hat{u}(\xi) = \frac{1}{2(-\mathrm{j})^{k-1}|\xi|^k}\int\limits_{|x|=1}\delta^{(k-1)}(\langle\omega, x\rangle)\,\mathbf{e} \llcorner u, \tag{5}$$

$$\widehat{f}(\xi) = \frac{1}{2\mathrm{j}^{k-1}|\xi|^k}\int\limits_{|x|=1,\,\langle x,\omega\rangle=0}\langle\omega, \partial_x\rangle^{k-1} f(x)\,\Theta, \tag{6}$$

where $\omega = \xi/|\xi|$, and Θ is the Euclidean volume form on a unit $n-2$ sphere. The same is true for any even $k > 0$ and odd f. In both cases $\hat{f}$ is even.

Proof Let u be even and k be odd. The form $\mathbf{e} \llcorner u$ is even, we can change x to $-x$ in (1) and take the sum. In the right hand side we get an integral with the kernel

$$\begin{aligned}
&(k-1)!\mathrm{j}^{-k}\left[(\langle\xi, x\rangle - i0)^{-k} + (-\langle\xi, x\rangle - i0)^{-k}\right] \\
&= \mathrm{j}^{-k}|\xi|^{-k}\langle\omega, \partial_x\rangle^{k-1}\left[(\langle\omega, x\rangle - i0)^{-1} - (\langle\omega, x\rangle + i0)^{-1}\right] \\
&= \mathrm{j}^{1-k}|\xi|^{-k}\langle\omega, \partial_x\rangle^{k-1}\delta(\langle\omega, x\rangle) = \mathrm{j}^{1-k}|\xi|^{-k}\delta^{(k-1)}(\langle\omega, x\rangle).
\end{aligned}$$

This implies (5). Let now u be odd and k be even. In this case we take the difference of values of (1) at x and at $-x$. The corresponding kernel is the same with the opposite sign, which completes the proof of (5). To prove (6) we apply the following formula for integration by parts on the sphere

$$\int_{\mathrm{S}^{n-1}} \langle \omega, \partial_x \rangle\, g(x)\, h(x)\, \Omega = -\int_{\mathrm{S}^{n-1}} g(x)\, \langle \omega, \partial_x \rangle\, h(x)\, \Omega, \tag{7}$$

where g and h are homogeneous generalized functions in V such that $\deg g + \deg h = 1 - n$. We apply (7) to $h(x) = \delta^{k-j-1}(\langle \omega, x \rangle)$ and $g(x) = \langle \omega, \partial_x \rangle^{j-1} f$ for $j = 1, \ldots, k-1$ and obtain (6) since $\mathbf{e} \llcorner u = f\Omega$ and $\delta(\langle \omega, x \rangle)\, \Omega = \Theta$.

Now we check (7). The function $\langle \omega, \partial_x \rangle (gh)$ is homogeneous of degree $-n$, hence the form $a \doteqdot \langle \omega, \partial_x \rangle (gh)\, \Omega$ is closed in $V \backslash 0$ and

$$\int_{\mathrm{S}^{n-1}} a = \int_M a$$

for any arbitrary manifold M that is homologically equivalent to S^{n-1} in $V \backslash 0$. Let M be a cylinder parallel to the vector ω. Obviously $\int_M a = 0$. □

Corollary 4 *For any even natural k and any C^k smooth even homogeneous distribution $u = f\,\mathrm{d}x$ of degree k, we have*

$$\widehat{f}(\xi) = \frac{1}{\mathrm{j}^k\, |\xi|^k} \int_{|x|=1} \langle \omega, \partial_x \rangle^{k-1} f(x)\, \frac{\Omega}{\langle \omega, x \rangle} = \frac{(k-1)!}{\mathrm{j}^k} \int_{|x|=1} f(x)\, \frac{\Omega}{\langle \xi, x \rangle^k},$$

for any $\xi \neq 0$. The same is true for any natural odd k and odd f. In both cases $\hat{f}$ is even.

A proof can be given on the same lines as (6).

