Reconstruction from a sampling of circle integrals in $\mathbf{SO}(3)$

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Abstract New reconstruction methods for a function on the rotation group from nonredundant data of integrals over circles are proposed. Range conditions for the integrals are discussed.

Key words: rotation group, circle integral, pole density function, Funk transform, quaternion, range condition, characteristic surface, Goursat problem

1 Introduction

Reconstruction of a function on the group $\mathbf{SO}(3)$ from data of circular integrals is a mathematical model for the quantitative texture analysis of polycrystalline materials by means of X-ray or neutron diffraction data, see [3]. The integrand is called in this context "orientation distribution function (ODF)" and the mean of integrals over a union of two orthogonal circles - "pole density function (PDF)". which is obtained from X-ray diffraction experiments. The problem of texture analysis is to extract information on ODF from knowledge of PDF. Several methods of reconstruction an ODF from data of PDF are known since sixties: expansion in spherical harmonics [15], [3], a Funk-type inversion formula [11],[4], backprojection inversion [2] and inversion by singular integral operator [8]. These methods work also for reconstruction of a function on the group from data of its circle integrals instead of PDF, but anyway they were applied only to the complete 4D data. The complete data is however not technically attainable and redundant. We describe two explicit methods that allow to reconstruct a function from some nonredundant samplings of circle integrals. We discuss the range conditions for the Funk transform.

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2 Rotations and quaternions

We write a quaternion q in the form $q_0 + \mathbf{q}$; the number $q_0 = \mathrm{Re} q \in \mathbb{R}$ is called the real part and $\mathbf{q} \in \mathbb{R}^3$ is the imaginary part of q. For an arbitrary quaternion

q the linear map

$$G(q): h \mapsto qhq^*,\ q^* = q_0 - \mathbf{q}$$

transforms any pure imaginary quaternion h to a pure imaginary one. This map can be written in the form:

$$G(q)\,\mathbf{h} = \left(q_0^2 - |\mathbf{q}|^2\right)\mathbf{h} + 2q_0\mathbf{q}\times\mathbf{h} + 2\langle\mathbf{q},\mathbf{h}\rangle\,\mathbf{q},\ \mathbf{h}\in\mathbf{S}^2 \tag{1}$$

For any unit q the transform $G(q)$ preserves the norm and it is easy to check that $\det G(q) = 1$, hence $G(q) \in \mathbf{SO}(3)$. Any unit non-real quaternion $q = q_0 + \mathbf{q}$ represents the rotation about the axis $\mathbf{q}/|\mathbf{q}|$ by the angle $\varphi = 2\arccos q_0$ and $\mathrm{tr}G(q) = 4q_0^2 - 1$.

The quaternion field $\mathbb{H}$ is a 4-dimensional Euclidean space with the scalar product $\langle q, r\rangle = \mathrm{Re}(qr^*)$, $\|q\|^2 = \sum_i q_i^2$. The subspace $\mathbb{H}_0 = \{q; q_0 = 0\}$ is the space of pure imaginary quaternions. The operator of multiplication by a unit quaternion p is a rotation in $\mathbb{H}$ since $\langle pq, pr\rangle = \langle p^*pq, r\rangle = \langle q, r\rangle$. Let $\mathbb{S}$ be the unit sphere of all unit quaternions; the map $G : \mathbb{S} \to \mathbf{SO}(3)$ given by (1) has quadratic components. It is two-fold, since the quaternions q and $-q$ generates the same orthogonal transform. Let $\mathbb{S}_+$ be the hemisphere $\{q \in \mathbb{S},\ q_0 > 0\}$. The sphere $\mathbb{S} \cap \mathbb{H}_0$ of all unit imaginary quaternions is the boundary of $\mathbb{S}_+$. Consider the central projection, see Fig.1:

$$\pi : \mathbb{S}_+ \to 1 + \mathbb{H}_0, q \mapsto \frac{q}{q_0} = 1 + \frac{\mathbf{q}}{q_0} \tag{2}$$

We have the following commutative diagram

$$\begin{tikzcd} & \mathbb{S} \arrow[dl, "G"'] \arrow[d] & \\ \mathbf{SO}(3) \arrow[r, phantom, "\cong"] & \mathbb{S}/\mathbb{Z}_2 \arrow[r, phantom, "\cong"] & \mathbb{RP}_3 \\ & \mathbb{S}_+ \arrow[ul, "G"] \arrow[u] \arrow[r, "\pi"] & 1+\mathbb{H}_0 \arrow[u] \end{tikzcd} \tag{3}$$

where $\mathbb{Z}_2 = \{1, -1\}$ is the subgroup of $\mathbb{S}$. This diagram shows that the manifold of the group $\mathbf{SO}(3)$ is isomorphic to the projective 3-space. Moreover the group possesses the canonical metric that is isometric to the standard metric of $\mathbb{RP}_3$.

