

# Associative deformations of complex analytic spaces

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**Abstract.** A theory of associative deformations is developed for general complex analytic spaces. Deformation quantization and commutative deformation are particular cases of this concept. Deformation cohomology and obstruction are studied. It is proved that any compact analytic space has a formal versal associative deformation.

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## 1 Introduction

A general concept of quantization was developed by F. Berezin [3]. The conception of deformation-quantization was elaborated in [2] in the spirit of deformation philosophy was promoted by M. Flato since the early seventies. A global formal quantization of a smooth Poisson manifold was constructed by De Wilde-Lecomte [6], Omori-Maeda-Yoshioka [14] and Fedosov [7] in the symplectic case and by Kontsevich [12] in the general case. Another approach based on Kontsevich's local formula was used by Cattaneo-Felder-Tomassini [5]. For algebraic manifolds, the problem of global quantization is discussed in [13]. Yekutieli [19] has adapted the method of [5] for construction of quantization for a class of Poisson algebraic manifolds.

The similarity of deformation-quantizations and deformations in the commutative geometry was stressed by Sternheimer [18]. The unifying concept is the notion of *associative* deformation. Note the pioneering paper of Gerstenhaber [9] on associative deformations of commutative algebras. He did few steps paving a bridge to Kodaira-Spencer's theory [10].

Our objective is to develop the theory of associative deformations for the category of complex analytic spaces. The basic notions: flatness, pull back, infinitesimal deformation, extension are taken from Grothendieck's commutative theory [11], which generalizes Kodaira-Spencer's approach. We focus on the formal theory of associative deformations. The basic construction is the deformation complex of a complex analytic space  $X$ , where the universal quadratic equation of deformation is evaluated. In particular, the differential of a deformation is a linear map to the first cohomology. Any obstruction to extension of a deformation is evaluated in the second cohomology. The cohomology of the deformation complex is expressed in terms of the cohomology of the Hochschild cochain complex of polydifferential operators in the structure sheaf. We shall see that the differential of any associative deformation canonically splits into a sum of a commutative deformation (the Kodaira-Spencer class) and of a 'pure' quantization of  $X$  (the 'Poisson' class). A formal versal associative deformation always exists provided the first global cohomology of  $X$  has finite dimension.

## 2 Complex analytic spaces and deformations

We remind some basic notions of the theory of complex analytic spaces.

**Analytic algebras and germs.** Let  $\mathbb{C}$  be the field of complex numbers and  $\mathbb{C}\{z_1, \dots, z_n\}$ ,  $n \geq 0$  be the algebra of convergent power series of complex variables  $z_1, \dots, z_n$  with coefficients in  $\mathbb{C}$ . This algebra is local with the maximal ideal  $\mathfrak{m}$  of series with zero free term. A regular analytic algebra  $\mathcal{O}$  is a  $\mathbb{C}$ -algebra isomorphic to  $\mathbb{C}\{z_1, \dots, z_n\}$  for some  $n$ . An analytic algebra  $\mathbf{A}$  is by definition a quotient  $\mathcal{O}/\mathfrak{l}$  of a regular analytic algebra  $\mathcal{O}$ . It is always a local algebra with the maximal ideal  $\mathfrak{m}(\mathbf{A}) = \mathfrak{m}(\mathcal{O})/\mathfrak{l}$ . An analytic *germ* is a germ  $(S, s)$  of a closed set  $S \subset \mathbb{C}^n$  at a point  $s \in S$ , endowed with an analytic algebra  $\mathcal{O}_S \cong \mathbb{C}\{z_1, \dots, z_n\}/\mathfrak{l}$ , where the coordinate functions  $z_1, \dots, z_n$  vanish at the point  $s$  and  $\mathfrak{l}$  is the ideal generated by some elements  $f_1, \dots, f_m \in \mathbb{C}\{z_1, \dots, z_n\}$  such that the set  $f_1(z) = \dots = f_m(z) = 0$  coincides with  $S \cap U$  for a neighborhood  $U$  of  $s$ . If  $S = \{s\}$ ,  $\mathcal{O}_S$  is called Artin algebra; the equivalent condition is  $\mathfrak{m}^k(\mathcal{O}_s) = 0$  for some  $k$ . In particular, a point  $\circ$  endowed with the algebra  $\mathbb{C}$  is an Artin germ (simple point).

A mapping of germs  $(R, \mathcal{O}_R) \rightarrow (S, \mathcal{O}_S)$  is a continuous map of germs  $f : (R, \circ) \rightarrow (S, \circ)$  together with a homomorphism  $\mathbb{C}$ -algebras  $\phi : \mathcal{O}_S \rightarrow \mathcal{O}_R$  such that  $\phi(a)(s) = a(f(s))$  for any  $s \in S$  and  $a \in \mathcal{O}_S$ . The morphism  $(f, \phi)$  is called *flat*, if  $\mathcal{O}_R$  is flat  $\mathcal{O}_S$ -module via  $\phi$ . A map  $(f, \phi)$  is called *imbedding*, if  $\phi$  is a surjection. Differently speaking,  $\mathcal{O}_S$  is an extension of  $\mathcal{O}_R$  by means of the ideal  $\mathfrak{J} = \text{Ker } \phi$ . The mapping  $(f, \phi)$  is called *small extension*, if  $\mathfrak{m}(\mathcal{O}_S)\mathfrak{J} = 0$ . For any analytic germ  $(S, \circ; \mathbf{A})$  the subgerm  $S_1$  defined by the quotient algebra  $\mathbf{A}/\mathfrak{m}^2(\mathbf{A})$  is a small extension of the

simple point  $\{o\}$ . Any  $\mathbb{C}$ -derivation  $t : \mathbf{A} \rightarrow \mathbb{C}$  can be uniquely lifted to  $\mathbf{A}/\mathfrak{m}^2(\mathbf{A})$ . The set of all derivations is dual to the space  $\mathfrak{m}(\mathbf{A})/\mathfrak{m}^2(\mathbf{A})$ ; it is called the tangent space to  $S$  and is denoted  $T(S)$ . There always exists a (non canonical) imbedding of germs  $S \rightarrow T(S)$ .

**Complex analytic spaces.** For an open set  $U \subset \mathbb{C}^n$  we denote  $\mathcal{O}_U$  the sheaf of germs of holomorphic functions in  $U$ , that is  $\mathcal{O}_{U,w} = \mathbb{C}\{z_1 - w_1, \dots, z_n - w_n\}$ . A *complex analytic space* (c.a.s.) is a topological space  $X$  together with a sheaf  $\mathcal{O}_X$  of analytic algebras, such that for any point  $x \in X$  there exist a neighborhood  $Y$ , a proper imbedding  $\phi : Y \rightarrow U \subset \mathbb{C}^n$  for some  $n$ , where  $U$  is open in  $\mathbb{C}^n$ , and a surjective morphism of sheaves  $\pi_U : \mathcal{O}_U \rightarrow \phi_*(\mathcal{O}_X|Y)$ , whose kernel is the sheaf of ideals generated by some holomorphic functions  $f_1, \dots, f_m$  in  $U$ . Below, we always assume that any c.a.s. is a union of countable set of compact sets.

A morphism of complex spaces is a pair:  $(f, \phi)$ , where  $f : X \rightarrow Y$  is a continuous map and  $\phi : f^*(\mathcal{O}_Y) \rightarrow \mathcal{O}_X$  is a homomorphism of sheaves of  $\mathbb{C}$ -algebras on  $X$ . It is called flat, if for any point  $x \in X$  the homomorphism  $\phi_x : \mathcal{O}_{Y,y} \rightarrow \mathcal{O}_{X,x}$ ,  $y = f(x)$  makes  $\mathcal{O}_{X,x}$  a flat module over  $\mathcal{O}_{Y,y}$ .

**Deformation.** Let  $(X, \mathcal{O}_X)$  be a c.a.s. and  $(S, \mathcal{O}_S)$  be an Artin germ. An *associative deformation* (briefly, deformation) of  $X$  with the base  $S$  is called any  $\mathcal{O}_S$ -sheaf  $\mathcal{A}$  on  $X$  endowed with an associative multiplication  $\mathcal{O}_S$ -bilinear operation  $\mu : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ , together with an isomorphism  $\alpha : \mathcal{O}_X \cong \mathcal{A} \otimes_{\mathcal{O}_S} \mathbb{C}$  of sheaves of  $\mathbb{C}$ -algebras such that

(i)  $\mathcal{O}_S$ -sheaf  $\mathcal{A}$  is flat,

(ii) the sheaf morphism  $\mu \otimes \mathbb{C} : \mathcal{A} \otimes_{\mathcal{O}_S} \mathbb{C} \times \mathcal{A} \otimes_{\mathcal{O}_S} \mathbb{C} \rightarrow \mathcal{A} \otimes_{\mathcal{O}_S} \mathbb{C}$  is isomorphic to the (commutative) multiplication in  $\mathcal{O}_X$  via  $\alpha$ . This implies that each fiber  $\mathcal{A}_x$  is a local  $\mathbb{C}$ -algebra with the maximal two-side ideal  $\mathfrak{M}_x = \rho^{-1}(\mathfrak{m}_x)$ , where  $\mathfrak{m}_x$  is the maximal ideal in  $\mathcal{O}_x$  and  $\rho : \mathcal{A} \rightarrow \mathcal{O}_X$  is the surjection generated by  $\alpha$ .

(iii)  $\mu$  is continuous in the sense, that for any compact set  $K \subset X$  and a number  $k$  there exists a number  $l$  such that for any point  $x \in K$  we have  $\mu(a, b) \in \mathfrak{M}_x^k$  provided  $ab \in \mathfrak{M}_x^l$ .

(iv) For any point  $x \in X$  there exist a neighborhood  $Y$ , a proper embedding  $\phi : Y \rightarrow U$  as in the previous definition and a surjective morphism of  $\mathcal{O}_S$ -sheaves  $\pi_S : \mathcal{O}_{U \times S} \rightarrow \phi_*(\mathcal{A}|Y)$  such that  $\text{Ker } \pi_S$  is a coherent  $\mathcal{O}_{U \times S}$ -subsheaf and the diagram commutes

$$\begin{array}{ccccc} \mathcal{O}_{U \times S} & \xrightarrow{\pi_S} & \phi_*(\mathcal{A}|Y) & \rightarrow & 0 \\ \rho \downarrow & & \rho|Y \downarrow & & \\ \mathcal{O}_U & \xrightarrow{\pi_U} & \phi_*(\mathcal{O}_X|Y) & \rightarrow & 0 \end{array}, \quad (1)$$

where  $\pi_U$  is as above and the morphism  $\rho$  is natural.

**Equivalence.** We say that deformations  $(X, S, \mathcal{A}, \mu_{\mathcal{A}})$  and  $(X, S, \mathcal{B}, \mu_{\mathcal{B}})$  of  $X$  with the same base  $S$  are equivalent, if there exist a isomorphism of sheaves of

$\mathcal{O}_S$ -algebras  $\varepsilon : \mathcal{B} \rightarrow \mathcal{A}$  on  $X$  such that  $\mu_{\mathcal{A}}(\varepsilon(a), \varepsilon(b)) = \varepsilon\mu_{\mathcal{B}}(a, b)$ .

**Pull back.** Let  $F = (X, S, \mathcal{A}, \mu)$  be a deformation and  $\phi : (R, \mathcal{O}_R) \rightarrow (S, \mathcal{O}_S)$  be a morphism of germs. The tensor product  $\mathcal{B} \doteq \mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{O}_R$  is a  $\mathcal{O}_R$ -sheaf and  $\pi_R = \pi_S \otimes \mathcal{O}_R : \mathcal{O}_{U \times R} \rightarrow \phi_*(\mathcal{B}|_Y)$  is a  $\mathcal{O}_R$ -surjection. The operation  $\mu_R = \mu \otimes \mathcal{O}_R$  is well defined in  $\mathcal{B}$ , is associative and fulfils the equation  $\mu_R \otimes \mathbb{C} = \mu \otimes \mathbb{C}$ . The quadruple  $(X, R, \mathcal{B}, \mu_R)$  is a deformation of  $X$ ; It is called *pull back* of  $F$  by  $\phi$  and is denoted  $\phi^*(F)$ .

**Versal deformation.** A *formal* germ  $\hat{S}$  is the direct limit a sequence of imbeddings of Artin germs:  $\phi_k : S_k \hookrightarrow S_{k+1}$ ,  $k = 0, 1, 2, \dots$ ,  $S_0 = \{\circ\}$ . A morphism of formal germs  $\phi : \hat{S} \rightarrow \hat{T}$  is defined as a commutative diagram of morphisms of Artin germs  $\phi_k : S_k \rightarrow T_{\alpha(k)}$ ,  $k = 1, 2, \dots$ , where  $\alpha$  is monotone function on the set  $\mathbb{N}$  of natural numbers. A *formal* deformation of a space  $X$  with the formal base  $\hat{S}$  is the inverse limit  $\hat{F}$  of a sequence of deformations  $F_k$  of  $X$ , where the base of  $F_k$  is  $S_k$  and there is equivalence  $\phi_k^*(F_{k+1}) \simeq F_k$  for  $k \geq 1$ . A formal deformation  $F$  is called *versal*, if for any deformation  $G = (X, \hat{R}, \mathcal{B}, \nu)$  with a formal base  $\hat{R}$  there exist a morphism of formal germs  $\phi : \hat{R} \rightarrow \hat{S}$  and an equivalence  $\varepsilon : F \simeq \phi^*(G)$  over  $\hat{R}$ . The versal deformation is called *universal*, if the morphism  $\phi$  is uniquely defined.