In particular, for any function f with compact support in V and any point $a \in V$ the distribution $\mathrm{L}f(a, v)\, \mathrm{d}v$ is even and homogeneous of degree $n-1$ with respect to direction vector. If n is odd, then by Corollary 4

$$F_{v \mapsto \xi}\left(\mathrm{L}f(a, v)\, \mathrm{d}v\right) = \frac{(n-2)!}{\mathrm{j}^{n-1}} \int_{|v|=1} \mathrm{L}f(a, v)\, \frac{\Omega}{\langle \xi, v \rangle^{n-1}}$$

for any $\xi \neq 0$. This is a proof of an equation appeared in Smith [19] $(n = 3)$. For even n, a similar equation follows from Corollary 3.

$$F_{v \mapsto \xi}\left(\mathrm{L}f(a, v)\, \mathrm{d}v\right) = \frac{1}{2\mathrm{j}^{k-1}\, |\xi|^k} \int_{|v|=1, \langle v, \omega \rangle = 0} \langle \omega, \partial_v \rangle^{k-1}\, \mathrm{L}f(a, v)\, \Theta.$$

3 Self-Duality of Integral Transforms

Let E^n be an Euclidean space of dimension n and $x_1, \ldots, x_n$ be an Euclidean coordinate system. The volume form in E is equal to $\mathrm{d}x = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$ and the restriction of the form $\omega(E) = \mathbf{e} \lrcorner \, \mathrm{d}x$ to the unit sphere $\mathrm{S}(E^n)$ is equal to the volume form of the sphere. For any vector subspace $L \subset E$, we supply the unit sphere $\mathrm{S}(L)$ in L with the volume form $\omega(L)$. A projective (Funk) integral transform of an even function f in $\mathrm{S}(E^n)$ is defined by integrals

$$\mathrm{M}f\,(\mathrm{S}(L)) = \frac{1}{2} \int\limits_{\mathrm{S}(L)} f\,\omega(L).$$

We remind some points of [17] and give more details concerning self-duality of the projective transform M.

Theorem 5 *For an arbitrary homogeneous generalized function f in E^n of degree $-k$, $k < n$, the equation holds*

$$\int\limits_{\mathrm{S}(L^{\perp})} \hat{f}\,\omega\left(L^{\perp}\right) = \int\limits_{\mathrm{S}(L)} f\,\omega(L),$$

where L is an arbitrary subspace of E^n of dimension k and $L^{\perp}$ is its polar, where $\hat{f} = F(f\mathrm{d}x)$ is a homogeneous generalized function of degree $k - n$.

A proof is given in [17] Theorem 5.10.

The affine (Radon) integral transform of an integrable function f in an Euclidean space E is defined by integrals

$$\mathrm{R}f\,(A) = \int\limits_{A} f\mathrm{d}V\,(A),$$

where A is an arbitrary affine subspace of E^n and $\mathrm{d}V\,(A)$ is the Euclidean volume form in A. Let E^+ be an Euclidean space of dimension $n+1$ with coordinates $x_0, \ldots, x_n$. Consider an imbedding $\varepsilon : E^n \to E^+$, $x \mapsto (1, x)$. Let A be an affine k plane in E and A^+ be the linear envelope of $\varepsilon(A)$. Consider the hemisphere $\mathrm{S}_+\left(A^+\right) = \mathrm{S}\left(A^+\right) \cap \{x_0 > 0\}$. The central projection $\pi : \mathrm{S}_+\left(A^+\right) \to A$ is invertible and

$$\frac{\mathrm{d}V_S\left(y, \mathrm{S}\left(A^+\right)\right)}{\mathrm{d}V_E\,(x, A)} = d^+(A)\,|(1, x)|^{-k},\; d^+(A) = \mathrm{dist}_{E^+}(L, 0),$$

where $x = \pi(y)$, $\mathrm{d}V_S, \mathrm{d}V_E$ are volume densities in $\mathrm{S}\left(A^+\right)$ and in A, respectively. Factorization of the right hand side implies an equation

$$\mathrm{M}g\left(\mathrm{S}\left(A^+\right)\right) = d^+(A)\,\mathrm{R}f\,(A), \tag{8}$$

where f is an arbitrary integrable function in A and $g(y) = |(1, x)|^k f(x)$ is defined $\mathrm{S}_+\left(A^+\right)$. This equation was found by Kurusa [13] in an equivalent form.