3 Circles and planes

Let $\mathbf{S}^2$ be the unit sphere in an Euclidean space where the group $\mathbf{SO}(3)$ is represented. A circle $\mathbf{C}(\mathbf{h}, \mathbf{y})$ in the group $\mathbf{SO}(3)$ is defined by two points $\mathbf{h}, \mathbf{y} \in \mathbf{S}^2$ so that

$$\mathbf{C}(\mathbf{h}, \mathbf{y}) = \{g \in \mathbf{SO}(3); \mathbf{h} = g\mathbf{y}\}.$$

The circle $\mathbf{C}(\mathbf{h}, \mathbf{y})$ can be parametrized by $g = g_{\mathbf{h}} R(\varphi)\, g_{\mathbf{y}}^{-1}$ where

$$R(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix},\ 0 \le \varphi < 2\pi$$

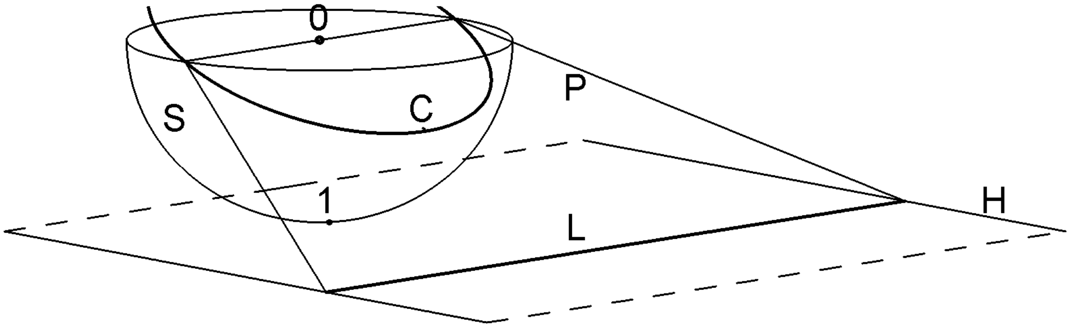

Figure 1: Central projection in the quaternion space

and $g_{\mathbf{h}}, g_{\mathbf{y}}$ are some rotation such that $g_{\mathbf{y}}\mathbf{n} = \mathbf{y}$, $g_{\mathbf{h}}\mathbf{n} = \mathbf{h}$ where $\mathbf{n} = (0, 0, 1)$. The measure $\mathrm{d}\varphi$ on $\mathbf{C}(\mathbf{h}, \mathbf{y})$ does not depend on the choice of $g_{\mathbf{h}}$ and $g_{\mathbf{y}}$. The manifold Σ of all circles in the group is parameterized by the product $\mathbf{S}^2 \times \mathbf{S}^2$ where the opposite points are identified since $\mathbf{C}(\mathbf{h}, \mathbf{y}) = \mathbf{C}(-\mathbf{h}, -\mathbf{y})$.

Proposition 1 *For any central 2-plane P in $\mathbb{H}$ the image of the circle $P \cap \mathbb{S}$ by G is a circle in $\mathbf{SO}(3)$ and vice versa. The map G is conformal with the conformal coefficient* 2.

Proof. Choose a quaternion a such that $\mathrm{Re}(aq) = 0$ for $q \in P \cap \mathbb{S}$. Set $q' = aq$ and write the equation of the circle in the form $\mathrm{Re}q' = 0$, $q' = q'(\varphi) = r\cos\varphi + s\sin\varphi$ with some unit orthogonal vectors $r, s \in \mathbb{S}_1$. Take the frame $(\mathbf{r}, \mathbf{s}, \mathbf{t})$, $\mathbf{t} = \mathbf{r} \times \mathbf{s}$ and write the matrix of the rotation $G(q')$ in this frame by means of (1):

$$G(q'(\varphi)) = \begin{pmatrix} \cos 2\varphi & -\sin 2\varphi & 0 \\ \sin 2\varphi & \cos 2\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{4}$$

We see that the point $G(q')$ runs over a circle two times faster than the point q'. The same is true for the elements $G(q(\varphi)) = G(a^*)\, G(q'(\varphi))$.

To prove the second statement we note that any great circle $\mathbb{C}$ in $\mathbb{S}$ is a geodesic and the image $\mathbf{C} = G(\mathbb{C})$ is a circle in the group which is a geodesic curve in the standard metric. Therefore G is conformal and the conformal coefficient equals 2 according to (4). Let now $\mathbf{C}$ be a circle in the group. It is a geodesic and G is conformal, hence the set $\mathbb{C} = G^{-1}(\mathbf{C})$ is also a geodesic. This implies that $\mathbb{C}$ is a great circle. □

Corollary 2 *Any circle $\mathbf{C}$ in $\mathbf{SO}(3)$ is transformed by (2) to a projective line in $\mathbb{RP}_3$ and to a straight line in $\mathbb{H}_0$.*

The curve $\mathbb{C}(\mathbf{h}, \mathbf{y}) = G^{-1}(\mathbf{C}(\mathbf{h}, \mathbf{y}))$ is a great circle in $\mathbb{S}$ that covers the circle $\mathbf{C}(\mathbf{h}, \mathbf{y})$ twice for any $\mathbf{h}, \mathbf{y} \in \mathbf{S}^2$.

Proposition 3 *For any $\mathbf{h}, \mathbf{y} \in \mathbf{S}^2$ the circles $\mathbf{C}(\mathbf{h}, \mathbf{y})$ and $\mathbf{C}(-\mathbf{h}, \mathbf{y})$ are contained in orthogonal 2-planes L_+, L_- in $\mathbb{H}$.*

We call such circles orthogonal.