**Linearization.** Let  $\Delta$  be the analytic germ  $(\circ, \mathcal{O}_\Delta)$ , with the algebra  $\mathcal{O}_\Delta = \mathbb{C}\{z\}/(z^2)$ . The tangent space to  $\Delta$  is spanned by the vector  $d/dz : a + bz \mapsto b$ . For a c.a.s.  $X$  we denote by  $D(X)$  the set of equivalence classes of deformations of  $X$  with the base  $\Delta$ . This set has a natural structure of  $\mathbb{C}$ -vector space. Any morphism of analytic germs  $\delta : \Delta \rightarrow S$  defines a derivation  $t : \mathcal{O}_S \rightarrow \mathbb{C}$ , which is the composition of the homomorphism  $\mathcal{O}_S \rightarrow \mathcal{O}_D$  and of the vector  $d/dz$ . Vice versa, any derivation  $t : \mathcal{O}_S \rightarrow \mathbb{C}$  can be obtained from a morphism  $\delta$ . For an arbitrary deformation  $F$  of  $X$  the pull back  $\delta^*(F)$  a deformation with the base  $D$ , which defines an element of  $D(X)$ . It has sense of the ‘derivative’ of  $F$  in the direction  $t$ . The rule  $\delta \mapsto \delta(F)$  determines a linear map  $DF : T(S) \rightarrow D(X)$ , which we call the *differential* or the Kodaira-Spencer-Poisson (KSP) class of  $F$ . For any versal deformation  $F$  the differential  $DF$  must be surjective. A versal deformation  $F$  is called minimal, if  $DF$  is bijective.

### 3 Globalization

In this section we reproduce some constructions of [15]. A *free graded differential analytic* sheaf-algebra is a graded sheaf  $\mathcal{R}^* = \bigoplus_{k \leq 0} \mathcal{R}^k$  with a differential  $d$  of degree 1 such that

(i)  $\mathcal{R}^0$  is the sheaf of holomorphic functions on the open unit polydisc  $\mathbb{D}^N \subset \mathbb{C}^N$  for some  $N$ ,

(ii)  $\mathcal{R}^*$  has a structure of graded commutative algebra,

(iii) a system of homogeneous sections of  $\mathcal{R}$  is distinguished; it consists of the set  $\{z_1, \dots, z_N\}$  of coordinate functions in  $\mathbb{C}^N$  and of the set  $E$  of free generators of  $\mathcal{R}^*$  over  $\mathcal{R}^0$ , that is  $\mathcal{R}^* = \mathcal{R}^0[E]$ . Each generator  $e \in E$  has negative grading and the number of generators is finite in each grading.

(iv)  $d$  is a derivative that is  $d(ab) = d(a)b + (-)^{|a|}ad(b)$  for any homogeneous  $a \in \mathcal{R}^{|a|}$  and any  $b \in \mathcal{R}^*$ . This implies that  $d$  vanishes on  $\mathcal{R}^0$  and is a  $\mathcal{R}^0$ -morphism.

Let FGDA be the category of free graded differential analytic sheaf-algebras. A morphism of the category FGDA is an injective homomorphism of graded differential algebras  $\varphi : p^*(\mathcal{R}[E], d) \rightarrow (\mathcal{S}[F], d')$  such that  $\varphi(E) \subset F$ , where  $p : \mathbb{C}^N \rightarrow \mathbb{C}^M$  is a coordinate projection.

Let  $\mathcal{O}$  be a coherent sheaf of analytic algebras defined in a neighborhood of the closure of the polydisc  $\mathbb{D}^N$ . It always has a resolution  $(\mathcal{R}^*, d)$  in  $\mathbb{D}^N$ , which is an object of FGDA (called Tyurina resolution). The algebra of sections  $R \doteq \Gamma(\mathbb{D}^N, \mathcal{R})$  is a  $R^0$ -free resolution of the algebra  $\Gamma(\mathbb{D}^N, \mathcal{O})$ .

**Polyhedrons.** A *polyhedron* in a c.a.s.  $X$  is a relatively compact subset  $P \subset X$  together with a proper holomorphic imbedding  $\phi_P : Y \hookrightarrow U$ , where  $\bar{P} \subset Y$ ,  $\mathbb{D}^N \Subset U \subset \mathbb{C}^N$  such that  $P = \phi^{-1}(\mathbb{D}^N)$ . The map  $\phi_P$  is called the barrier mapping of  $P$ . If  $P, Q$  are polyhedrons in  $X$ , the intersection  $P \cap Q$  is again a polyhedron with the barrier map  $\phi_P \times \phi_Q$ . A polyhedral covering of the space  $X$  is a countable locally finite covering  $P = \{P_\alpha, \alpha \in \mathbb{N}\}$  that fulfils the condition: for any  $\alpha \in \mathbb{N}$ , the domain  $Y_\alpha$  of barrier map  $\phi_{P_\alpha} : Y_\alpha \rightarrow \mathbb{C}^{N(\alpha)}$  contains all the polyhedrons  $P_\beta$ ,  $\beta \in \mathbb{N}$  such that  $\bar{P}_\alpha \cap \bar{P}_\beta \neq \emptyset$ . Any c.a.s.  $X$  (which is assumed  $\sigma$ -compact) has a polyhedral covering  $P$ , moreover for an open covering  $V$  of  $X$  we can choose a finer polyhedral covering of  $X$ .

A *simplex* of the covering  $P$  is an ordered subset  $A = \{\alpha_0, \dots, \alpha_d\} \subset \mathbb{N}$  such that the polyhedron  $P_A \doteq \bigcap_{\alpha \in A} P_\alpha$  is not empty. We use the notation  $\dim A = d$  and always assume that  $\alpha_0 < \dots < \alpha_d$ . The barrier map of this polyhedron is equal to  $\phi_A \doteq \times_A \phi_{P_\alpha} : \bigcap_A Y_\alpha \rightarrow \times_A \mathbb{C}^{N(\alpha)}$ . The *nerve*  $N$  of the covering  $P$  is the category of all simplices of the covering and of all inclusions  $B \subset A$ .

**Resolution.** Let  $(X, \mathcal{O})$  be a c.a.s. and  $P$  its polyhedral covering with the nerve  $N$ . We call a *resolution* of  $X$  on  $P$  any covariant functor  $\mathcal{R} : N \rightsquigarrow \text{FGDA}$  such that

(v) for any simplex  $A$  the sheaf  $\mathcal{R}(A)$  is a resolution of  $\phi_* (\mathcal{O}|_{P_A})$ .

For any simplex  $A \in N$  we have  $\mathcal{R}(A) = \mathcal{R}^0[E(A)]$ , where  $E(A)$  is a distinguished set of generators of negative grading. Moreover, for any subsimplex  $i_A^B : B \subset A$  we have the inclusion  $\mathcal{R}(i_A^B)(E(B)) \subset E(A)$ . Any element  $e \in E(A)$  that does not belong to the union  $\bigcup \mathcal{R}(i_A^B)E(B)$  taken over all proper subsets  $B \subset A$ , is called *proper* generator. We can assume that  $|e| \leq -\dim A$  for any proper generator  $e$ .

Given a resolution  $\mathcal{R}$  of a space  $X$ , we define the functor  $R \doteq \Gamma(\mathcal{R})$  from the nerve  $N$  to the category GDA of graded differential  $\mathbb{C}$ -algebras by the formula:  $R(A) = \Gamma(\mathbb{D}^{N(A)}, \mathcal{R}(A))$ , for any simplex  $A$ . The morphism  $R(i_A^B)$  is generated by  $\mathcal{R}_A^B$  for any subsimplex  $B \subset A$ .

**Theorem 3.1** [15] *For an arbitrary c.a.s.  $X$  and any locally finite polyhedral covering  $P$  of  $X$  there exists a resolution  $\mathcal{R}$  of  $X$  on  $P$ .*

## 4 The deformation complex

Let  $\mathcal{S} = (\mathcal{H}[E], d)$ ,  $\mathcal{S}' = (\mathcal{H}'[E'], d')$  be FGDA-sheaves on polydiscs  $\mathbb{D} \subset \mathbb{C}^N$ ,  $\mathbb{D}' \subset \mathbb{C}^M$ , respectively. The *analytic* tensor product of these sheaves is the sheaf  $\mathcal{S} \odot \mathcal{S}' = \mathcal{H} \odot \mathcal{H}'[E \cup E']$ , where  $\mathcal{H} \odot \mathcal{H}'$  means the sheaf of holomorphic functions on  $\mathbb{D} \times \mathbb{D}'$ . The product  $\mathcal{S} \odot \mathcal{S}'$  is again a FGDA-sheaf with the grading  $||[a|b]| = |a| + |b|$  and the differential  $d|1 + 1|d'$ , where we denote  $[a|b] = a \otimes b$ . We use the notation  $\mathcal{S}^{\odot n}$  for the multiple analytic tensor product, and  $[a_1|\dots|a_n]$  for the product of elements  $a_1, \dots, a_n$ ,  $a_i \in \mathcal{S}$ . The canonical bornology in  $S \doteq \Gamma(\mathbb{D}^N, \mathcal{S})$  is defined as follows: a  $B \subset S$  is called *bounded*, if functions  $f \in B$  are uniformly bounded on any compact  $K \subset \mathbb{D}^N$ . The similar construction is applied to any term of the graded algebra  $S[E]$ , which is a free  $S$ -module of finite rank. Set  $S' = \Gamma(\mathbb{D}^M, \mathcal{S}')$  and define the analytic tensor products  $S \odot S' \doteq \Gamma(\mathbb{D}^N \times \mathbb{D}^M, \mathcal{S} \odot \mathcal{S}')[E \cup E']$ . The natural injection  $S[E] \otimes S'[E'] \rightarrow S[E] \odot S'[E']$  has dense image. Remind that a linear operator acting in bornological spaces is called bounded if it maps any bounded set to a bounded set.

Let  $P = \{P_\alpha, \alpha \in \mathbb{N}\}$  be a polyhedral covering of  $X$  with the nerve  $N$ ,  $\mathcal{R}$  be a resolution of  $X$  on this covering. We define the sequence of functors  $R^{\odot n} : N \rightsquigarrow$  GDA, where we define  $R^{\odot n}(A) = R(A)^{\odot n}$  for any simplex  $A \in N$ . Given a linear  $N$ -map  $\phi : R^{\odot n+1} \rightarrow R$ , we define degree  $\deg \phi = n$  (shifted by -1 Hochschild degree) and the *grading*  $|\phi|$  by the rule:  $|\phi| = |\phi(\alpha)| - |\alpha|$  for any homogeneous  $\alpha = [a_1|\dots|a_n] \in R^{\otimes n+1} \subset R^{\odot n+1}$ , where  $|\alpha| = |a_1| + \dots + |a_n|$ . We extend  $\phi$  to a linear operator  $\Phi$  defined on the bigraded space  $R^{\otimes*}$  by the formula

$$\Phi[a_1|\dots|a_{n+k+1}] = \sum_{i=0}^k (-)^{\sigma} [a_1|\dots|a_i|\phi([a_{i+1}|\dots|a_{i+n+1}])|a_{i+n+2}|\dots|a_{k+n+1}], \quad (2)$$

where  $\sigma = in + (|a_1| + \dots + |a_i|)|\phi|$ . We set  $\Phi[a_1|\dots|a_m] = 0$  for  $m \leq n$ . If  $\delta$  is bounded with respect to the canonical bornology induced from  $R^{\odot*}$ , it can be uniquely extended to a bounded operator  $\Phi : R^{\odot n+k} \rightarrow R^{\odot k}$ ,  $k = 2, 3, \dots$ . The operator  $\Phi$  has bigrading (degree and grading)  $(-n, |\phi|)$ . In particular, the extension of

the differential  $d : \mathbb{R} \rightarrow \mathbb{R}$  is the operator

$$d[a_1|\dots|a_m] = \sum (-)^{|a_1|+\dots+|a_i|} [a_1|\dots|a_i|da_{i+1}|a_{i+2}|\dots|a_m].$$

of bidegree  $(0, 1)$ . The extension of the multiplication operator  $\partial : \mathbb{R}^{\otimes 2} \rightarrow \mathbb{R}$ ,  $\partial(a|b) = ab$  by the rule (2) is the standard chain differential of bidegree  $(1, 0)$

$$\partial[a_1|\dots|a_m|a_n] = \sum_1^{n-1} (-)^i [a_1|a_2|\dots|a_{i-1}|a_i a_{i+1}|a_{i+2}|\dots|a_n]$$

without boundary terms. Its extension to the space  $\mathbb{R}^{\odot n}$ ,  $n \geq 2$  looks as follows

$$\partial_n \alpha(z^1, \dots, z^n) = \sum_{i=0}^{i=n} (-)^i \alpha(z^1, \dots, z^{i-1}, z^i, z^i, z^{i+1}, \dots, z^n). \quad (3)$$