Definition For a k plane $A \subset E^n\backslash 0$, the set of solutions x of the system of equations

$$\langle x, y\rangle + 1 = 0, \quad y \in A$$

is a $n-k-1$ plane denoted $A^0 \subset E^n\backslash 0$. We call this plane the *polar* of A. The double polar space of A coincides with A. For any k plane $A \subset E$, the linear envelope of $\varepsilon\left(A^0\right)$ is the orthogonal complement of the linear envelope of $\varepsilon(A)$. Let f be a function in E^n such that

$$\left(1 + |x|^n\right) f(x) \in L_1\left(E^n\right). \tag{9}$$

Fix an integer $k,\ 0 < k < n$ and define the function in $E^+\backslash 0$

$$g(x_0, x) \doteqdot |x_0|^{-n-1} |(x_0, x)|^k f\left(\frac{x}{x_0}\right), \quad g(0, x) = 0.$$

It is even and homogeneous of degree $k-n-1$. Distribution $g\mathrm{d}x_0 \wedge \mathrm{d}x$ is homogeneous of degree k and by (9) it is locally integrable in E^+. The Fourier transform $\tilde{g}(\xi_0, \xi) = F(g\mathrm{d}x_0 \wedge \mathrm{d}x)$ is an even homogeneous generalized function of degree $-k$ in the dual space $\left(E^+\right)^* \cong E^+$. We call the function

$$f^0(\xi) \doteq |(1, \xi)|^{k-n-1} \tilde{g}(1, \xi) \tag{10}$$

the k-polar function of f. By Proposition 1, any function f can be recovered from its polar function by

$$f(x) = \frac{(n-k)!}{|(1, x)|^k} \int\limits_{S(E^+)} \left(\frac{|(\xi_0, \xi)|}{2\pi i\,(\xi_0 + \langle \xi, x\rangle - i0)}\right)^{n+1-k} |\xi_0|^{-n-1}\ f^0\left(\frac{\xi}{\xi_0}\right) \mathbf{e} \lrcorner\, (\mathrm{d}\xi_0 \wedge \mathrm{d}\xi), \tag{11}$$

where $\mathbf{e}$ is the Euler field in E^+. Integral (11) is well defined as a generalized function.

Theorem 6 *Let f be a function in E^n satisfying (9). For an arbitrary affine k space $A \subset E^n$, $0 < k < n$, we have*

$$d^{1/2}\left(A^0\right) \mathrm{R} f^0\left(A^0\right) = d^{1/2}(A)\, \mathrm{R} f(A), \tag{12}$$

where f^0 is the k-polar function of f and $d(G) \doteq \operatorname{dist}(G, \{0\})$ for a set $G \subset E^n\backslash 0$.

For a proof, we apply Theorem 5 to g and A^+ and note that $d(A)\, d\left(A^0\right) = 1$. See more details in [17].

4 A Reconstruction from Line Integrals

Theorem 7 *Let $0 < k < n-1$, Δ be a variety of dimension k of affine $k-1$ subspaces of $E^n\backslash 0$ and $\Sigma_k(\Delta)$ be the variety of all k planes B that contains a $k-1$*

plane $A \in \Delta$. If $\Sigma_k(\Delta)$ is complete in $E^n\backslash 0$, then any function f in E satisfying (9) can be reconstructed from data of $\mathrm{R}_k f(B)$, $B \in \Sigma_k(\Delta)$.

Remark The variety $\Sigma_k(\Delta)$ has dimension $n-k+k=n$, hence integral data $\mathrm{R}_k f(B)$ are not redundant.

Proof Inclusion $A \subset B$ implies $B^0 \subset A^0$ and B^0 is an arbitrary hypersurface in a $n-k$ plane A^0. By (8) and (12), we know all integrals $\mathrm{R}_{n-k-1} f^0\left(B^0\right)$, and we can recover the function f^0 in A^0 by the Radon-John formulas. We claim that the union of all $n-k$ planes A^0 is equal to $E^n\backslash 0$. By the assumption the union of all A is complete. This implies that for any hyperplane H in $E^n\backslash 0$, there exists a plane $A \subset H$. This yields $H^0 \subset A^0$, which means that any point $H^0 \in E^n\backslash 0$ is contained in the union of all A^0. It follows that function f^0 can be recovered in $E^n\backslash 0$. By (10) we reconstruct the function $\hat{g}$, by the inverse Fourier transform we determine g and f. □

Theorem 8 *Let Γ be a closed non-contractible curve in the projective closure* P^n *of E^n. Any smooth function f with compact support in E^n can be reconstructed from integrals* $\mathrm{L}f(L) = \mathrm{R}f(L)$ *for lines L that meet Γ.*