Proof. (See another proof in [12]) First, consider the case $\mathbf{y} = \mathbf{h}$. The circle $\mathbb{C}(\mathbf{h}, \mathbf{h})$ consists of quaternions $q = q_0 + \mathbf{q}$ where $\mathbf{q} = t\mathbf{h}$ for $t^2 + q_0^2 = 1$, hence L_+ is the plane of the quaternions $q = q_0 + t\mathbf{h}$ with arbitrary q_0 and t. The circle $\mathbf{C}(-\mathbf{h}, \mathbf{h})$ consists of rotations about an axis $\mathbf{r}$ orthogonal to $\mathbf{h}$, hence $\mathbb{C}(-\mathbf{h}, \mathbf{h})$ is the set of such pure imaginary quaternions $\mathbf{r}$ and $L_-$is the plane in $\mathbb{H}$ orthogonal to all $q = q_0 + t\mathbf{h}$. The planes L_+ and L_- are obviously orthogonal which proves the proposition for the special case. Take an arbitrary $\mathbf{y} \in \mathbf{S}^2$ and choose a unit quaternion p such that $G(p)\,\mathbf{h} = \mathbf{y}$. Then $\mathbf{C}(\pm\mathbf{h}, \mathbf{y}) = \mathbf{C}(\pm\mathbf{h}, G(p)\,\mathbf{h}) = G(p)\,\mathbf{C}(\pm\mathbf{h}, \mathbf{h})$, hence $\mathbb{C}(\pm\mathbf{h}, \mathbf{y}) = p\mathbb{C}(\pm\mathbf{h}, \mathbf{h})$. The operator of multiplication by p is a rotation in $\mathbb{H}$. Therefore the statement holds for arbitrary $\mathbf{y}$. $\square$

Projective planes in the group can be described as follows (M. Eastwood). The set $\mathbf{S}_e$ of all rotations through π radians is obviously isomorphic to a projective plane. Any projective plane in $\mathbf{SO}\left(3\right)$ a set of the form $\mathbf{S}_a = a\mathbf{S}_e$ for some $a \in \mathbf{SO}\left(3\right)$.

Proposition 4 *For any $a \in \mathbf{SO}\left(3\right)$ the surface*

$$\mathbf{S}_a \doteq \left\{g \in \mathbf{SO}\left(3\right),\ \operatorname{tr}\left(a^* g\right) = -1\right\}$$

is a projective plane. Vice versa, any projective plane has such form for some a.

Proof. Set $a = 1$; any rotation g_π by π satisfies $\mathrm{tr}g_\pi = -1$. Vice versa, suppose that $\mathrm{tr}g = -1$. The transform g is a rotation about a vector $\mathbf{v}$ by an angle φ. Then $\mathrm{tr}g = 1 + 2\cos\varphi = -1$ hence $\varphi = \pi$. For arbitrary a and $g \in \mathbf{S}_a$ we have $g' \doteq a^* g \in \mathbf{S}_1$, hence $g = ag' \in a\mathbf{S}_1 = \mathbf{S}_a$. Prove the inverse statement. Proposition 1 yields that for any plane $\mathbf{P} \subset \mathbf{SO}\left(3\right)$ the set $\mathbb{P} = G^{-1}\left(\mathbf{P}\right)$ is a big 2-sphere in $\mathbb{S}$. It can be defined by the equation $\operatorname{Re}\left(a^* g\right) = 0$ for some unit quaternion a which is equivalent to $\operatorname{tr}\left(a^* g\right) = -1$. $\square$

The equation $\mathbb{S}_r = \left\{q \in \mathbb{S}; \operatorname{Re}\left(r^* q\right) = 0\right\}$ for an arbitrary $r \in \mathbb{S}$ defines a big 2-sphere in $\mathbb{S}$. Any big sphere is given by this equation for some unit quaternion a. In particular, $\mathbb{S}_1 = \mathbb{S} \cap \mathbb{H}_0$.

Proposition 5 *For an arbitrary $r \in \mathbb{S}$ we have $G\left(\mathbb{S}_r\right) = \mathbf{S}_{G(r)}$.*

Proof. For the unit quaternion r we have by (1) $G\left(\mathbb{S}_1\right) = \mathbf{S}_1$. For an arbitrary $r \in \mathbb{S}$ and $q \in \mathbb{S}_r$ we have $r^* q \in \mathbb{S}_1$ hence, $q = rr^* q \in r\mathbb{S}$, that is $\mathbb{S}_r = r\mathbb{S}_1$ and $G\left(\mathbb{S}_r\right) = G\left(r\right) G\left(\mathbb{S}_1\right) = G\left(r\right)\mathbf{S}_1 = \mathbf{S}_{G(r)}$. $\square$

4 Funk transform and inversion

Remind the classical Minkowski-Funk's result [13],[5],[6]: an even function f of the unit sphere $\mathbf{S}^2$ can be uniquely reconstructed from knowledge of integrals over great circles C

$$Mf\left(\mathrm{C}\right) = \int_{\mathrm{C}} f \mathrm{d}\varphi$$

Choose an arbitrary point $s \in \mathbf{S}^2$ and take the average over small circles

$$F\left(s, \phi\right) = \frac{1}{2\pi} \int_{\mathrm{dist}(\mathrm{C}, s) = \phi} Mf\left(\mathrm{C}\right) \mathrm{d}\psi \tag{5}$$

Then

$$f\left(s\right) = -\frac{1}{\pi} \int_0^{\pi/2} \frac{\mathrm{d}F\left(s, \varphi\right)}{\sin\varphi} + \frac{1}{\pi} F\left(s, \frac{\pi}{2}\right) \tag{6}$$

For a bounded function f on $\mathbf{SO}\,(3)$ the Funk transform [1] is an even function on $\mathbf{S}^2 \times \mathbf{S}^2$ defined by

$$Mf\,(\mathbf{h},\mathbf{y}) = \int_{\mathbf{C}(\mathbf{h},\mathbf{y})} f\,(g)\,\mathrm{d}\varphi = \int_0^{2\pi} f\left(g_{\mathbf{h}} R\,(\varphi)\, g_{\mathbf{y}}^{-1}\right)\mathrm{d}\varphi.$$

The pole density function of f is measured in experiments and coincides with the mean

$$Pf\,(\mathbf{h},\mathbf{y}) = \frac{1}{2}\left[Mf\,(\mathbf{h},\mathbf{y}) + Mf\,(-\mathbf{h},\mathbf{y})\right]$$

Any inversion operator R for the Funk transform can be applied to the pole density function. The function $F = RPf$ has the same Fourier coefficients of even degree as f but all coefficients of odd degree vanish (the degree of a Fourier coefficient is the degree of the corresponding Wigner spherical function [11],[2], that is of the corresponding irreducible representation). Odd Fourier coefficients can not be determined from any X-ray diffraction experiment according to the Bunge-Roe theory [3].