Consider the bigraded space  $\text{Hom}(\mathbb{R}^{\odot*}, \mathbb{R})$  of bounded linear mappings  $f : \mathbb{R}^{\odot*} \rightarrow \mathbb{R}$  (called cochains). Introduce the total degree  $\text{Deg } f = \text{deg } f + |f|$ . Define composition of cochains by the formula  $(f \circ g)(a) = f(G(a))$  for  $a \in \mathbb{R}^{\odot*}$ , where  $G$  is the extension of  $g$  as in (2). We have  $\text{deg}(f \circ g) = \text{deg } f + \text{deg } g$  and  $|(f \circ g)| = |f| + |g|$ . The commutator  $[f, g] = f \circ g - (-)^{\text{Deg } f \text{ Deg } g} g \circ f$  is a graded Lie bracket operation with respect to the total degree, since it fulfils the Jacobi identity

$$[f, [g, h]] = [[f, g], h] + (-)^{\text{Deg } f \text{ Deg } g} [f, [g, h]]$$

In particular, the commutator  $(-)^{\text{deg } f} [\partial, \cdot]$  is equal to the standard differential  $\partial^*$ . The Jacobi identity implies the equation

$$\partial^* [f, g] = [\partial^* f, g] + (-)^{\text{Deg } f} [f, \partial^* g]. \quad (4)$$

Let  $\text{Diff}(\mathbb{R}^{\odot*}, \mathbb{R})$  be the subspace of  $\text{Hom}(\mathbb{R}^{\odot*}, \mathbb{R})$ , whose elements are polydifferential operators. This subspace is stable under the operators  $\partial, d$ , so  $\text{Diff}$  is a bicomplex. Consider the total complex  $\text{Tot} \doteq \text{Tot}^*(\mathcal{R})$  of the bicomplex  $\text{Diff}(\mathbb{R}^{\odot*}, \mathbb{R})$  with the total degree  $\text{Deg}$  and the differential

$$[D, f] = (-)^{\text{deg } f} [d, f] + [\partial, f] = (-)^{\text{deg } f} ([d, f] + \partial^* f)$$

where  $D = (-)^{\text{deg } f} d + \partial$ . We call it the deformation complex of  $X$ . The bracket operator applied to polydifferential operators is again a polydifferential operator, which implies that  $\text{Tot}$  is a graded Lie algebra.

## 5 The global cohomology

The graded space  $QT^*(\mathcal{R}) \doteq H^*(\text{Tot}^*(\mathcal{R}), D)$  is called the *associative* cohomology of  $\mathcal{R}$ . By (4), the bracket is inherited in the deformation cohomology and keeps satisfying the Jacobi identity. Therefore the deformation cohomology has structure of graded Lie algebra.

**Proposition 5.1** *For any c.a.s.  $X$  and any polyhedral coverings  $P$  and  $P'$  and respective resolutions  $\mathcal{R}$  and  $\mathcal{R}'$  of a  $X$  there exists a homotopy of graded differential Lie algebras  $\text{Tot}^*(\mathcal{R}) \sim \text{Tot}^*(\mathcal{R}')$ . It follows that the graded Lie algebra  $QT^*(\mathcal{R})$  does not depend in a natural sense on choice of the covering and of the resolution of  $X$ .*

◀ Sketch of a proof. Take the covering  $Q = P \cup P'$  and construct a resolution  $\mathcal{S}$  on  $Q$  together with a  $N$ -morphism  $r : \mathcal{R} \odot \mathcal{R}' \rightarrow \mathcal{S}$  with values in the category FGDA. The resolvent  $\mathcal{S}$  is generated by the set of all generators of  $\mathcal{R}$  and  $\mathcal{R}'$ , which is completed by additional generators to fulfil the condition (v) of Sec. 3 as in [15]. We call  $\mathcal{S}$  a product of  $\mathcal{R}$  and  $\mathcal{R}'$ . Consider the morphism  $r(\text{id} \odot 1) : \mathcal{R} \rightarrow \mathcal{R} \odot \mathcal{R}' \rightarrow \mathcal{S}$ , which is well defined as a functor  $N \rightsquigarrow \text{FGDA}$ . It generates a homomorphism of differential Lie algebras  $\sigma : \text{Tot}^*(\mathcal{R}) \rightarrow \text{Tot}^*(\mathcal{S})$  defined as follows:  $\sigma q(A) = r(\text{id} \odot 1) q(A)$  for any simplex of  $P$  and  $\sigma q(A) = 0$  otherwise. This is a homotopy of complexes. This together with the homotopy  $\text{Tot}^*(\mathcal{R}') \sim \text{Tot}^*(\mathcal{S})$  completes the proof. ▶

Below we use the notation  $QT^*(X)$  for this algebra. Consider the  $\mathcal{O}$ -sheaf  $\mathcal{H}^*(\mathcal{O})$ , whose stalks are *analytic* Hochschild cohomology modules  $\mathcal{H}^*(\mathcal{O}_x)$ ,  $x \in X$ . For calculations see [8] (polynomial case) and [17]. The analytic sheaves  $\mathcal{H}^k$ ,  $k \geq 0$  are always coherent.

**Theorem 5.2** *There exists a spectral sequence  $E$  with the second term*

$$E_2^{pq} = H^p(X, \mathcal{H}^{q+1}(\mathcal{O}))$$

*that converges to the deformation cohomology.*

◀ Introduce the decreasing filtration  $\text{Tot}^{(p)}$ ,  $p = 0, 1, \dots$  in  $\text{Tot}(\mathcal{R})$ , where  $\text{Tot}^{(p)}$  means the space of polydifferential  $N$ -operators  $h : R^{\odot*} \rightarrow R$  that vanish on all simplices  $A$ ,  $\dim A < p$ . This filtration is invariant with respect to the differential  $d$ . Consider the spectral sequence  $E$  generated by this filtration. By arguments of [15]

$$E_0^{p*} = \text{Tot}^{(p)} / \text{Tot}^{(p+1)} = \bigoplus_{\dim A=p} \text{Diff}(R(A)^{\odot*}, R(A)), p \geq 0.$$

It follows

$$E_1^{p*} \cong \bigoplus_{\dim A=p} H^*(\text{Diff}(R(A)^{\odot*}, R(A)), [d, \cdot])$$

**Lemma 5.3** *We have  $H^*(\text{Diff}(\mathbb{R}(A)^{\odot*}, \mathbb{R}(A)), [d, \cdot]) \cong \Gamma(X_A, \mathcal{H}^{*+1})$ .*

By this Lemma we have  $E_1^{p*} \cong \bigoplus_{\dim A=p} \Gamma(X_A, \mathcal{H}^{*+1})$ ,  $d_1 = \delta$ . This yields  $E_2^{pq} \cong H^p(X, \mathcal{H}^{q+1})$  and completes the proof of Theorem.  $\blacktriangleright$

$\blacktriangleleft$  Proof of Lemma 5.3. The augmentation  $\pi : (\mathbb{R}(A), d) \rightarrow \mathcal{O}(A) \doteq \Gamma(X_A, \mathcal{O})$  generates the morphism of complexes

$$(\text{Diff}(\mathbb{R}(A)^{\odot*}, \mathbb{R}(A)), [d, \cdot]) \rightarrow (\text{Diff}(\mathbb{R}(A)^{\odot*}, \mathcal{O}(A)), d^*) \quad (5)$$

which induces a cohomology isomorphism, since  $\pi$  does so. Consider the increasing filtration  $\text{Diff}_q(A)$ ,  $q = 0, -1, \dots$  in the left-hand side, where  $\text{Diff}_q(A)$  consists of operators  $f$  that vanish on all elements  $a \in \mathbb{R}(A)^{\odot*}$ ,  $|a| \leq q$ . Let  $E'$  be the corresponding spectral sequence; we have  $E_0 = (\text{Diff}(\mathbb{R}(A)^{\odot*}, \mathcal{O}(A)), \partial^*)$  and  $E'_1 = H^*(\text{Diff}(\mathbb{R}(A)^{\odot*}, \mathcal{O}(A)), \partial^*)$ . The differential  $d_1$  in  $E'_1$  acts by  $f \mapsto fd$ . By Theorem 11.1 the right-hand side is isomorphic to  $\wedge^* \text{Der}(\mathbb{R}(A), \mathcal{O}(A))$ . Therefore

$$E'_2 \cong H^*(\wedge^* \text{Der}(\mathbb{R}(A), \mathcal{O}(A)), d_1), \quad (6)$$

where

$$\begin{aligned} & d_1(t_1 \wedge \dots \wedge t_n)[a_1 | \dots | a_n] \\ &= \sum_i (-1)^{|a_1| + \dots + |a_{i-1}|} t_1(a_1) \dots t_{i-1}(a_{i-1}) t_i(d(a_i)) t_{i+1}(a_{i+1}) \dots t_n(a_n). \end{aligned}$$

The similar construction in [17] (Theorem 7.1) starts from the complex

$$(\text{Hom}(\mathbb{R}(A)^{\odot*}, \mathcal{O}(A)), d_1), \quad (7)$$

where  $\text{Hom}$  means the space of bounded linear operators with respect to the canonical bornology in the algebras  $\mathbb{R}(A)^{\odot n}$  and  $\mathcal{O}(A) \cong \text{Cok } d : \mathbb{R}^{-1}(A) \rightarrow \mathbb{R}^0(A)$ . The natural injection of the right-hand side of (5) to this complex is well-defined, since any polydifferential operator is bounded with respect to the canonical bornology. A filtration was defined in (7) that coincides with the above filtration in (5). Moreover, according to (6), the second term of our spectral sequence coincides in the second term with the sequence of [17]. Therefore that the cohomology of the right-hand side of (6) is isomorphic to the cohomology of (7). By [17], the latter is isomorphic to  $\Gamma(X_A, \mathcal{H}^{*+1})$ , which completes the proof of Lemma.  $\blacktriangleright$

**Corollary 5.4** *For any compact c.a.s.  $X$  the deformation cohomology has finite  $\mathbb{C}$ -dimension.*

$\blacktriangleleft$  The cohomology spaces of the sheaves  $\mathcal{H}^n(\mathcal{O}_X)$ ,  $n = 1, 2, \dots$  have finite dimension, since the analytic Hochschild cohomology sheaves are coherent, see [17]. Therefore the term  $E_2^{pq}$  has finite dimension for each  $p$  and  $q$ . This property holds also for the limit  $E_\infty$ .  $\blacktriangleright$

## 6 Associative cohomology versus commutative cohomology

A derivation in the resolution  $\mathcal{R}$  is an arbitrary linear  $\mathbb{N}$ -morphism  $t : \mathcal{R} \rightarrow \mathcal{R}$  such that  $[\partial, t] = 0$ . The space  $\text{Der}(\mathcal{R})$  of all derivations is a graded Lie-subalgebra in  $\text{Tot}(\mathcal{R})$  with the differential  $[d, \cdot]$ , which agrees with the differential  $D$  in  $\text{Tot}(\mathcal{R})$ . The imbedding of complexes  $\text{Der}(\mathcal{R}) \rightarrow \text{Tot}(\mathcal{R})$  generates a linear map of graded Lie algebras  $t^* : T^*(X) \rightarrow \text{QT}^*(X)$ , where  $T^*(X) \doteq H^*(\text{Der}(\mathcal{R}))$  is a graded Lie algebra called the tangent cohomology. It plays the same role for commutative deformations of c.a.s. [15] as  $\text{QT}^*(X)$  for associative deformations and has an expression in terms of the graded sheaf  $\mathcal{T}^* = \bigoplus_{k \geq 0} \mathcal{T}^k$  (called Tyurina cohomology) similar to Theorem 5.2. The Tyurina cohomology is isomorphic to the analytic Harrison cohomology  $\mathcal{H}\text{arr}^{*-1}$  with shifted grading [16]. The latter is the cohomology of the subcomplex of cochains that vanish on all shuffle products.

**Degrees 0 and 1.** The sheaf  $\mathcal{T}^0 \cong \mathcal{H}\text{arr}^1 = \mathcal{H}^1$  coincides with the sheaf of germs of holomorphic tangent fields on  $X$  and the global cohomology coincide:  $\text{QT}^0(X) = H^0(X, \mathcal{T}^0) = T^0(X)$ . For degree 1 we have

$$\text{gr}_1 \text{QT}^1(X) = H^1(X, \mathcal{T}^0), \quad \text{gr}_0 \text{QT}^1(X) = \tilde{H}^0(X, \mathcal{T}^1) \oplus H^0(X, \mathcal{Q}),$$

where  $\text{gr}$  means the grading in the cohomology induced by the filtration  $\{\text{Tot}^{(p)}\}$  in the deformation complex,  $\mathcal{T}^1$  is the symmetric and  $\mathcal{Q}$  denotes the skew-symmetric direct components of  $\mathcal{H}^2$ ,  $\tilde{H}^0(X, \mathcal{T}^1)$  stands for the kernel of  $d_2 : H^0(X, \mathcal{T}^1) \rightarrow H^2(X, \mathcal{T}^0)$ . This results in the following commutative diagram with exact rows.

$$\begin{array}{ccccccc} 0 & \rightarrow & H^1(X, \mathcal{T}^0) & \rightarrow & T^1(X) & \rightarrow & \tilde{H}^0(X, \mathcal{T}^1) & \rightarrow & 0 \\ & & \parallel & & t^1 \downarrow & & \downarrow & & \\ 0 & \rightarrow & H^1(X, \mathcal{T}^0) & \rightarrow & \text{QT}^1(X) & \rightarrow & \tilde{H}^0(X, \mathcal{T}^1) \oplus \Gamma(X, \mathcal{Q}) & \rightarrow & 0 \end{array}$$

(for the upper row see [15], Theorem 4.1). The mapping  $t^1$  is obviously injective. By the next theorem, the mapping sequence canonically splits, hence  $\text{QT}^1(X) \cong T^1(X) \oplus \Gamma(X, \mathcal{Q})$ .