Proof The statement follows from Theorem 7, if we show that the variety $\Sigma(\Gamma)$ of all lines that meet Γ is complete in P^n. By the condition, the class of Γ in the fundamental group $\pi_1(\mathrm{P}^n)$ does not vanish. We have $\pi_1(\mathrm{P}^n) \cong \mathbb{Z}_2$ hence the class of Γ is equal to the class of a projective line $\Lambda \subset \mathrm{P}^n$. Let F be the set of all pairs (x, H) such that $x \in H$ and H is a hyperplane in P^n. Consider the set

$$B(\Gamma) = \cup\left\{B_P = \{(x, H),\ x \in P \subset H\},\quad P \in \Sigma(\Gamma)\right\}.$$

Let $\pi_\Gamma : B(\Gamma) \to F$ be the projection such that $\pi_\Gamma(P, (x, H)) = (x, H)$. This map is proper, since $\Sigma(\Gamma)$ is compact. Therefore the number $\deg \pi_\Gamma (\mathrm{mod}\, 2)$ is well defined and $\deg \pi_\Gamma = \deg \pi_\Lambda$, since Γ and Λ are homotopically equivalent. We have $\deg \pi_\Lambda = 1$, since any point $(x, H) \in F$, $x \in \mathrm{P}^n\backslash\Lambda$ is covered by π_Λ only once. This yields $\deg \pi_\Gamma = 1$ and implies surjectivity which implies completeness of $\Sigma(\Gamma)$. □

Orlov's result [14,15] is the special case of Theorem 8, where Γ is a curve in $\mathrm{P}^3\backslash E^3$. The curve Γ is not contained in an affine part of P^n, since it is non-contractible. In the next section, we focus on reconstruction from data of ray transform with a source trajectory Γ in an affine space.

5 Compact Source Sets

Tuy [20] proposed a method of reconstruction of a function f of three variables. It can be formulated as follows:

Theorem 9 *Let Γ be a curve in and Euclidean space E^3 with a piecewise C^1 parameterization $y = y(s)$, $y'_s \neq 0$, $s \in [0, 1]$ that satisfies the completeness condition for a point $x \in E^3\backslash\Gamma$: almost any plane through x meets Γ at a point y transversely.*

Then arbitrary real function $f \in C^2\left(E^3\right)$ *with compact support in* $E^3\backslash\Gamma$ *can be reconstructed at x from ray transform g by*

$$f(x) = \frac{1}{2\pi i}\int\limits_{\xi\in \mathrm{S}^2} \frac{G_s(y,\xi)}{\left\langle y_s', \xi\right\rangle}\omega(\xi), \tag{13}$$

where ω *is the volume form on the unit sphere* S^2,

$$G_s(y,\xi) = \frac{\partial G(y,\xi)}{\partial s},\ G(y,\xi) = F_{v\to\xi}(g(y,v)\,\mathrm{d}v)$$

and, for any $\xi \in \mathrm{S}^2$, *a point* $y \in \Gamma$ *in (13) is chosen in such a way that* $\langle y-x,\xi\rangle = 0,\ \left\langle y_s',\xi\right\rangle \neq 0$.

Note that the Fourier transform G is defined in the sense of §2. We rearrange this formula in terms of convergent integrals. First, take a bounded function $\varepsilon : \Gamma\times \mathrm{S}^2 \to \mathbb{R}$ fulfilling

$$\sum_{y\in\Gamma,\langle y,\xi\rangle=p} \varepsilon(y,\xi) = 1 \tag{14}$$

for any pair (p,ξ) such that there exists at least one point $y\in\Gamma$ such that $\langle y,\xi\rangle = p$ and $\left\langle y_s',\xi\right\rangle \neq 0$. We assume that $\varepsilon(y,\xi) = 0$ except for a finite number of points y, and call ε *distribution* function. Such a function can be easily constructed for any smooth curve Γ.