5 Reconstruction in the rotation group

We show here two methods of reconstruction of a function on $\mathbf{SO}\,(3)$ from nonredundant data of circle integrals.

Theorem 6 *Let* Γ *be a closed noncontractible plane curve in* $\mathbf{SO}\,(3)$*. Any continuous function* f *in* $\mathbf{SO}\,(3)$ *can be reconstructed from data of integrals over circles that meet* Γ *that is of circles* $\mathbf{C}\,(\mathbf{h},g\mathbf{h})$*,* $\mathbf{h} \in \mathbf{S}^2, g \in \Gamma$.

Remark. The variety of all such $\mathbf{C}$ has dimension 3 hence the data are not redundant.

Proof. Let G be the map as in (3); the lifting $\mathrm{f}\,(s) \doteq f\,(G\,(s))$ is an even function on the sphere $\mathbb{S}$. Let P be an arbitrary 3-subspace of $\mathbb{H}$ that contains the plane curve $G^{-1}\,(\Gamma)$. The intersection $P \cap \mathbb{S}$ is a 2-sphere.

Lemma 7 *Any great circle* $\mathbb{C}$ *in* $P \cap \mathbb{S}$ *has a common point with* $G^{-1}\,(\Gamma)$.

Proof of Lemma. It is sufficient to check that the circle $\mathbf{C} = G\,(\mathbb{C})$ has a common point with Γ. The intersection index $\mathrm{ind}\,(\mathbf{C},\Gamma)$ in the projective plane $G\,(P \cap \mathbb{S})$ is well defined mod 2. It is equal to 1 since both curves are noncontractible. $\square$

We have for any point $s \in P \cap \mathbb{S}$

$$\mathrm{f}\,(s) = -\frac{1}{\pi}\int_0^{\pi/2} \frac{\mathrm{d}F\,(s,\varphi)}{\sin\varphi} + \frac{1}{\pi}F\left(s,\frac{\pi}{2}\right)$$

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[1] We use this name instead of the term Radon transform used in [2]. Reconstruction of an even function on a sphere from data of big circle integrals is due to P. Funk, see references below.

where $F(s,\varphi)$ is the average of of Mf over all circles $\mathbb{C}$ with the distance φ from s. By Proposition 1

$$F(s,\varphi)=\frac{1}{2\pi}\int_{\mathrm{dist}(\mathbb{C},s)=\varphi}M\mathrm{f}(\mathbb{C})\,\mathrm{d}\psi=\frac{1}{2\pi}\int_{\mathrm{dist}(\mathbf{C},G(s))=\varphi/2}Mf(\mathbf{C})\,\mathrm{d}\psi$$

The function f is now reconstructed in the sphere $P\cap\mathbb{S}$, hence f is known on the projective plane $G(P\cap\mathbb{S})$. It is sufficient to check now that the planes $G(P\cap\mathbb{S})$ cover the whole group $\mathbf{SO}(3)$. Take an arbitrary $g\in\mathbf{SO}(3)$ and consider a 3-space P in $\mathbb{H}$ that contains the circle $\mathbb{C}$ and a point $s\in\mathbb{S}$ such that $G(s)=g$. Then $G(P\cap\mathbb{S})$ contains $\mathbf{C}$ and g. □

Theorem 8 *Let Γ be a closed noncontractible curve in $\mathbf{SO}(3)$. Any continuous function f defined on the group can be effectively reconstructed from only knowledge of the integrals $Mf(\mathbf{h},-g\mathbf{h})$ for $\mathbf{h}\in\mathbf{S}^2$ and $g\in\Gamma$.*

Proof. Fix $g\in\Gamma$ and consider the function $f_g(a)=f(ga)$. We have $Mf_g(\mathbf{h},-\mathbf{h})=Mf(\mathbf{h},-g\mathbf{h})$ and the right-hand side is known for any vector $\mathbf{h}\in\mathbf{S}^2$ by the condition. Therefore we know all the integrals

$$Mf_g(\mathbf{h},-\mathbf{h})=\int_{\mathbf{C}(\mathbf{h},-\mathbf{h})}f_g\mathrm{d}\varphi=\int_{\mathbb{C}(\mathbf{h},-\mathbf{h})}G^*(f_g)\,\mathrm{d}\varphi$$

For any $\mathbf{h}\in\mathbf{S}^2$ the circle $\mathbb{C}(\mathbf{h},-\mathbf{h})\doteq G^{-1}(\mathbf{C}(\mathbf{h},-\mathbf{h}))$ is contained in $\mathbb{S}_1$ and any great circle in $\mathbb{S}_1$ has such form. The even function $G^*(f_g)$ can be reconstructed in the plane $\mathbb{S}_1$ by means of (5,6). Therefore the function f_g is known in the plane $\mathbf{S}_1$ and f is determined in $\mathbf{S}_{g^*}$. Check that the planes $\mathbf{S}_{g^*}$ cover the group $\mathbf{SO}(3)$ as g runs over Γ. Consider the map

$$\sigma_\Gamma:\Gamma\times\mathbf{S}_1\to\mathbf{SO}(3),\ (\gamma,a)\mapsto a^*\gamma$$