**Theorem 6.1** *The canonical morphism of graded Lie algebras  $t^* : T^*(X) \rightarrow \text{QT}^*(X)$  is injective and admits a canonical linear splitting.*

A proof will be given in the last Section.

**Degree 2.** By Theorem 5.2, we have

$$\begin{aligned} \text{gr}_0 \text{QT}^2 &= \text{Ker } d_3 : \text{Ker} \{ d_2 : \Gamma(X, \mathcal{H}^3) \rightarrow H^2(X, \mathcal{H}^2) \} \rightarrow H^3(X, \mathcal{T}^0), \\ \text{gr}_1 \text{QT}^2 &= H^1(X, \mathcal{Q}) \oplus \text{Ker } d_2 : H^1(X, \mathcal{T}^1) \rightarrow H^3(X, \mathcal{T}^0), \\ \text{gr}_2 \text{QT}^2 &= H^2(X, \mathcal{T}^0), \end{aligned}$$

where  $\text{gr}_2 T^2 = \text{gr}_2 QT^2$ ,  $\text{gr}_1 T^2$  coincides with the first summand in  $\text{gr}_1 QT^2$  and  $\text{gr}_0 T^2$  is a direct summand in  $\text{gr}_0 QT^2$ .

## 7 The basic equation

The basic equation of a c.a.s.  $X$  is  $D^2 = 0$ , which is equivalent to the system  $\partial^2 = 0$ , (associativity),  $d^2 = 0$  (by the construction) and  $[d, \partial] = 0$ , which means that  $d$  is a derivation. Let  $(S, \mathcal{O}_S)$  be an analytic germ and  $\mathfrak{m}(S)$  be the maximal ideal in  $\mathcal{O}_S$ . Consider a sum  $q = q_0 + q_1 + q_2 + \dots \in \text{Tot}^1(\mathcal{R}) \otimes \mathcal{O}_S$ , where  $q_k : \mathcal{R}^{\otimes k+1} \rightarrow \mathcal{R} \otimes \mathfrak{m}(S)$  is a N-functor of bidegree  $(k, 1 - k)$ ,  $k \geq 0$ , whose values are polydifferential operators. We extend both to endomorphisms of  $\mathcal{R}^{\odot*}$  by the rule (2) and write the equation of the deformation  $X$

$$(D + q)^2 = 0. \quad (8)$$

It is equivalent to the system equations, where we write  $t = q_0$

$$(d + t)^2 = 0, \quad (9)$$

$$[\partial + q_1, d + t] = 0, \quad (10)$$

$$(\partial + q_1)^2 + [q_2, d + t] = 0, \quad (11)$$

$$[\partial + q_1, q_2] + [q_3, d + t] = 0, \quad (12)$$

...

which are bihomogeneous components of (8) of bidegree  $(0, 2)$ ,  $(1, 1)$ ,  $(2, 0)$ ,  $(3, -1)$ , ..., respectively.

**Proposition 7.1** *Let  $(X, \mathcal{O}_X)$  be an arbitrary c.a.s. and  $\mathcal{R}$  be a resolution of  $X$ . For an arbitrary Artin germ  $S$  any solution  $q$  of (8) in  $\text{Tot}^1(\mathcal{R}) \otimes \mathfrak{m}(S)$  defines a deformation  $F$  of  $X$  over  $S$ .*

◀ By (9) the sum  $d + t$ ,  $t \doteq q_0$  is a differential in  $\mathcal{R} \otimes \mathcal{O}_S$ . For any simplex  $\Delta \in \mathbb{N}$  the complex  $(\mathcal{R} \otimes \mathcal{O}_S, d + t)$  is acyclic in negative degrees. We consider the sheaf  $\mathcal{A}_\Delta$  defined on the polyhedron  $P_\Delta = \cap \{P_\alpha, \alpha \in \Delta\}$  such that  $\phi_\Delta^*(\mathcal{A}_\Delta) \cong H^0(\mathcal{R}(\Delta) \otimes \mathcal{O}_S, d + t)$ . Due to functoriality of the construction, these sheaves glue together to a global sheaf  $\mathcal{A}$  on  $X$ . The sheaf  $\mathcal{A}$  has a well defined structure of  $\mathcal{O}_S$ -module since the differential  $d + t$  commutes with the action of  $\mathcal{O}_S$ . The sheaf  $\mathcal{A}$  is flat over  $\mathcal{O}_S$  since the differential  $d + t$  is acyclic in negative degrees and obviously  $\mathcal{A} \otimes_{\mathcal{O}_S} \mathbb{C} \cong \mathcal{O}_X$ . For any  $\alpha \in \mathbb{N}$  the value of the functor  $q_1$  on the simplex  $\{\alpha\}$  is a bidifferential operator  $q_1(\alpha) : \mathcal{R}_S(\alpha) \otimes \mathcal{R}_S(\alpha) \rightarrow \mathcal{R}_S(\alpha)$  of grading 0. According to (10) we have

$$(d + t) (\partial + q_1^{-1}) + (\partial + q_1^0) (d + t) = 0, \quad (13)$$

where  $q_1^j$  denotes the restriction of  $q_1(\alpha)$  on the term  $(\mathcal{R}(\alpha) \otimes \mathcal{R}(\alpha))^j$ . We have the exact sequence of sheaves

$$(\mathcal{R}_S(\alpha) \otimes_{\mathbb{C}} \mathcal{R}_S(\alpha))^{-1} \xrightarrow{d+t} (\mathcal{R}_S(\alpha) \otimes_{\mathbb{C}} \mathcal{R}_S(\alpha))^0 \rightarrow \mathcal{A} \otimes_{\mathbb{C}} \mathcal{A}|_{P_\alpha \times P_\alpha} \rightarrow 0$$

and the equation (13) implies that  $\partial + q_1(\alpha)$  induces a sheaf-morphism  $\mathcal{Q}(\alpha) : \delta_\alpha^*(\mathcal{A} \otimes \mathcal{A}) \rightarrow \mathcal{A}$ , where  $\delta_\alpha : P_\alpha \rightarrow P_\alpha \times P_\alpha$  is the diagonal map. This morphism commutes with action of the algebra  $\mathcal{O}_S$ . It is continuous in  $\mathfrak{M}$ -adic topology, since  $q_1$  is a polydifferential operator. By (11) this morphism is an associative product  $\mu_\alpha$  in the  $\mathcal{O}_S$ -sheaf  $\mathcal{A}|_{P_\alpha}$ . Now we check that  $\mu_\beta = \mu_\alpha$  in  $\mathcal{A}_\Delta$  for arbitrary 1-simplex  $\Delta = \{\alpha, \beta\}$ . Remind that the algebra  $\mathcal{R}^0(\Delta)$  contains the subspace  $\mathcal{R}^0(\alpha) \otimes \mathcal{R}^0(\beta)$ . Therefore the bidifferential operator  $q_1(\Delta)$  is defined for any pair of arguments  $a, b \in \mathcal{R}^0(\alpha), a', b' \in \mathcal{R}^0(\beta)$ . Suppose that  $\pi(a) = \pi(a')$ , where  $\pi = \pi_S$  as in (1). Then we have  $a - a' = (d + t_\Delta)c$  for some  $c \in \mathcal{R}^{-1}(\Delta)$ , whereas  $(d_\Delta + t_\Delta)(b) = 0$ . The equation (10) evaluated on the simplex  $\Delta$ , yields

$$\begin{aligned} & (\partial + q_1(\alpha)) [a|b] - (\partial + q_1(\Delta)) [a'|b] \\ &= (\partial + q_1(\Delta)) [a - a'|b] = (\partial + q_1(\Delta)) [(d_\Delta + t_\Delta)c|b] \\ &= (\partial + q_1(\Delta)) (d_\Delta + t_\Delta) [c|b] = (d_\Delta + t_\Delta) (\partial + q_1(\Delta)) [c|b]. \end{aligned}$$

The image of the right-hand side in  $\mathcal{A}_\Delta$  vanishes, which implies that  $\mu_\alpha(\pi(a), \pi(b)) = \mu_\Delta(\pi(a'), \pi(b))$ . By the similar arguments we come to the equation  $\mu_\Delta(\pi(a'), \pi(b)) = \mu_\beta(\pi(a'), \pi(b'))$ , q.e.d. This leads us to the conclusion that  $(X, S, \mathcal{A})$  is an associative deformation.  $\blacktriangleright$

**Theorem 7.2** *Let  $X$  be an arbitrary c.a.s. and  $S$  be an arbitrary Artin germ. For any deformation  $F$  of  $X$  over  $S$  one can find a polyhedral covering  $P$  and a resolution  $\mathcal{R}$  of  $X$  on  $P$  and a solution  $q \in \text{Tot}^1(\mathcal{R}) \otimes \mathfrak{m}(S)$  of (8) that defines this deformation in the sense of Proposition 7.1.*

$\blacktriangleleft$  Let  $(X, S, \mathcal{A})$  be an associative deformation. We can find a locally finite covering  $Y$  of  $X$  such that for any  $Y \in Y$  there is commutative diagram like (1). Choose a finer polyhedral covering  $P$  and a resolution  $\mathcal{R}$  of  $X$  on this covering. Then for any vertex  $\alpha \in \mathbb{N}$  the diagram (1) is constructed for  $Y = P_\alpha$  and  $U = \mathbb{D}^{N(\alpha)}$ . We include this diagram in a commutative diagram

$$\begin{array}{ccccccc} \dots & \xrightarrow{d_S} & \mathcal{O}_{U \times S}^m & \xrightarrow{d_S} & \mathcal{O}_{U \times S} & \xrightarrow{\pi_S} & \phi_*(\mathcal{A}|Y) \rightarrow 0 \\ & & \rho \downarrow & & \rho \downarrow & & \rho_Y \downarrow \\ \dots & \xrightarrow{d} & \mathcal{O}_U^m & \xrightarrow{d} & \mathcal{O}_U & \xrightarrow{\pi} & \phi_*(\mathcal{O}_X|Y) \rightarrow 0 \end{array}, \quad (14)$$

where the bottom line is the complex  $\mathcal{R}(\alpha)$  and the upper line is the sheaf  $\mathcal{R}_S(\alpha) = \mathcal{R}(\alpha) \otimes \mathcal{O}_S$  with a differential  $d_S$  to be constructed. Let  $\mathbf{e}_1, \dots, \mathbf{e}_m$  be the set of generators of  $\mathcal{R}(\alpha)$  of grading -1. We can find some sections  $f_1, \dots, f_m$  of the sheaf  $\text{Ker } \pi_S$  such that  $\rho(f_i) = d(\mathbf{e}_i), i = 1, \dots, m$ . This follows from flatness of  $\mathcal{O}_S$ -sheaf  $\mathcal{A}$ . Define the  $\mathcal{O}_{U \times S}$ -morphism  $d_S : \mathcal{O}_{U \times S}^n \rightarrow \mathcal{O}_{U \times S}$  by the formula  $d_S(\mathbf{e}_i) = f_i$  and extend  $d_S$  as derivation to the graded subalgebra  $\mathcal{S}^1(\alpha) \subset \mathcal{R}_S(\alpha)$  generated over  $\mathcal{O}_{U \times S}$  by the elements  $\mathbf{e}_1, \dots, \mathbf{e}_m$ . Next, we take generators  $\mathbf{e}'_1, \dots, \mathbf{e}'_n$  of  $\mathcal{R}(\alpha)$  of grading -2 and set  $d_S(\mathbf{e}'_i) = g_i$ , where  $g_i \in \text{Ker } d_S$  and  $\rho(g_i) = d(\mathbf{e}'_i), i = 1, \dots, n$ . We extend the morphism  $d_S$  as derivation to the  $\mathcal{O}_{U \times S}$ -algebra  $\mathcal{S}^2(\alpha)$  generated over  $\mathcal{O}_{U \times S}$  by the generators of grading -1 and -2 and so on. This yields constructions of the upper line, which is a free graded-commutative algebra  $\mathcal{R}_S(\alpha)$  with the differential  $d_S$ . This algebra is a resolution of the sheaf  $\phi_*(\mathcal{A}|Y)$ , since the product  $\mathcal{R}_S(\alpha) \otimes \mathbb{C}$  is a resolution of the sheaf  $\phi_*(\mathcal{O}_X|Y) \cong \phi_*(\mathcal{A}|Y) \otimes \mathbb{C}$  and the algebra  $\mathcal{O}_S$  is Artin. Consider the projection  $p : U \times S \rightarrow U$  and the injective morphism of sheaves  $\iota : p^*(\mathcal{O}_U) \rightarrow \mathcal{O}_{U \times S}$ . We keep the same notation  $d$  for the differential  $\iota d \rho$  in the sheaf  $\mathcal{R}_S(\alpha)$ . Now we can write  $d_S = d + t(\alpha)$ , where the morphism  $t(\alpha)$  is a derivation in  $\mathcal{R}_S(\alpha)$  such that  $\rho t = 0$ . The equation  $(d + t(\alpha))^2 = 0$  coincides with (9) for the simplex  $\Delta = \{\alpha\}$ . We extend this derivation to a N-functor  $t$  on proper generators of  $\mathcal{R}(\Delta)$ ,  $\dim \Delta = 2, 3, \dots$  in the same way and extend it on the algebra  $\mathcal{R}(\Delta)$  as a derivation, that is  $[\partial, t] = 0$ .