Theorem 10 *Let* Γ *be a curve in* E^3 *with a piecewise* C^2 *parameterization* $y = y(s)$ *and* ε *be an arbitrary distribution function. Suppose that* Γ *satisfies the completeness condition for a point* $x\in E^3\backslash\Gamma$. *Then an arbitrary function* $f\in C^2\left(E^3\right)$ *with compact support in* $E^3\backslash\Gamma$ *can be reconstructed at x from its ray transform g by*

$$f(x) = -\frac{1}{8\pi^2}\int\limits_{\langle y-x,\xi\rangle=0} \frac{\varepsilon(y,\xi)}{\left\langle y_s',\xi\right\rangle}\int\limits_{\langle v,\xi\rangle=0} g_{s;\xi}(y,v)\,\mathrm{d}\theta\omega(\xi). \tag{15}$$

Here $\mathrm{d}\theta$ *is the angular measure on the circle* $\{v;\langle v,\xi\rangle = 0\}$ *and*

$$g_{s;\xi}(y,v) = \left\langle \xi, \frac{\partial}{\partial v}\right\rangle \frac{\partial}{\partial s} g(y,v).$$

Remark If ε is an even distribution function that is $\varepsilon(y,-\xi) = \varepsilon(y,\xi)$, the integrand is an even function hence the circle $\{\langle v,\xi\rangle = 0\}$ can be replaced by any half-circle.

Proof Take $p = \langle x,\xi\rangle$ in (14) and write (13) in the form

$$f(x) = \frac{1}{2\pi i}\int\limits_{\xi\in\mathrm{S}^2}\sum_{y;\langle y,\xi\rangle=\langle x,\xi\rangle}\varepsilon(y,\xi)\frac{G_s(y,\xi)}{\left\langle y_s',\xi\right\rangle}\omega(\xi) = \frac{1}{2\pi i}\int\limits_{\mathrm{O}\backslash\mathrm{C}}\varepsilon(y,\xi)\frac{G_s(y,\xi)}{\left\langle y_s',\xi\right\rangle}\omega(\xi), \tag{16}$$

where $\mathrm{O} \doteqdot \{(y, \xi), \langle y - x, \xi\rangle = 0\}$ and $\mathrm{C} \doteqdot \left\{(y, \xi) \in \mathrm{O}; \left\langle y_s', \xi\right\rangle = 0\right\}$. The set O is a smooth oriented manifold of dimension 2. Consider the projection $\pi : \mathrm{O} \to \mathrm{S}$, $(y, \xi) \mapsto \xi$. By Sard's theorem the set $\mathrm{C} \subset \mathrm{S}$ of critical values of π has zero measure. For any $\xi \in \mathrm{S}\backslash\mathrm{C}$ the set $\mathrm{O}_\xi \doteqdot \pi^{-1}(\xi)$ is non empty by the completeness condition and $\left\langle y'(s), \xi\right\rangle \neq 0$ for any choice $y = y(s) \in \mathrm{O}_\xi$ since ξ is not a critical value. It follows that we can neglect C in (16). The Fourier transform G of the homogeneous distribution $g(y, v)\,\mathrm{d}v$ of degree 2 can be taken from Proposition 1:

$$G(y, \xi) = -\frac{1}{4\pi^2} \int\limits_{\mathrm{S}^2} \frac{g(y, v)\,\omega(v)}{(\langle \xi, v\rangle - i0)^2}, \quad \xi \neq 0.$$

Substituting in (13) we obtain an integral with the complex valued kernel

$$f(x) = -\frac{1}{8\pi^3 i} \int\limits_{\mathrm{O}} \varepsilon(y, \xi) \frac{\omega(\xi)}{\left\langle y_s', \xi\right\rangle} \int\limits_{\mathrm{S}^2} g_s(y, v) \frac{\omega(v)}{(\langle \xi, v\rangle - i0)^2},$$

and take its real part:

$$\begin{aligned} f(x) &= -\frac{1}{8\pi^3} \int\limits_{\mathrm{O}} \varepsilon(y, \xi) \frac{\omega(\xi)}{\left\langle y_s', \xi\right\rangle} \int\limits_{\mathrm{S}^2} g_s(y, v)\, \mathrm{Im}\,(\langle \xi, v\rangle - i0)^{-2}\,\omega(v) \\ &= \frac{1}{8\pi^2} \int\limits_{\mathrm{O}} \varepsilon(y, \xi) \frac{\omega(\xi)}{\left\langle y_s', \xi\right\rangle} \int\limits_{\mathrm{S}^2} g_s(y, v)\,\delta'(\langle \xi, v\rangle)\,\omega(v) \\ &= -\frac{1}{8\pi^2} \int\limits_{\mathrm{O}} \varepsilon(y, \xi) \frac{\omega(\xi)}{\left\langle y_s', \xi\right\rangle} \int\limits_{\langle v, \xi\rangle = 0} g_{s;\xi}(y, v)\,\mathrm{d}\theta(v). \end{aligned}$$