This is a continuous proper map and the degree $\deg\sigma_\Gamma$ of this map is well defined modulo 2 since Γ and $\mathbf{S}_1$ are connected manifolds (see e.g. [7]). Any noncontractible curve Γ can be homotopically deformed to a circle $\mathbf{C}_0$ since the fundamental group of $\mathbf{SO}(3)$ is isomorphic to $\mathbb{Z}_2$ and is generated by an arbitrary circle. Therefore $\deg\sigma_\Gamma=\deg\sigma_{\mathbf{C}_0}$ We show below that $\deg\sigma_{\mathbf{C}_0}=1$ hence $\deg\sigma_\Gamma=1$ for any Γ. This implies that σ_Γ is surjective. □

Lemma 9 *We have $\deg\sigma_{\mathbf{C}}=1$ for any circle $\mathbf{C}$.*

Proof of Lemma. We can suppose that $\mathbf{C}$ is the group of rotations about a point $\mathbf{n}\in\mathbf{S}^2$. For an arbitrary $g\in\mathbf{SO}(3)$ we consider an arc joining $\mathbf{n}$ and $g(\mathbf{n})$ and take the middle point $\mathbf{m}$ of the arc. Let a be the rotation by π about the point $\mathbf{m}$. We have $ag\mathbf{n}=\mathbf{n}$, hence $\gamma=ag$ is a rotation about $\mathbf{n}$, that is $\gamma\in\mathbf{C}$. Thus $g=a^*\gamma$ that means that $\sigma_\Gamma(\gamma,a)=g$. The rotation a is unique if $g(\mathbf{n})\neq-\mathbf{n}$ whereas the equation $g(\mathbf{n})=-\mathbf{n}$ implies that $g\in\mathbf{S}_1$. In other words, each point $g\in\mathbf{SO}(3)\setminus\mathbf{S}_1$ has only one preimage. It follows that $\deg\sigma_\Gamma=1$. □

6 Another method of reconstruction

Theorem 6 can be obtained by a more constructive method.

Proposition 10 *[14]Let π be the map as (2); for any $k = 2, 3, 4$, an arbitrary central k-plane P in $\mathbb{H}$ and any point $s \in P \cap \mathbb{S}_+$ the equation holds*

$$\frac{\mathrm{d}V_{\mathbb{S}}\left(s, P \cap \mathbb{S}_+\right)}{\mathrm{d}V_E\left(x, Q\right)} = [Q] \left\|x\right\|^{-k} \tag{7}$$

where $\mathrm{d}V_{\mathbb{S}}$ is the volume element in the sphere $\mathbb{S}_+$, $\mathrm{d}V_E$ is the Euclidean volume element on $\mathbb{H}_0$, $[Q]$ denotes the distance from $Q \doteq P \cap (1 + \mathbb{H}_0)$ to the origin $0 \in \mathbb{H}$ and $\|x\|$ is the length of the vector $x = \pi(s)$ in $\mathbb{H}$.

The factorization of the right-hand side is important for below calculations.

Proposition 11 *If $f \in L_2(\mathbf{SO}(3))$ and the integrals $Mf(\mathbf{h}, g\mathbf{h})$ are known for all $\mathbf{h} \in \mathbf{S}^2$ and some $g \in \mathbf{S}_1$ then the Fourier transform $F\phi(\xi)$ of the function*

$$\phi(x) = \|x\|^{-2} \mathrm{f}(s), s = \pi^{-1}(x) = \|x\|^{-1}(1 + x), \ x \in \mathbb{H}_0$$

can be reconstructed in all points $\xi \in \mathbb{R}^3$ such that $g\xi = -\xi$.

Proof. By Proposition 10 the equation holds

$$\int_L \phi(x)\,\mathrm{d}l = [L]^{-1} \int_{\pi^{-1}(L)} \mathrm{f}\mathrm{d}\varphi$$

for any line L in $\mathbb{H}_0$ where $\mathrm{f}(s) = f(G(s))$. Take an arbitrary orthogonal frame (q, r, s) in $\mathbb{H}_0$ such that $G(s) = g$. For any real a and b the line $L(a, b) = \{x; a = \langle q, x\rangle, b = \langle r, x\rangle\}$ is contained in the plane $H_a = \{x \in \mathbb{H}_0; \langle q, x\rangle = a\}$ and the projective closure $P(a, b)$ of $L(a, b)$ contains the improper point $\infty \cdot s$. The circle $\mathbb{C}_{a,b} = \pi^{-1}(P(a, b))$ contains the point s hence the circle $G(\mathbb{C}(a, b))$ contains g. Therefore this circle is equal to $\mathbf{C}(\mathbf{h}, g\mathbf{h})$ for some $\mathbf{h}$ and the integral of f over $\mathbb{C}_{a,b}$ is known for all a, b. The function ϕ is integrable in $\mathbb{H}_0$ and for any plane H_a the integral

$$\int_{H_a} \phi(x)\,\mathrm{d}H = \int_{\mathbb{R}} \mathrm{d}b \int_{L(a,b)} \phi(x)\,\mathrm{d}l = \int_{\mathbb{R}} \frac{\mathrm{d}b}{\left(1 + a^2 + b^2\right)^{1/2}} \int_{\mathbb{C}_{a,b}} \mathrm{f}(s)\,\mathrm{d}\varphi$$
$$= \int_{\mathbb{R}} \frac{\mathrm{d}b}{\left(1 + a^2 + b^2\right)^{1/2}} Mf(\mathbf{h}, g\mathbf{h})$$