The multiplication  $\mu$  is a bidifferential operator since of the condition (ii), Sec.2. The multiplication  $(a, b) \mapsto \mu(\pi_S(a), \pi_S(b))$  is also a bidifferential operator  $\mathcal{O}_{U \times S} \otimes \mathcal{O}_{U \times S} \rightarrow \phi_*(\mathcal{A}|Y)$ . By Lemma 7.3 below we can find a bidifferential operator  $q^0 : \mathcal{O}_{U \times S} \otimes \mathcal{O}_{U \times S} \rightarrow \mathcal{O}_{U \times S}$  such that  $\pi_S q^0[a|b] = \mu(\pi_S(a), \pi_S(b))$ . We have  $\pi_S q^0[d_S a|b] = 0$  for any  $a \in \mathcal{R}_S^{-1}(\alpha)$ . Therefore we can find a bidifferential operator  $q^{-1} : \mathcal{R}^{-1}(\alpha) \odot \mathcal{R}^0(\alpha) \rightarrow \mathcal{R}^{-1}(\alpha)$  such that  $q^0[d_S a|b] = d_S q^{-1}[a|b]$  for any  $b \in \mathcal{R}^0(\alpha)$ . We extend this operator on the sheaf  $\mathcal{R}^0(\alpha) \odot \mathcal{R}^{-1}(\alpha)$  to fulfil the symmetric equation. The direct sum of these two sheaves is the term  $[\mathcal{R}(\alpha) \odot \mathcal{R}(\alpha)]^{-1}$  of the sheaf  $\mathcal{R}(\alpha)^{\odot 2}$ . Next, we determine an operator  $q_{-2} : (\mathcal{R}(\alpha) \odot \mathcal{R}(\alpha))^{-2} \rightarrow \mathcal{R}^{-2}(\alpha)$  that satisfies the equation  $q^{-1}[d_S a|b] + (-)^{|a|} q^{-1}[a|d_S b] = d_S q^{-2}[a|b]$  and so on. This equation (10) follows, if we set  $q_1(\alpha) = q^0 \oplus q^{-1} \oplus q^{-2} \oplus \dots$ . The equation  $\pi_S(\partial + q_1)^2[a|b|c] = 0$  holds for arbitrary elements  $a, b, c \in \mathcal{R}^{\odot 3}(\alpha)$ , due to associativity of  $\mu$ . We apply again Lemma 7.3 and we construct the term  $q_2(\alpha)$  that satisfies (11). To fulfil (12) we note that the operator  $r \doteq [\partial, q_2]$  has negative grading and by (11), (10)

$$[d + t, r] = [[d + t, \partial + q_1], q_2] + [\partial + q_1, [d + t, q_2]] = -[\partial + q_1, (\partial + q_1)^2] = 0.$$

We check that the equation  $r = [d + t, q_3]$  has a solution  $q_3 \in \text{Diff}^{-2}(\mathbb{R}^{\odot 4}(\alpha), \mathbb{R}(\alpha))$ . First we solve the equation  $(d + t)q_3^0 = r^0$  on the zero grading term  $[\mathbb{R}^{\odot 4}(\alpha)]^0$ . It

is possible by Lemma 7.3, since  $(d+t)r^0 = 0$ . The morphism  $r' = r - [d+t, q_3^0]$  vanishes on  $[\mathbb{R}^{\odot 4}(\alpha)]^0$  and the restriction  $r'^{-1}$  of  $r'$  on the term of grading -1 satisfies  $(d+t)r'^{-1} = 0$  and so on. This completes construction of the solution  $p = t + q_1 + q_2 + \dots$  on an arbitrary vertex  $\alpha \in \mathbb{N}$ .

Take an arbitrary simplex  $\Delta = (\alpha, \beta)$  of dimension 1 and consider the diagram like (14), where  $U = P(\Delta)$ . We define the operator  $t(\Delta)$  on the subsheaf  $\mathcal{S}(\Delta) \doteq \mathcal{R}(\alpha) \otimes_{\mathbb{C}} \mathcal{R}(\beta)$  is generated by the generators of  $\mathcal{R}(\alpha)$  and  $\mathcal{R}(\beta)$ . Therefore the differential  $t(\Delta)$  is uniquely defined in  $\mathcal{S}(\Delta)$  as a N-derivation. Then we extend this differential step by step it to all proper generators of the algebra  $\mathcal{R}(\Delta)$  and arrive to a derivation  $d$  defined on the graded algebra  $\mathcal{S}(\Delta)[E_\Delta]$ , where  $E_\Delta$  stands for the set of proper generators of  $\mathcal{R}(\Delta)$ . This subalgebra is dense in  $\mathcal{R}(\Delta)$  and  $t$  is uniquely extended to a derivation in  $\mathcal{R}(\Delta)$ . Next, we extend  $t$  to proper generators of  $\mathcal{R}(\Delta)$ ,  $\dim \Delta = 2$ , using the arguments of [15], and so on.

We define the morphism  $q_1 \doteq q_1(\Delta)$  as follows

$$q_1[a|b] = q_1(\alpha)[a|b], \quad q_1[a'|b'] = q_1(\beta)[a'|b']$$

for any elements  $a, b \in \mathcal{R}(\alpha)$  and  $a', b' \in \mathcal{R}(\beta)$ , respectively. Further we apply Lemma 7.3 to define the bidifferential operator  $q_1^0 : \mathcal{R}^0(\alpha) \otimes \mathcal{R}^0(\beta) \rightarrow \mathcal{R}(\Delta)$  in such a way that  $\pi_s q_1^0[a|b'] = \mu(\pi_s(a), \pi_s(b'))$  and extend it on the product  $\mathcal{R}^0(\beta) \otimes \mathcal{R}^0(\alpha)$  to fulfil the equation  $\pi_s(\partial + q_1^0)[a'|b] = \mu(\pi_s(a'), \pi_s(b))$ . Thus the value of  $\pi_s q_1^0[a|b]$  depends only on the product  $\mu(\pi_s(a), \pi_s(b))$  for arbitrary  $a, b \in \mathcal{R}^0(\alpha) \otimes \mathcal{R}^0(\beta) \subset \mathcal{R}^0(\Delta)$ . If  $\pi_s(a) = \pi_s(a')$ , we have  $a - a' = (d+t)c$  for some  $c \in \mathcal{R}^{-1}(\Delta)$  and vice versa. This yields

$$\pi_s(\partial + q_1^0)(d+t)[c|b] = \pi_s(\partial + q_1^0)[a|b] - \pi_s(\partial + q_1^0)[a'|b] = 0.$$

Because of  $\text{Ker } \pi_s = \text{Im } (d+t)$ , we can construct by Lemma 7.3 a bidifferential operator

$$q_1^{-1} : \mathcal{R}^{-1}(\Delta) \otimes \mathcal{R}^0(\Delta) \rightarrow \mathcal{R}^{-1}(\Delta)$$

such that the equation  $(\partial + q_1^0)(d+t) + (d+t)(\partial + q_1^{-1}) = 0$  holds. We extend the operator  $q_1^{-1}$  to the term  $\mathcal{R}^0(\Delta) \otimes \mathcal{R}^{-1}(\Delta)$  by relating the values of  $q_1^0$  on the products  $[a|b]$  and  $[a|b']$  tied by the relation  $\pi_s(b) = \pi_s(b')$ . Then we step down to the level of grading -2 and so on.

On the next step, we take an arbitrary simplex  $\Delta = \{\alpha, \beta, \gamma\}$  of dimension 2. The operator  $q_1(\Gamma)$  is already constructed on the products  $\mathcal{R}(\Gamma) \otimes \mathcal{R}(\Gamma)$  for any 1-simplex  $\Gamma \subset \Delta$  and we only need to define it for products  $[a|b]$ , where  $a$  or  $b$  is a proper generator  $\mathbf{e}$  of  $\mathcal{R}(\Delta)$ . The values  $q_1^{-2}(\Delta)(\mathbf{e}, b)$  and  $q_1^{-2}(\Delta)(a, \mathbf{e})$  are to be chosen to 'tie' the values of  $q_1^{-1}$ , which are already determined, and so on. ►

**Lemma 7.3** *Let  $\phi : \mathcal{F} \rightarrow \mathcal{G}$  be a surjective morphism of coherent sheaves defined on an open polydisc  $\mathbb{D} \subset \mathbb{C}^N$  and  $\mathcal{E}$  be a free coherent sheaf in  $\mathbb{D}$ . Any polydifferential operator  $\alpha : \mathcal{E}^{\otimes n} \rightarrow \mathcal{G}$  can be lifted to a polydifferential operator  $\beta : \mathcal{E}^{\otimes n} \rightarrow \mathcal{F}$ .*

◀ Let  $\mathbf{e}_1, \dots, \mathbf{e}_r$  be generators of  $\mathcal{E}$ . Take a set of monomials  $p_1, \dots, p_n$  in  $\mathbb{C}^N$  whose order is less or equal to  $\text{ord } \alpha$ . The product  $p = [p_1 \mathbf{e}_{i(1)} | \dots | p_n \mathbf{e}_{i(n)}]$  is a section of the sheaf  $\mathcal{E}$ . Because of the map  $\phi : \Gamma(\mathbb{D}, \mathcal{F}) \rightarrow \Gamma(\mathbb{D}, \mathcal{G})$  is surjective, we can find a section  $f$  of the sheaf  $\mathcal{F}$  such that  $\phi(f) = \alpha(p)$  and set  $\beta(p) = f$ . There is a unique polydifferential operator  $\beta : \mathcal{E}^{\otimes n} \rightarrow \mathcal{F}$  of order  $\leq \text{ord } \alpha$ , which agrees with these data. ▶

**Corollary 7.4** *If  $\dim \text{QT}^1(X) < \infty$ , there is a natural linear bijection  $\text{QT}^1(X) = D(X)$ . Moreover, any deformation  $F$  of  $X$  with a small base  $S$  such that  $DF : T(S) \rightarrow \text{QT}^1(X)$  is a bijection, is universal in the category of deformations with small bases.*

We omit a proof which is straightforward.

**Proposition 7.5** *Let  $q, Q \in \text{Tot}^1(\mathcal{R}) \otimes \mathfrak{m}(S)$  be solutions of (8) such that  $Q = q \bmod J$  for an small ideal  $J \subset \mathcal{O}_S$ , and  $F, F'$  be respective deformations of  $X$  over  $S$ . If an isomorphism  $\varepsilon : F \rightarrow F'$  is fixed such that  $\varepsilon = 1 \bmod J$ , then there exists a cochain  $v \in \text{Tot}^0(\mathcal{R}) \otimes J$  such that  $q - Q = [D, v]$ .*

◀ Let  $\mathcal{A}$  be the structure sheaf of  $F$ . Consider the diagram

$$\begin{array}{ccccccccc} 0 & \rightarrow & \mathcal{O}_X \otimes J & \rightarrow & \mathcal{A} & \rightarrow & \mathcal{A} \otimes \mathcal{O}_S/J & \rightarrow & 0 \\ & & \parallel & & \theta \swarrow & & \varepsilon \downarrow & & \parallel \\ 0 & \rightarrow & \mathcal{O}_X \otimes J & \rightarrow & \mathcal{A} & \rightarrow & \mathcal{A} \otimes \mathcal{O}_S/J & \rightarrow & 0 \end{array},$$

where the lines are exact, since  $\mathcal{A}$  is flat over  $\mathcal{O}_S$ . We have  $\varepsilon = 1 + \theta$ , where  $\theta$  is a differential operator such that  $\theta = 0 \bmod J$ . It is a derivation, since  $\varepsilon$  is an automorphism of the sheaf-algebra  $\mathcal{A}$ . Therefore it can be lifted to a differential operator  $u^0$  in  $\mathcal{R}^0(\alpha) \otimes J$  for any  $\alpha \in \mathbb{N}$ . Write  $t = q_0$ ,  $T = Q_0$ . The operator  $u^0$  acts from  $\text{Im}(d + t^{-1})$  to  $\text{Im}(d + T^{-1})$ , (where  $t^j$  means the restriction of  $t$  to  $\mathcal{R}^j$  etc.). This implies that there exists a sheaf-morphism  $u^{-1}$  in  $\mathcal{R}^{-1}(\alpha)$  such that  $(d + T^{-1})(1 + u^{-1}) = (1 + u^0)(d + t^{-1})$ , where the compositions of  $t$  and  $u$  vanish, since of  $J^2 = 0$ . This yields the identity  $T^{-1} - t^{-1} = u^0 d - d u^{-1} = -[d, u^0 + u^{-1}]$ . In the same way, we can construct morphisms  $u^k$  in  $\mathcal{R}^{-k}$  that fulfil the equations  $(d + T^{-k})(1 + u^{-k}) = (1 + u^{1-k})(d + t^{-k})$ ,  $k = 2, 3, \dots$  They can be written in the short form  $[d, u_0] = t - T$ , where  $u_0 = u^0 + u^{-1} + \dots : \mathcal{R} \rightarrow \mathcal{R}$ . The identity