We applied here equation $\mathrm{Im}\,(t - i0)^{-2} = -\pi\delta'(t)$ and noted that $\delta(\langle \xi, v\rangle)\,\omega(v) = \mathrm{d}\theta$ is the angular measure on the circle $\{v; \langle v, \xi\rangle = 0\}$. This yields (15). □

6 A Quantitative Support Theorem

The famous Helgason's support theorem of 1964 [6] is

Theorem 11 *Let $f \in C(E^n)$ satisfy the following conditions*

(i) For each $\delta \geq 0$ $|x|^\delta f(x)$ is bounded.
(ii) There exists a constant $A > 0$ such that $\mathrm{R}f(H) = 0$ for any hyperplane L with distance $> A$ from the origin.
Then $f(x) = 0$ for $|x| > A$.

See Quinto [18] for a survey of related results. We state a version of this Theorem.

Theorem 12 *Let f be a continuous function in $E^2\backslash B$ that fulfils conditions (i) for some $\delta > 1$ and (ii) for any line $L \subset E^2\backslash B$ where B is a compact set. Then for any circle S that encloses B, we have*

$$\int_S f(x)\, p(x)\, \mathrm{d}s = 0 \tag{17}$$

for any polynomial p in E^2 of degree $< \delta$. In other words, Fourier expansion of $f|_S$ does not contain frequencies $< \delta$.

This statement implies Helgason's theorem for the case $n = 2$, since by (i) f is orthogonal to any polynomial on S. It follows that f vanishes on any circle S, hence $f \equiv 0$ in $E^2 \backslash B$. The case $n > 2$ follows by the descent method.

Remark Condition (i) can not be much weakened, since the function $f(z) = \operatorname{Re} z^{-k}$, $k \geq 2$ fulfils (i) for $\delta = k$ and (ii) for $A = 0$ but $fp > 0$ for $p = \operatorname{Re} z^k$.

Proof We assume that S is the unit circle and expand f in harmonics:

$$f(x) = \sum_{-\infty}^{\infty} \exp(ik\theta)\, f_k(r),\ \ r = |x| \geq 1, \quad \theta = \arg(x_1, x_2).$$

The coefficients

$$f_k(r) = \frac{1}{2\pi} \int_0^{2\pi} f(\exp(i\theta)\, r) \exp(-ik\theta)\, \mathrm{d}\theta$$

are continuous functions in $[1, \infty)$ and satisfy

$$|f_k(r)| \leq Cr^{-\delta} \tag{18}$$

for any k and a constant C. Condition (ii) is fulfilled for each term of this series, which yields

$$\int_a^{\infty} f_k(r) \frac{\cos\left(k \cos^{-1}(a/r)\right)}{\left(1 - (a/r)^2\right)^{1/2}} \mathrm{d}r = 0$$

for any $a \geq 1$. Following Cormack's method [1] we multiply by

$$\frac{\cosh\left(k \cosh^{-1}(a/s)\right)}{\left((a/s)^2 - 1\right)^{1/2}} \frac{\mathrm{d}a}{a},$$

and integrate for $1 \leq s \leq a$

$$\int_s^{\infty} \frac{\cosh\left(k \cosh^{-1}(a/s)\right)}{\left((a/s)^2 - 1\right)^{1/2}} \left(\int_a^{\infty} \frac{\cos\left(k \cos^{-1}(a/r)\right)}{\left(1 - (a/r)^2\right)^{1/2}} f_k(r)\, \mathrm{d}r \right) \frac{\mathrm{d}a}{a} = 0. \tag{19}$$