is also known. By Proposition 10,

$$\int_{H_a} \phi(x)\,\mathrm{d}H = \frac{1}{1 + a^2} \int_{\mathbb{S}_{a+q}} \|x\|^3 \phi(s)\,\mathrm{d}S = \frac{1}{1 + a^2} \int_{\mathbb{S}_{a+q}} \left(\frac{\|x\|^2}{1 + a^2 + b^2}\right)^{1/2} \mathrm{f}(s)\,\mathrm{d}S \tag{8}$$

where $\mathbb{S}_{a+q} = \pi^{-1}(H_a)$. We have $\|x\|^2 = 1 + a^2 + b^2 + c^2$, $c = \langle s, x\rangle$ and

$$\left(\frac{\|x\|^2}{1+a^2+b^2}\right)^{1/2} = \left(1 - \frac{c^2}{\|x\|^2}\right)^{-1/2}$$

The right hand side is bounded except for a neighborhood of the point γ where the estimate $O\left(\mathrm{dist}^{-1/2}(t,s)\right), t \in \mathbb{S}_{a+q}$ holds. This estimate is square integrable and integral (8) converges. Therefore the Fourier transform

$$F(\phi)(\xi) = \int \exp(-2\pi\imath a\tau)\, \mathrm{d}a \int_{H_a} \phi \mathrm{d}H$$

is known for the vector $\xi = \tau q$ for any $\tau \in \mathbb{R}$ and any unit vector q orthogonal to s. The set of such vectors ξ is equal to the polar of the vector s. The later coincides with the kernel of the matrix $g + e$ which equals $2s \otimes s$ since of (1). This means that $g\xi = -\xi$ and vice versa. $\square$

Theorem 12 *Let* Γ *be a closed noncontractible plane curve in* $\mathbf{SO}(3)$. *Any function* $f \in L_2(\mathbf{SO}(3))$ *can be reconstructed from data of integrals over circles that meet* Γ *that is of circles* $\mathbf{C}(\mathbf{h}, g\mathbf{h})$, $\mathbf{h} \in \mathbf{S}^2, g \in \Gamma$.

Proof. Suppose that $\Gamma \subset \mathbf{S}_a$ for some $a \in \mathbf{SO}(3)$. Introduce the new quaternion variable $r = aq$. The plane $\mathbb{S}_a$ is now given by the equation $r_0 = 0$. Consider the central projection $\pi_a : \mathbb{S}_+ \to 1 + \mathbb{H}_0$, $q \mapsto 1 + \mathbf{r}/\mathbf{r}_0$, where $\mathbb{S}_+$ is the hemisphere in $\mathbb{S}$ with the equator $\mathbb{S}_a$. The function $\phi_a(x) = \|x\|^{-2} \mathrm{f}(s)$, $x = \pi_a(s)$ is defined in $\mathbb{H}_0$ similarly to ϕ.

Lemma 13 *For any* $\xi \in \mathbb{R}^3$ *there exists at least one point* $\mathbf{r} \in \pi^{-1}(\Gamma)$ *such that* $\langle \mathbf{r}, \xi\rangle = 0$.

According to Proposition 11 the Fourier transform $F(\phi_a)$ is known at any point $\xi \in \mathbb{R}^3$ and can find ϕ_a by means of the inverse Fourier transform. Finally we have $\mathrm{f}(s) = \|\pi_a(s)\|^2 \phi_a(\pi_a(s))$. $\square$

Proof of Lemma. Take an arbitrary point $\xi \neq 0$ and consider 2-sphere $P_\xi = \{\mathbf{r} \in \mathbb{S}_1; \langle \mathbf{r}, \xi\rangle = 0\}$ in $\mathbb{S}$. The (non-orientable) intersection index $\mathrm{ind}(P_\xi, \Gamma)$ is well defined in $\mathbb{S}$ as an integer mod 2. Check that $\mathrm{ind}(P_\xi, \Gamma) = 1$. There exists a homotopy $\Gamma \sim \mathbf{C}$ for a circle $\mathbf{C}$ since the curve Γ is not contractible. We have $\mathrm{ind}(P_\xi, \Gamma) = \mathrm{ind}(P_\xi, \mathbf{C})$ since the index is homotopically invariant. Therefore we need only check the statement for the circle. Write a equation $\mathbf{r} = \cos\varphi \mathbf{r}_1 + \sin\varphi \mathbf{r}_2,\ 0 \leq \varphi < \pi$ for $\mathbf{C}$ and have $\langle \mathbf{r}, \xi\rangle = \cos\varphi \langle \mathbf{r}_1, \xi\rangle + \sin\varphi \langle \mathbf{r}_2, \xi\rangle$. This function vanishes just for one value of the angle φ unless ξ is orthogonal to both $\mathbf{r}_1, \mathbf{r}_2$. This yields $\mathrm{ind}(P_\xi, \mathbf{C}) = 1$. $\square$

7 Range conditions for the Funk transform

The well known John equation [9] provides a range condition for the line integral transform in $\mathbb{R}^3$ at least for fast decreasing functions. To my best knowledge the case of the group $\mathbf{SO}(3)$ was first studied in the theory of texture analysis. For an arbitrary $f \in L_2(\mathbf{SO}(3))$ the pole density function is an even function $P \in L_2\left(\mathbf{S}^2 \times \mathbf{S}^2\right)$ which satisfies the ultrahyperbolic equation

$$\Delta_{\mathbf{h}} P(\mathbf{h}, \mathbf{y}) = \Delta_{\mathbf{y}} P(\mathbf{h}, \mathbf{y}) \tag{9}$$

where Δ denotes the angular part of the Laplace operator on 3-space. Vice versa any function $P \in L_2\left(\mathbf{S}^2 \times \mathbf{S}^2\right)$ that fulfils (9) is equal to the pole density function of some $f \in L_2(\mathbf{SO}(3))$ [1], [2].