$\varepsilon\mu(a|b) = \mu(\varepsilon(a), \varepsilon(b))$ , where  $\mu(a|b) = ab + \pi q^0(a|b)$ ,  $a, b \in \mathbb{R}$ , implies the equation

$$u^0(ab) + q^0(a|b) = Q^0(a|b) + u^0(a)b + au^0(b) - dv^0(a|b)$$

for some bidifferential operator  $v^0 : \mathbb{R}^{\otimes 2} \rightarrow \mathbb{R}$ . It can be written in the form  $q^0 - Q^0 = [\partial, u^0] + dv^0$ . Because of the cochains  $q$  and  $Q$  fulfils (10) we have the following equation on  $(\mathbb{R}^{\otimes 2})^{-1}$

$$\begin{aligned} [\partial, u^0] d &= [d, [\partial, u_0]] = -[\partial, [d, u_0]] = -[\partial, t^{-1} - T^{-1}] \\ &= [d, q_1 - Q_1] = d(q^{-1} - Q^{-1}) + (q^0 - Q^0) d, \end{aligned}$$

where we use the equation  $[D, q - Q] = 0$ , which follows from (8), since  $\mathfrak{m}J = 0$ . This yields

$$d(q^{-1} - Q^{-1} - v^0 d) = (q^0 - Q^0 - [\partial, u^0] - dv^0) d = 0.$$

Therefore there exists a differential operator  $v^{-1} : \mathbb{R}^{\otimes 2} \rightarrow \mathbb{R}$  such that  $|v^{-1}| = -1$  and  $q_1 - Q_1 - [\partial, u^{-1}] - v^0 d = -dv^{-1}$ , that is  $q_1 - Q_1 = [\partial, u^{-1}] - [d, v^0 + v^{-1}]$ . Then we move to the space  $(\mathbb{R}^{\otimes 2})^{-2}$  and construct an operator  $v^{-2}$  of bidegree  $(2, -2)$  and so on. The series  $u_1 = v^0 + v^{-1} + v^{-2} + \dots$  solves the equation  $q_1 - Q_1 = [\partial, u_0] - [d, u_1] = [D, u_0 + u_1]$ . Next we construct the term  $v_2$  of bidegree  $(2, -2)$  and so on. The cochain  $v = v_0 + v_1 + v_2 + \dots$  is a solution to the equation  $q - Q = [D, v]$ .

►

Let  $L$  be a graded Lie  $\mathbb{Q}$ -algebra and  $v \in L$  be an even element such that the operator  $[v, \cdot]$  is nilpotent. The series

$$\exp(v) : p \mapsto p + [v, p] + \frac{1}{2} [v, [v, p]] + \frac{1}{6} [v, [v, [v, p]]] + \dots$$

is finite for any  $p \in L$  and defines a transformation in  $L$ .

**Proposition 7.6** *For any even nilpotent element  $v \in L$  the transformation  $\exp(v)$  is an automorphism of  $L$ , that is  $[\exp(v)p, \exp(v)q] = \exp(v)[p, q]$  for any  $p, q \in L$ .*

◀ A proof is elementary. ▶

**Proposition 7.7** *If  $q \in \text{Tot}^1(\mathcal{R}) \otimes \mathfrak{m}$  is a representation of a deformation  $F$  over an Artin germ  $S$  and  $v \in \text{Tot}^0(\mathcal{R}) \otimes \mathfrak{m}$ , then  $Q \doteq \exp(v)(D + q) - D$  is a representation of an equivalent deformation  $G \simeq F$ .*

◀ A proof can be done by straightforward arguments. ▶

## 8 Obstructions

**Theorem 8.1** *Let  $F : X \rightarrow S$  be a deformation of a c.a.s.  $X$  with an Artin base  $S$ . For any small extension  $\phi : S \rightarrow V$ , there is defined a map of c.a.s.  $\text{Ob}(F, \phi) : V \rightarrow \text{QD}^2(X)$  such that*

- (i) *vanishing of  $\text{Ob}(F, \phi)$  is necessary and sufficient for existence of a deformation  $G : Y \rightarrow V$  such that  $\phi^*(G) \simeq F$ ,*
- (ii)  *$\text{Ob}(F, \phi) = \text{Ob}(F, \psi\phi)$  holds for an arbitrary imbedding  $\psi : V \rightarrow V'$  that is identical on  $S$  such that  $\psi\phi$  is a small extension.*
- (iii)  *$\text{Ob}(F, \phi) = \text{Ob}(F', \phi)$  for any equivalent deformation  $F' \simeq F$  over  $S$ .*

The map  $\text{Ob}(F, \phi)$  is called *obstruction* to extension of the deformation  $F$  to  $V$ . The obstruction always vanishes on  $S$ , since of (ii).

◀ Fix a resolution  $\mathcal{R}$  of  $X$  and a bijection  $\mathcal{O}_V \cong \mathcal{O}/I$ , where  $\mathcal{O}$  is a regular analytic algebra, let  $J$  be the kernel of the surjection  $\mathcal{O} \rightarrow \mathcal{O}_S$ . Take a representation  $q \in \text{Tot}^1 \otimes \mathfrak{m}$  of the deformation  $F$ , where  $\mathfrak{m}$  is the maximal ideal in  $\mathcal{O}$ . We have  $(D + q)^2 \in \text{Tot}^2 \otimes J$ , since of (8). By the Jacobi identity, we have  $[e, [e, e]] = 0$  for any cochain  $e \in \text{Tot}^1$ . This yields

$$0 = [D + q, [D + q, D + q]] = 2 [D + q, (D + q)^2] = 2 [D, (D + q)^2] \pmod{\mathfrak{m}J},$$

since the term  $[q, (D + q)^2]$  vanishes  $\pmod{\mathfrak{m}J}$ . We have  $\mathfrak{m}J \subset I$ , since the extension is small, which implies that the square  $(D + q)^2$  is a cocycle  $\pmod{I}$ ; we denote by  $\text{Ob}(F, \phi) \in \text{QT}^2 \otimes J$  the cohomology class of this cocycle.

Check the properties of the obstruction. (i) If  $\text{Ob}(F, \phi) = 0$ , there exists an cochain  $r \in \text{Tot}^1 \otimes I$  such that  $(D + q)^2 = -[D, r]$  and we have

$$(D + q + r)^2 = (D + q)^2 + [D, r] + [q, r] = (D + q)^2 + [D, r] = 0 \pmod{I},$$

since  $[q, r] = 0$  modulo the ideal  $\mathfrak{m}J \subset I$ . Therefore the cochain  $p = q + r$  represents a deformation  $G$  over  $S$  which is an extension of  $F$ . Vice versa, if there exists an extension  $G$  of  $F$ , we can take a representation  $p$  of  $G$  instead of  $q$ . The bracket  $(D + p)^2$  vanishes  $\pmod{I}$ , since  $p$  fulfils (8) over  $V$ .

(ii) Let  $q \in \text{Tot}^1 \otimes \mathfrak{m}(V')$  be a cochain whose restriction on  $S$  represents the deformation  $F$ . We have  $\text{Ob}(F, \psi\phi) = \text{cl}(D + q)^2$ , whereas the cochain  $(1 \otimes \psi^*)(D + q)^2$  represents the obstruction  $\text{Ob}(F, \phi)$ .

(iii) Let  $F \simeq F'$  and  $q, q'$  be some representation of these deformations on resolutions  $\mathcal{R}, \mathcal{R}'$  of  $X$  on some coverings  $P, P'$ . We take the product  $\mathcal{S}$  of the resolutions  $\mathcal{R}, \mathcal{R}'$  on the covering  $P \cup P'$  in the sense of Proposition 5.1 and lift  $q$  to a cochains  $s$  that represent  $F$  in the complex  $\text{Tot}^1(\mathcal{S})$ . It can be seen from

the construction that  $\text{cl}(D + q)^2 = \text{cl}(D + s)^2$ . Therefore we can assume that  $q$  and  $q'$  are representations of  $F$  in  $\text{Tot}^1(\mathcal{R})$  for the same resolvent  $\mathcal{R}$ . We show that  $\text{cl}(D + q)^2 = \text{cl}(D + q')^2 \pmod{\mathfrak{l}}$ , where  $\mathfrak{l}$  is the ideal of the extension  $\phi$ . The cochains  $q$  and  $q'$  coincide over the point  $\{\circ\} \subset S$ , since both vanish at this point. Choose a tower of small extensions  $\{\circ\} = S_0 \hookrightarrow S_1 \hookrightarrow \dots \hookrightarrow S$ ; let  $S_k$  be the largest of them, where  $q$  and  $q'$  coincide. We use induction on  $k$  and denote by  $\mathfrak{K}$  is the ideal of extension  $S_k \rightarrow S_{k+1}$ . By Proposition 7.5 we have  $D + q' = \exp(v)(D + q)$ , where  $v \in \text{Tot}^0 \otimes \mathfrak{K}$  and the exponent is well defined since  $v = 0 \pmod{\mathfrak{m}(S_k)}$ . We have  $\exp(v) = 1 + v \pmod{\mathfrak{m}\mathfrak{K}}$ , hence  $q' - q = [D, v] \pmod{\mathfrak{m}\mathfrak{K}}$ . Now we define the cochain  $p \in \text{Tot}^1 \otimes \mathfrak{m}$  from the equation  $D + p = \exp(v)(D + q)$ . By Proposition 7.6 we have  $(D + p)^2 = \exp(v)(D + q)^2 \pmod{\mathfrak{l}}$ , which implies  $(D + p)^2 = (D + q)^2$ , since  $\exp(v) = 1 \pmod{\mathfrak{m}}$ . On the other hand, we have  $p = q + [D, v] = q' \pmod{\mathfrak{m}\mathfrak{K}}$ , which yields  $p = q'$  over  $S_{k+1}$ . By induction we conclude that  $\text{cl}(D + p)^2 = \text{cl}(D + q')^2$  and finally  $\text{cl}(D + q)^2 = \text{cl}(D + q')^2$ . This makes the general inductive step and completes the proof.  $\blacktriangleright$

**Quadratic obstruction.** Given a vector space  $V$  and a cocycle  $q = t + q_1 + \dots \in \text{Tot}^1 \otimes V$  there is defined a deformation  $F_1$  of  $X$  with the base  $S_1 \subset V$ ,  $\mathcal{O}_{S_1} \doteq \mathcal{O}(V)/\mathfrak{m}^2$ , since all linear terms in (9),(10),(11) vanish and quadratic terms are neglected. The terms  $(q_0)^2, [q_0, q_1], (q_1)^2$  define the obstruction map  $\text{Ob}(F_1, S_2) : S_2 \rightarrow \text{QT}^2(X)$ , where  $S_2$  is the germ with the algebra  $\mathcal{O}(V)/\mathfrak{m}^3$ . We call it *quadratic obstruction*. If  $V = \text{QT}^1(X)$  and  $DF$  is bijective, then this obstruction is a universal quadratic mapping  $\text{Obs} : \text{QT}^1(X) \rightarrow \text{QT}^2(X)$ . By Theorem 5.2 it generates the graded quadratic map  $\text{gr } \text{QT}^1(X) \rightarrow \text{gr } \text{QT}^2(X)$ . The latter is a sum of the canonical bilinear mappings  $H^p(X, \mathcal{H}^k) \times H^q(X, \mathcal{H}^l) \rightarrow H^{p+q}(X, \mathcal{H}^{k+l-1})$ ,  $p+k = q+l = 2$ , which combine the bracket operation in  $\mathcal{H}^*$  with the cup-product in cohomology. In the case  $p = k = q = 1$  it is just Kodaira's quadratic map.

## 9 Construction of a versal deformation

**Theorem 9.1** *For an arbitrary c.a.s.  $X$  such that  $\dim \text{QT}^1(X) < \infty$  there exists a formal minimal versal deformation  $F : \hat{X} \rightarrow \hat{S}$ . The base is the kernel of a formal map  $B : \text{QT}^1(X) \rightarrow \text{QT}^2(X)$  such that  $B(0) = 0$ ,  $dB(0) = 0$ , and  $d^2B$  is equal to the universal quadratic obstruction.*

*If  $\text{QT}^0(X) = 0$ , any minimal versal deformation is universal.*

$\blacktriangleleft$  **1.** We will construct a sequence of deformations  $F_k = (X, \mathcal{A}_k)$  with Artin bases  $S_k$ , where  $\{\circ\} = S_0 \hookrightarrow S_1 \hookrightarrow S_2 \hookrightarrow \dots$  is a tower of small extensions such that the restriction of  $F_{k+1}$  to  $S_k$  is equivalent to  $F_k$ . Fix a polyhedral covering  $P$  and a resolution  $\mathcal{R}$  of  $X$  on  $P$ . We denote by  $Z^1(\text{Tot})$  the space of 1-cocycles in

$\text{Tot} = \text{Tot}(\mathcal{R})$  and abbreviate the notation  $Q^* \doteq \text{QT}^*(X)$ . Denote  $\mathbf{B} \doteq \mathcal{O}(Q^1, 0)$  and set  $\mathcal{O}_{S_1} = \mathbf{B}/\mathfrak{m}^2$ , where  $\mathfrak{m}$  is the maximal ideal in  $\mathbf{B}$ . Choose a linear section  $p_1 : Q^1 \rightarrow Z^1(\text{Tot})$  of the canonical projection  $Z^1(\text{Tot}) \rightarrow Q^1$ . We can write it as a cochain  $p_1 \in \text{Tot}^1 \otimes \mathfrak{m}/\mathfrak{m}^2$ . The equation  $[D, p_1] = 0$  implies (8) mod  $\mathfrak{m}^2$ . According to Proposition 7.1 the element  $p_1$  defines a deformation  $F_1$  of  $X$  over  $S_1$ .