The integral of the absolute value is bounded by

$$C_k \int_s^\infty \int_s^r \frac{(a/s)^{|k|-1}}{\left((a/s)^2-1\right)^{1/2}\left(1-(a/r)^2\right)^{1/2}} \mathrm{d}a \; |f_k(r)| \,\mathrm{d}r \le C_k \int_1^\infty |f_k(r)| \, r^{|k|-1} \log r \mathrm{d}r$$

since $\cosh\left(k \cosh^{-1}(\cdot)\right)$ is the Chebyshev polynomial of degree $|k|$ and the inner integral equals $O\left((r/s)^{|k|}\log(r/s)\right)$. Now (18) implies that integral (19) is absolutely convergent in $[1, \infty)$ if $|k| < \delta$. Changing the order of integrations we get

$$\int_a^\infty f_k(r)\,\mathrm{d}r \int_b^r \frac{\cosh\left(k\cosh^{-1}(a/b)\right)}{\left((a/b)^2-1\right)^{1/2}} \frac{\cos\left(k\cos^{-1}(a/r)\right)}{\left(1-(a/r)^2\right)^{1/2}} \frac{\mathrm{d}a}{a} = 0.$$

The inner integral is equal to $\pi/2$ [1], which implies

$$\int_a^\infty f_k(r)\,\mathrm{d}r = 0$$

for any $a \ge 1$. Differentiation yields $f_k = 0$ for any k, $|k| < \delta$, and (17) follows for any polynomial of degree $< \delta$. $\square$

References

1. Cormack, A.M.: The Radon transform on a family of curves in the plane I. Proc. Am. Math. Soc. **83**, 325–330 (1981)
2. Finch, D.: Cone beam reconstruction with sources on a curve. SIAM J. Appl. Math. **45**, 665–673 (1985)
3. Gelfand, I.M., Graev, M.I.: Integral transformations connected with line complexes in a complex affine space (Russian). Dokl. Akad. Nauk SSSR **138**, 1266–1269 (1961)
4. Grangeat, P.: Mathematical framework of cone-beam 3D-reconstruction via the first derivative of the Radon transform. Lecture Notes in Mathematical, vol. 1497, pp. 66–97. Springer, Berlin (1991)
5. Haltmeier, M., Smith, K.T., Solmon, D.C., Wagner, S.L.: The divergent beam X-ray transform. Rocky Mountain J. Math. **10**(1), 253–283 (1980)
6. Helgason, S.: Integral Geometry and Radon Transforms. Springer, New York (2011)
7. Hörmander, L.: The Analysis of Linear Partial Differential Operators. Distribution Theory and Fourier Analysis. Springer, Berlin (1983)
8. John, F.: The fundamental solution of linear elliptic differential equations with analytic coefficients. Commun. Pure Appl. Math. **3**, 273–304 (1950)
9. Katsevich, A.: A general scheme for constructing inversion algorithms for cone beam CT. Int. J. Math. Math. Sci. **21**, 1305–1321 (2003)
10. Katsevich, A.: An improved exact filtered backprojection algorithm for spiral computed tomography. Adv. Appl. Math. **32**, 681–697 (2004)
11. Katsevich, A., Kapralov, M.: Filtered backprojection inversion of the cone beam transform for a general class of curves. SIAM J. Appl. Math. **68**, 334–353 (2007)
12. Kirillov, A.A.: On a problem of I M. Gel'fand. Soviet Math. Dokl. **2**, 268–269 (1961)
13. Kurusa, A.: The invertibility of the Radon transform on abstract rotational manifolds of real type. Math. Scand. **70**, 112–126 (1992)
14. Orlov, S.S.: Theory of 3D reconstruction: I conditions for a complete set of projections. Soviet Phys. Crystallogr. **20**, 312–314 (1975)
15. Orlov, S.S.: Theory of 3D reconstruction: II the recovery operator. Soviet Phys. Crystallogr. **20**, 429–433 (1975)

16. Pack, J.D., Noo, F.: Cone-beam reconstruction using 1D filtering along the projection of M-lines. Inverse Probl. **21**, 1105–1120 (2005)
17. Palamodov, V.: Reconstructive Integral Geometry. Birkhäuser, Basel (2004)
18. Quinto, E.T.: Helgason's support theorem and spherical Radon transforms. Contemp. Math. **464**, 249–264 (2008)
19. Smith, B.D.: Image reconstruction from cone-beam projections: necessary and sufficient conditions and reconstruction methods. IEEE Trans. Med. Imag. **4**, 14–25 (1985)
20. Tuy, H.K.: An inversion formula for cone-beam reconstruction. SIAM J. Appl. Math. **43**, 546–552 (1983)
21. Zhuang, T., Chen, G.-H.: New families of exact fan-beam and cone-beam image reconstruction formulae via filtering the backprojection image of differentiated projection data along singly measured lines. Inverse Probl. **22**, 991–100 (2006)