On the other hand, a range condition for the Funk transform on the rotation group can be obtained by direct reduction from the John equation in terms of quaternion coordinates. Let Σ be the variety of great circles in $\mathbb{S}$. To introduce local coordinates in Σ we use 3-plane sections $\Sigma_k = \{s \in \mathbb{S}_+, s_k = 0\}, k = 0, 1, 2, 3$ of the hemisphere $\mathbb{S}_+$. Fix two of them say Σ_0 and Σ_1. Let $\alpha = \alpha(\mathbb{C})$, $\beta = \beta(\mathbb{C})$ be the intersection points of a circle $\mathbb{C}$ with the spheres Σ_0 and Σ_1. The coordinates α, β are well defined for circles $\mathbb{C}$ transversal to both spheres. Denote by $|\alpha \wedge \beta|$ the area of the parallelogram formed by the vectors α and β.

Proposition 14 *The John equation for the Funk transform on* $\mathbf{SO}(3)$ *reads for the chart* $\Sigma_0 \times \Sigma_1$ *as follows*

$$J_{01} \frac{\Phi(\alpha, \beta)}{|\alpha \wedge \beta|} = 0 \tag{10}$$

where Φ *is a function defined on* Σ *and*

$$J_{01} = \det \begin{pmatrix} \frac{\partial}{\partial \alpha_2} - \alpha_2 E_\alpha - \alpha_2 & \frac{\partial}{\partial \alpha_3} - \alpha_3 E_\alpha - \alpha_3 \\ \frac{\partial}{\partial \beta_2} - \beta_2 E_\beta - \beta_2 & \frac{\partial}{\partial \beta_3} - \beta_3 E_\beta - \beta_3 \end{pmatrix} \tag{11}$$

where

$$E_\alpha = \sum_{p \neq 0} \alpha_p \frac{\partial}{\partial \alpha_p}, E_\beta = \sum_{q \neq 1} \beta_q \frac{\partial}{\partial \beta_q}$$

For any other chart $\Sigma_k \times \Sigma_l$ *the John equation looks as (10) with the operator* J_{01} *replaced by* J_{kl}*.*

Note that the differential operators in (11) commute and are tangent to the spheres Σ_0 and Σ_1 respectively.

See [10] for a similar result.

Proof. John's equation for a function ϕ in $\mathbb{R}^3$ and the family of lines

$$L(a, b) = \{x = (0, b_2, b_3) + (1, a_2, a_3)t, t \in \mathbb{R}\} \tag{12}$$

reads as follows

$$\left(\frac{\partial^2}{\partial a_2 \partial b_3} - \frac{\partial^2}{\partial a_3 \partial b_2}\right) A^{-1} X(a, b) = 0 \tag{13}$$

where

$$X(a, b) = \int_{L(a,b)} \phi(x) \, \mathrm{d}l = A \int_{\mathbb{R}} \phi(t, b_2 + a_2 t, b_3 + a_3 t) \, \mathrm{d}t$$

is the X-ray transform and $A = \left(1 + a_2^2 + a_3^2\right)^{1/2}$. Let π be the central projection as in (2). The equation (7) for a line $L \subset 1 + \mathbb{H}_0$ reads

$$\frac{\mathrm{d}s}{\mathrm{d}l} = \frac{\mathrm{d}V_S\left(s, \mathbb{C}\right)}{\mathrm{d}V_E\left(x, L\right)} = [L]\left\|x\right\|^{-2}$$

where $\mathbb{C} = \pi^{-1}(L)$ is a circle in $\mathbb{S}$, $x = \pi(s)$, $\mathrm{d}l, \mathrm{d}s$ are length elements and $[L]$ is the distance from L to the origin in $\mathbb{H}$. It follows that for any bounded function f on $\mathbb{S}_+$ and any circle $\mathbb{C}$

$$\int_{\mathbb{C}} \mathrm{f}(s)\,\mathrm{d}s = [L] \int_L \phi(x)\,\mathrm{d}l$$

where $\phi(x) = s_0^{-2}\mathrm{f}(s)$. Combining this equation with (13) we come up to (10). $\square$

Remark. The equation (10) is of the ultrahyperbolic type since the principal symbol is a quadratic form of signature (2,2). It is not a hyperbolic operator and the Cauchy problem in whichever noncharacteristic directions is not well posed in a space of nonanalytic functions.

The equations (10) look similar to (9) and vanishes on the image of the Funk transform. In fact it vanishes for any pole density function Φ too a fact proved in the next section.