Suppose now that a deformation  $F_k$  and a sequence of extensions  $S_1, \dots, S_k$  are already done for some  $k > 0$  and  $\mathcal{O}_k = \mathbf{B}/\mathfrak{l}_k$  is the algebra of the germ  $S_k$ , where  $\mathfrak{l}_k$  is the ideal in  $\mathbf{B}$  generated by some polynomials in  $Q^1$  such that  $\mathfrak{m}\mathfrak{l}_{k-1} \subset \mathfrak{l}_k \subset \mathfrak{l}_{k-1}$ . We suppose also that the deformation  $F_k$  is represented by a cochain  $p_k \in \text{Tot}^1 \otimes \mathfrak{m}$ , such that  $p_k = p_{k-1} \text{ mod } \mathfrak{l}_{k-1}$ . By Theorem 8.1, the obstruction cochain  $(D + p_k)^2$  is  $D$ -closed mod  $\mathfrak{m}\mathfrak{l}_k$ . Therefore the obstruction class  $\text{cl}(D + p_k)^2$  is defined as an element of  $Q^2 \otimes \mathfrak{l}_k/\mathfrak{m}\mathfrak{l}_k$ . Choose a linear basis  $\tau_1, \tau_2, \dots$  in  $Q^2$  and write  $\text{cl}(D + p_k)^2 = \sum \tau_j \otimes e_j$ , for some monomials  $e_1, \dots, e_l \in \mathfrak{l}_k$  that span the quotient  $\mathfrak{l}_k/\mathfrak{m}\mathfrak{l}_k$ . Define the ideal  $\mathfrak{l}_{k+1} = \mathfrak{m}\mathfrak{l}_k + (e_1, \dots, e_l)$  in  $\mathbf{B}$  and set  $\mathcal{O}_{k+1} = \mathbf{B}/\mathfrak{l}_{k+1}$ , where  $(e_1, \dots, e_l)$  means the ideal by the monomials  $e_1, \dots, e_l$ . The germ  $S_{k+1}$  is defined by the algebra  $\mathcal{O}_{k+1}$ ; it is a small extension of  $S_k$ . We have  $\text{cl}(D + p_k)^2 = 0$  in  $S_{k+1}$ , that is  $(D + p_k)^2 = [D, r] \text{ mod } \mathfrak{l}_{k+1}$ , where  $r \in \text{Tot}^1 \otimes \mathfrak{l}_k$ . Set  $p_{k+1} = p_k - r$  and have

$$(D + p_{k+1})^2 = (D + p_k)^2 - [D, r] - [p_k, r] + (r)^2 = (D + p_k)^2 - [D, r] \text{ mod } \mathfrak{m}\mathfrak{l}_k,$$

since  $p_k = 0 \text{ mod } \mathfrak{m}$ . The right-hand side vanishes mod  $\mathfrak{l}_{k+1}$ , hence the cochain  $p_{k+1}$  fulfils (8) over  $S_{k+1}$ . By Theorem 7.2 it represents a deformation  $F_{k+1}$  of  $X$  over the base  $S_{k+1}$ . This completes the general step of the induction.

**2.** Check that the formal deformation  $\hat{F} = \lim_k \{F_k \rightarrow S_k\}$  is versal. Let  $G = (X, R, \mathcal{B}, \nu, \beta)$  be an arbitrary deformation of  $X$  over an Artin germ  $R$ . It is sufficient to check that the deformation  $G$  is induced from  $\hat{F}$ . We assume first that  $R$  is small. The differential  $DF = DF_1$  is bijective according to the above construction, Therefore we can write  $DG = DF\lambda$ , where  $\lambda : T(R) \rightarrow T(S_1) = T(S)$  is a linear mapping. It defines a morphism of germs  $\lambda : R \rightarrow S_1 \rightarrow S$  since  $R$  is small and  $\lambda^*(F) = \lambda^*(F_1) \simeq G$ . This means that the deformation  $F_1$  is universal in the category of deformations with small bases.

Suppose now that  $Q$  is an arbitrary Artin germ and there exist a small extension  $\theta : Q \rightarrow R$ , and a morphism of germs  $\phi : Q \rightarrow S_k$  for some  $k$  and an equivalence of deformations  $\varepsilon : \phi^*(\mathcal{A}_k) \simeq \theta^*(\mathcal{B})$  over  $Q$ , where  $\mathcal{A}_k$  is the sheaf of  $F_k$ . We will construct a morphism of germs  $\psi : R \rightarrow S_{k+1}$  such that the diagram

$$\begin{array}{ccc} Q & \xrightarrow{\theta} & R \\ \phi \downarrow & & \psi \downarrow \\ S_k & \hookrightarrow & S_{k+1} \end{array} \quad (15)$$

commutes and an extension  $\delta : \psi^*(\mathcal{A}_{k+1}) \simeq \mathcal{B}$  of  $\varepsilon$ . The morphism  $\delta$  will be bijective, since of flatness. This will imply the versality of  $F$ , since any Artin base  $R$  is on the

top of a tower of small extensions  $\{\circ\} \hookrightarrow R_1 \hookrightarrow R_2 \hookrightarrow \dots \hookrightarrow R$ . At the zero step the morphism  $\psi_0$  is trivial.

Take the cochain  $\mathfrak{p} \in \text{Tot}^1 \otimes \mathfrak{m}(S_k)$  that represents the deformation  $\mathcal{A}_k$  on a resolution  $\mathcal{R}$  of  $X$ . The element  $\phi^*(\mathfrak{p}) \in \text{Tot}^1 \otimes \mathfrak{m}(Q)$  represents the deformation  $\theta^*(\mathcal{B})$ . We may assume that the morphism  $\phi$  is given by some polynomial mapping  $\Phi : T(R) \rightarrow T(S_k) = Q^1$ . We have  $\Phi^*(\mathfrak{l}_k) \subset \mathfrak{J}$ , where  $\mathcal{O}_k = \mathbb{B}/\mathfrak{l}_k$  is the algebra of  $S_k$  and  $\mathcal{O}_Q = \mathcal{O}_R/\mathfrak{J}$  and  $\Phi^*(\mathfrak{m}\mathfrak{l}_k) \subset \mathfrak{m}\mathfrak{J} = 0$ , since the extension  $\theta$  is small. Consider the obstruction class  $\text{cl}(\Phi^*(D + \mathfrak{p}))^2 = \text{cl}(D + \Phi^*(\mathfrak{p}))^2$ . By Theorem 8.1 this class is equal to  $\phi^*(\text{cl}(D + \mathfrak{p})^2) \in Q^2 \otimes \mathfrak{J}$ . On the other hand, the class  $\phi^*(\text{cl}(D + \mathfrak{p})^2)$  vanishes, since  $\Phi^*(\mathfrak{p})$  has an extension  $\tilde{\mathfrak{p}}$  to  $R$ . Therefore  $\text{cl}(D + \Phi^*(\mathfrak{p}))^2 = 0$  over  $R$ . This implies  $\Phi^*(e_i) = 0$  for any monomial  $e_i$  as in part 1 of the proof. Therefore  $\Phi(\mathfrak{l}_{k+1}) = 0$ , since  $\Phi$  vanishes on the ideal  $\mathfrak{m}\mathfrak{l}_k$ . It follows that  $\Phi$  defines a map  $\psi : R \rightarrow S_{k+1}$ , which makes the diagram (15) commutative. Now we consider the deviation  $\Delta\mathfrak{p} \doteq \tilde{\mathfrak{p}} - \Phi^*(\mathfrak{p}) \in \text{Tot}^1 \otimes \mathfrak{J}$ . We write it in the form  $\Delta\mathfrak{p} = (D + \tilde{\mathfrak{p}}) - (D + \Phi^*(\mathfrak{p}))$  and calculate the commutator over  $R$ :

$$(D + \tilde{\mathfrak{p}})\Delta\mathfrak{p} - \Delta\mathfrak{p}\Phi^*(D + \mathfrak{p}) = (D + \Phi^*(\mathfrak{p}))^2 + (D + \tilde{\mathfrak{p}})^2 = 0$$

The left-hand side is equal to  $[D, \Delta\mathfrak{p}] = 0$ , since other terms are contained in  $\mathfrak{m}\mathfrak{l} \subset \mathfrak{J}$ . Consider the class  $\text{cl}(\Delta\mathfrak{p}) \in Q^1 \otimes \mathfrak{J}$  and construct a polynomial mapping  $\omega : T(R) \rightarrow Q^1$  such that  $\text{cl}(\Delta\mathfrak{p}) = \mathfrak{p}_1\omega$ . It exists and unique, since  $\mathfrak{p}_1$  is bijective, and  $\omega = 0 \pmod{\mathfrak{J}}$ . Set  $\Psi = \Phi + \omega$  and have  $\text{cl}(\tilde{\mathfrak{p}} - \Psi^*(\mathfrak{p})) = \text{cl}(\tilde{\mathfrak{p}} - \Phi^*(\mathfrak{p}) - \mathfrak{p}_1\omega) = 0$ , since  $\mathfrak{p}_1$  induces the identity mapping in  $Q^1$ . It follows that  $\tilde{\mathfrak{p}} - \Psi^*(\mathfrak{p}) = [D, \mathfrak{v}]$  for some  $\mathfrak{v} \in \text{Tot}^0 \otimes \mathfrak{J}$ . Replace the cochain  $\tilde{\mathfrak{p}}$  by  $\mathfrak{p}' = \tilde{\mathfrak{p}} - [D, \mathfrak{v}]$  and have

$$D + \mathfrak{p}' = \exp(-\mathfrak{v})(D + \tilde{\mathfrak{p}}) = D + \tilde{\mathfrak{p}} - [D, \mathfrak{v}] = D + \Psi^*(\mathfrak{p}) \pmod{\mathfrak{m}\mathfrak{J}}$$

and  $\mathfrak{m}\mathfrak{J} = 0$ , since  $R$  is a small extension of  $Q$ . By Proposition 7.7, this equation provides an equivalence of deformations  $G \simeq \psi^*(F)$ .

**3.** Prove the last statement. Suppose that for some deformation  $G$  of  $X$  with the base  $R$  there exists two morphisms  $\phi, \psi : R \rightarrow T(S_k) = Q^1$  such that

$$\phi^*(F) \simeq G \simeq \psi^*(F). \tag{16}$$

We can assume that  $\phi = \psi$  on a subgerm  $Q$  of  $R$  such that  $R$  is a small extension of  $Q$  with an ideal  $\mathfrak{J}$ . We have then  $\psi = \phi + \delta$ ,  $\delta \in T(S) \otimes \mathfrak{J}$ . This yields  $\phi^*(\mathfrak{p}) = \psi^*(\mathfrak{p}) \pmod{\mathfrak{J}}$  and  $\psi^*(\mathfrak{p}) = \phi^*(\mathfrak{p}) + \mathfrak{p}_1\delta$  in  $R$ , where  $\mathfrak{p}_1 : Q^1 \rightarrow Z^1(\text{Tot})$  is as above. On the other hand, because of (16) and by Proposition 7.5 there exists an element  $\mathfrak{v} \in \text{Tot}^0 \otimes \mathfrak{J}$  such that the equation holds  $\psi^*(\mathfrak{p}) - \phi^*(\mathfrak{p}) = [D, \mathfrak{v}]$ . Comparing the differentials, we get  $\mathfrak{p}_1 \cdot (d\psi(\circ) - d\phi(\circ)) = [D, \mathfrak{v}]$ . The map  $\mathfrak{p}_1 = d\mathfrak{p}$  represents the differential of the deformation  $F$  and is injective, since the deformation is minimal.