8 Characteristic surfaces

The reconstructions given in Theorems 6, 8 and 12 give methods of solution of equation (10) with only one data on a 3D subvariety $Z \subset \Sigma$. This looks implausible for the second order equation. The explanation of paradox is that the variety Z is characteristic for the equation and the boundary value problem is similar to a Goursat problem but not to a Cauchy one. A more general statement is true

Proposition 15 *Let Γ be an arbitrary C^1-curve in $\mathbf{SO}(3)$. The manifold Z of all circles $\mathbf{C}$ that meet Γ is characteristic for the equation (10).*

Proof. We find a local equation $\Phi = 0$ for the manifold Z in Σ and check that the principal part of J_{kl} vanishes on $\mathrm{d}\Phi$. To simplify the calculations we replace $\mathbf{SO}(3)$ by its image in $\mathbb{H}_0$ by means of the map πG^{-1}. The images of circles are straight lines. We fix an arbitrary line $L_0 \subset \mathbb{H}_0$ and choose a coordinate system such that L_0 is parallel to $(1, 0, 0)$. Then the family of lines (12) with small $a = (a_2, a_3)$ covers a neighborhood of L_0 in the manifold Σ. We can assume that the curve Γ has the form $\{x; x_1 = \psi(x'), \phi(x') = 0, x' \doteq (x_2, x_3)\}$ for some C^1-functions ϕ, ψ, $\mathrm{d}\phi \neq 0$. Then the condition $x \in L(a, b) \cap \Gamma$ is equivalent to $\phi(x') = 0$, $x' = b + \psi(x')\,a$ where $b = (b_2, b_3)$. For small a this equation has a C^1-solution $x' = \xi(a, b)$ and the function

$$\Phi(a, b) \doteq \phi\left(b + \psi\left(\xi(a, b)\right) a\right)$$

vanishes on Z. Calculating the differential of the equation $\xi = b + \psi(\xi)\, a$ we get

$$\mathrm{d}\xi = \mathrm{d}b + \psi \mathrm{d}a + a \otimes \mathrm{d}\psi \mathrm{d}\xi,\ \ \mathrm{d}\xi = (e - a \otimes \mathrm{d}\psi)^{-1} (\psi \mathrm{d}a + \mathrm{d}b)$$

where e denotes the unit 2×2 matrix; the matrix $e - \mathrm{d}\psi \otimes a$ is invertible for small a. This yields

$$\begin{aligned} \mathrm{d}\Phi &= \mathrm{d}\phi\, (e - \mathrm{d}\psi \otimes a)^{-1} (\psi \mathrm{d}a + \mathrm{d}b) \\ &= \psi \mathrm{d}\phi\, (e - \mathrm{d}\psi \otimes a)^{-1} \mathrm{d}a + \mathrm{d}\phi\, (e - \mathrm{d}\psi \otimes a)^{-1} \mathrm{d}b = \mathrm{d}_a\Phi \mathrm{d}a + \mathrm{d}_b\Phi \mathrm{d}b \end{aligned}$$

We see that the form $\mathrm{d}_b\Phi$ does not vanish and $\mathrm{d}_a\Phi = \psi \mathrm{d}_b\Phi$ which yields

$$\mathrm{d}_a\Phi \wedge \mathrm{d}_b\Phi = 0 \tag{14}$$

This equation means that the manifold $\Phi = 0$ is characteristic for (10). □

Remark. Note that Γ need not to be a plane curve. Moreover, for an arbitrary smooth 2-manifold $M \subset \mathbf{SO}(3)$ the variety of circles tangent to M is also characteristic (see [14], Ch.7).

Proposition 16 *If a 3-surface Z of circles in $\mathbb{S}$ is characteristic, the surface of orthogonal circles $\mathbb{C}^{\intercal}, \mathbb{C} \in Z$ is also characteristic.*

Proof. The coordinates (α', β') of $\mathbb{C}^{\intercal}$ in the chart $\Sigma_i \times \Sigma_j$ are related to the coordinates (α, β) of $\mathbb{C}$ in $\Sigma_k \times \Sigma_l$, $(i, j, k, l) = (0, 1, 2, 3)$ as follows: $\alpha' = \alpha, \beta' = \beta$. Therefore the equation (10) immediately implies

$$J_{ij} \frac{\Phi(\alpha', \beta')}{|\alpha' \wedge \beta'|} = J_{kl} \frac{\Phi(\alpha, \beta)}{|\alpha \wedge \beta|} = 0.$$

□

Corollary 17 *The pole density function P fulfils always the John equations (10).*

Proof. The term $\Phi(\alpha, \beta) = Mf(\mathbf{h}, \mathbf{y})$ fulfils the equation (10) according to Proposition 14. The term $\Psi(\alpha, \beta) = Mf(-\mathbf{h}, \mathbf{y}) = Mf(\mathbf{h}, -\mathbf{y})$ also satisfies this equation since of Proposition 16. □

Remark. The condition (14) is local and is not sufficient for reconstruction from data in Z. A characteristic variety Z must be so large that the conormal bundle $N^*(Z)$ covers the phase space of the group. Otherwise only part of the wave front set of a function f can be reconstructed. The condition that Γ is noncontractible guarantees that the phase space is covered.

Proposition 18 *Let Γ be an arbitrary C^1-curve in $\mathbf{SO}(3)$. The manifold Z of all circles $\mathbf{C}(\mathbf{h}, -g\mathbf{h})$, $\mathbf{h} \in \mathbf{S}^2$, $g \in \Gamma$ is characteristic for the equation (10).*

Proof. By Proposition 3 the circle orthogonal to $\mathbf{C}(\mathbf{h}, -g\mathbf{h})$ is $\mathbf{C}(\mathbf{h}, g\mathbf{h})$. The family of circles $\mathbf{C}(\mathbf{h}, g\mathbf{h})$, $g \in \Gamma$ is characteristic according to the previous Proposition. Therefore the family $\{\mathbf{C}(\mathbf{h}, -g\mathbf{h}), g \in \Gamma\}$ is also characteristic in virtue of Proposition 16. □

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