Therefore  $d\psi(\circ) = d\phi(\circ)$  and also  $[D, v] = 0$ . The class of the cocycle  $v$  belongs to the space  $Q^0 \otimes J$ , which is trivial by the condition. Therefore  $v = [D, u]$  for some  $u \in \text{Tot}^{-1}(\mathcal{R}) \otimes \mathfrak{m}$ . This yields  $p_1\delta = \psi^*(p) - \phi^*(p) = [D, [D, u]] = 0$ , consequently  $\delta = 0$ .  $\blacktriangleright$

**Remark.** For any compact c.a.s.  $X$  there exists a minimal versal deformation  $f : Y \rightarrow R$  in the category of deformations of c.a.s. over all analytic germs (see [15] for a survey). It generates a formal deformation  $\hat{f} : \hat{Y} \rightarrow \hat{R}$ . By Theorem 9.1, there exists a formal versal deformation  $F = (X, \hat{S}, \mathcal{A}, \mu)$  and a formal mapping  $\phi : \hat{R} \rightarrow \hat{S}$  such that  $\phi^*(F)$  is equivalent to  $f$  in the formal sense, that is  $\phi^*(\mathcal{A}) \cong \mathcal{O}_{\hat{Y}}$ . Comparing the differentials of the deformations, we find:  $Df = DF \cdot d\phi$ , which implies that  $d\phi$  is injective and  $\phi$  is a formal imbedding. In fact, we can say more, there exists a unique simultaneous extension of  $F$  and  $f$  to a semi-formal versal deformation  $\Phi$  of  $X$  whose base is a formal neighborhood of the analytic germ  $R$ . We can not expect that there exists always a convergent versal associative deformation  $F$  over an analytic algebra. Indeed, the star-product given by the Groenewold-Moyal formula in  $\mathbb{C}^n$

$$a *_q b = \exp(q)(a, b) \tag{17}$$

for a Poisson bracket  $q$  with constant coefficients converges only in a very weak sense.

## 10 Examples

**K3-surfaces.** All complex analytic K3-surfaces (simply connected compact surfaces with trivial canonical class) form an irreducible family  $f : Y \rightarrow R$  of dimension  $\dim R = 20$ . This family is a universal commutative deformation for each fiber  $X_s = f^{-1}(s)$ . The deformation cohomology of each normal K3-surface is

$$\begin{aligned} H^0(X, T^0) &= 0, & H^1(X, T^0) &= \mathbb{C}^{20}, & H^0(X, \mathcal{Q}) &= \mathbb{C}, \\ H^2(X, T^0) &= 0, & H^1(X, \mathcal{Q}) &= 0, \end{aligned}$$

where  $T^0$  is the tangent sheaf on  $X$  and  $\mathcal{Q} = \wedge^2 T^0 \cong \Omega_X^2$ . We have  $QT^1(X) \cong \mathbb{C}^{21}$  and  $QT^2(X) = 0$ . By Theorem 9.1 there exists a formal versal deformation  $F : X \rightarrow \hat{S}$  of  $X$ , where  $\hat{S}$  is the formal completion of the germ  $(\mathbb{C}^{21}, 0)$ . The subgerm  $T^1(X) \cong \mathbb{C}^{20}$  is the base of a versal commutative deformation and the formal deformation  $F$  can be extended to a semi-formal deformation, whose base is a formal neighborhood of  $(\mathbb{C}^{20}, 0)$ . The value  $DF(q)$  for the complementary direction  $q$  is given by a generator of the space  $H^0(X, \mathcal{Q})$ , see an explicit form in [17].

Note that  $H^2(X, \mathcal{O}_X) \cong \mathbb{C}$  for any K3-surface. The role of the condition  $H^2(X, \mathcal{O}_X) = 0$  is discussed in [13], see also [19]. We can conclude that this condition is irrelevant to existence of a formal quantization.

**Complex tori.** Let  $\Lambda$  be a discrete lattice in  $\mathbb{C}^n$ ,  $n > 1$ ; the quotient  $X = \mathbb{C}^n/\Lambda$  is a complex  $n$ -torus. We have  $\mathcal{T}^0 \cong \mathcal{O}_X^n$ ,  $\mathcal{Q}_X = \wedge^2 \mathcal{T}^0$  for any  $\Lambda$ . Let  $\lambda_1, \dots, \lambda_{2n} \in \mathbb{C}^n$  be some vectors that generate  $\Lambda$ . The versal commutative deformation of  $X$  is given by variation of the first  $n$  vectors  $\lambda_1, \dots, \lambda_n$  as the last  $n$  vectors are frozen. The space  $\Gamma(X, \mathcal{Q})$  consists of tangent bivector fields  $q(s) = \sum_{i < j} s_{ij} \partial/\partial z_i \wedge \partial/\partial z_j$  with constant coefficients  $s_{ij}$ . Each field  $q$  is a Poisson bracket. It can be extended to a star-product in  $\mathcal{O}_X$  by means of the Groenewold-Moyal formula (17), which is translation invariant and can be lifted on  $\Lambda$ . Moreover, we can take the family of brackets  $q = q(s)$  in this formula and obtain a star product depending on a point  $s$ . This extension is a formal deformation-quantization of  $X$  with the base  $\Gamma(X, \mathcal{Q})$ . Finally we can combine both: the commutative deformation with the base  $\mathbb{T}^1 = H^1(X, \mathcal{T}^0)$  and the star-product depending on  $q \in \Gamma(X, \mathcal{Q})$ . This is a formal minimal versal deformation of  $X$  with the base  $\text{QT}^1$ , which has dimension  $n^2(n+1)/2$ . We see that the versal deformation is unobstructed despite the massive spaces  $\text{QT}^2 \cong H^2(X, \mathcal{T}^0)$  and  $H^2(X, \mathcal{O})$ . This deformation is universal, whereas the Lie group  $\text{QT}^0 \cong \mathbb{C}^n$  is not trivial. The reason is that the bracket representation of this group in  $\text{QT}^1$  is trivial.

## 11 Differential operators and Hochschild cohomology

Let  $\mathbb{R} = \mathbb{R}(U)$  be a regular analytic  $\mathbb{C}$ -algebra or the algebra of holomorphic functions in a domain  $U \subset \mathbb{C}^N$ . We define the analytic tensor product of such algebras as in Sec. 4. Consider the *analytic* bar-complex that consists of analytic algebras  $\text{Ca}_n(\mathbb{R}) = \mathbb{R} \odot \mathbb{R}^{\odot n} \odot \mathbb{R} = \mathbb{R}^{\odot n+2}$ ,  $n = 0, 1, 2, \dots$ . The subspace  $\text{C}_n(\mathbb{R}) \doteq \mathbb{R}^{\otimes n+2}$  is a  $\mathbb{R} \otimes \mathbb{R}$ -submodule and is dense in  $\text{Ca}_n(\mathbb{R})$ , which possesses a canonical bornology. The action of  $\mathbb{R} \otimes \mathbb{R}$  in  $\text{Ca}(\mathbb{R})$  is extended to the bounded action of the algebra  $\mathbb{R}^e = \mathbb{R} \odot \mathbb{R}$ . The explicit formula for this action is the following

$$\rho \cdot \alpha \mapsto \rho(z^0, z^{n+1}) \alpha(z^0, z^1, \dots, z^{n+1}), \quad \rho \in \mathbb{R}^e, \alpha \in \text{Ca}_n(\mathbb{R}).$$

The boundary operator  $\partial : \text{C}_n(\mathbb{R}) \rightarrow \text{C}_{n-1}(\mathbb{R})$  is bounded and is uniquely extended to a bounded  $\mathbb{R}^e$ -operator  $\partial_n : \text{Ca}_n(\mathbb{R}) \rightarrow \text{Ca}_{n-1}(\mathbb{R})$ ,  $n = 1, 2, \dots$ , which looks like (3) with two ‘border’ terms. Let  $\text{M}$  be a bornological  $\mathbb{R}$ -module and  $\text{Hom}_{\mathbb{R}^e}(\text{Ca}_n(\mathbb{R}), \text{M})$  denote the module of all bounded  $\mathbb{R}^e$ -morphisms  $f : \text{Ca}_n(\mathbb{R}) \rightarrow \text{M}$  with the standard differential  $\partial^* f = (-)^{\deg f} [\partial, f]$ . Any polydifferential operator  $f : \mathbb{R}^{\otimes n} \rightarrow \text{M}$  is bounded with respect to the canonical bornology and can be

uniquely extended to a  $\mathbb{R}^e$ -morphism  $\tilde{f} : \text{Ca}_n(\mathbb{R}) \rightarrow M$ , which is again a polydifferential operator. Therefore there is natural injective morphism  $j : \text{Diff}(\mathbb{R}^{\otimes*}, M) \rightarrow \text{Hom}_{\mathbb{R}^e}(\text{Ca}_*(\mathbb{R}), M)$ .

**Theorem 11.1** *Let  $\mathbb{R}$  be the algebra of holomorphic functions in a convex domain  $U \subset \mathbb{C}^N$  and  $M$  be a bornological  $\mathbb{R}$ -module. The morphism  $j$  generates isomorphism*

$$H^*(\text{Diff}(\mathbb{R}^{\otimes*}, M)) \cong H^*(\text{Hom}_{\mathbb{R}^e}(\text{Ca}_*(\mathbb{R}), M)) \cong \text{Der}^*(\mathbb{R}, M),$$

where  $\text{Der}^*(\mathbb{R}, M)$  means the module of skew polyderivations  $\mathbb{R}^{\otimes*} \rightarrow M$ .

**Remark.** The similar statement for the  $\mathbb{R}$ -algebra of  $C^\infty$ -functions of real variables is due to Cahen-Gutt-De Wilde [4]. Kontsevich [12] considered a polynomial algebra. The above proof works also in these cases.

◀ We have  $\text{Cok } \partial_1 \cong \text{Ca}_0(\mathbb{R})/\Delta \cong \mathbb{R}$ , where  $\Delta \subset \mathbb{R}^e$  is the ideal of the diagonal. This complex is split as a left  $\mathbb{R}$ -module by the left  $\mathbb{R}$ -homotopy morphism  $\sigma_n : \text{Ca}_n(\mathbb{R}) \rightarrow \text{Ca}_{n+1}(\mathbb{R})$ ,  $\sigma_n(\alpha(z^0, \dots, z^{n+1})) = (-)^{n-1} \alpha(z^0, \dots, z^{n+1})$ ; we have  $[\partial, \sigma] = 1$ . Fix a coordinate system  $z_1, \dots, z_N$  in  $\mathbb{C}^N$  and consider the Koszul complex  $K^* \doteq \wedge^*(\mathbb{R}^e)^N$ , which is a free  $\mathbb{R}^e$ -algebra with the generators  $\mathbf{e}_1, \dots, \mathbf{e}_N$  of degree 1 and the differential  $r$  which is a derivation of degree -1 given by  $r(\mathbf{e}_j) = w_j - z_j \in \mathbb{R}^e$ ,  $j = 1, \dots, N$ , where we denote  $z_j = z_j \otimes 1$ ,  $w_j = 1 \otimes z_j$ ,  $j = 1, \dots, N$ . The complex  $K^*$  is another resolution of the  $\mathbb{R}^e$ -module  $\mathbb{R}$ . The left  $\mathbb{R}$ -morphism in  $K^*$  given by

$$sa(z, w) = \sum_i \int_0^1 \frac{\partial a}{\partial w_i}(z, (1-t)z + t(w-z)) dt \wedge \mathbf{e}_i, \quad a \in K^*$$

is a splitting, since  $[r, s] = 1$ . Here we use convexity of  $U$ ; note that the integral has sense in the completed algebra  $\mathbb{R}^e$ , but not in  $\mathbb{R} \otimes \mathbb{R}$ . It follows that two complexes are homotopic. We give an explicit construction of the commutative diagram

$$\begin{array}{ccccccc} \dots & \text{Ca}_n(\mathbb{R}) & \xrightarrow{\partial_n} & \text{Ca}_{n-1}(\mathbb{R}) & \dots & \text{Ca}_2(\mathbb{R}) & \xrightarrow{\partial_2} & \text{Ca}_1(\mathbb{R}) & \xrightarrow{\partial_1} & \text{Ca}_0(\mathbb{R}) \\ & u_n \uparrow \downarrow v_n & & u_{n-1} \uparrow \downarrow v_{n-1} & & u_2 \uparrow \downarrow v_2 & & u_1 \uparrow \downarrow v_1 & & \parallel \\ \dots & K^n & \xrightarrow{r_n} & K^{n-1} & \dots & K^2 & \xrightarrow{r_2} & K^1 & \xrightarrow{r_1} & K^0 \end{array} \quad (18)$$

The vertical arrows are determined by the recurrent formulae

$$u_n = (\sigma_{n-1} u_{n-1} r_n)^e, \quad v_n = (s_{n-1} v_{n-1} \partial_n)^e, \quad n = 1, 2, \dots$$

where we set  $u_0 = v_0 = 1$ . The notation  $(\cdot)^e$  means that the left  $\mathbb{R}$ -morphisms  $\sigma_{n-1} u_{n-1} r_n : \mathbb{R} \odot \mathbb{R}^{\odot n} \rightarrow K^n$  and  $s_{n-1} v_{n-1} \partial_n : \wedge^n(\mathbb{R}^N) \rightarrow \text{Ca}_n(\mathbb{R})$  are extended to  $\mathbb{R}^e$ -morphisms by the following rule. Let  $\phi : L \rightarrow N$  be a  $\mathbb{R}$ -morphism acting

from a left  $R$ -module  $L$  to a  $R^e$ -module  $N$  as above. We consider its extension to a morphism of two-side  $R$ -modules  $\phi^e : L \otimes R \rightarrow N$  by  $\phi^e(l \otimes a) = \phi(l)a$ . The morphism  $\phi^e$  commutes with the left-right action of the algebra  $R \otimes R$ . In both cases  $\phi = \sigma_{n-1}u_{n-1}r_n$  and  $\phi = s_{n-1}v_{n-1}\partial_n$  the morphism  $\phi^e$  is bounded with respect to the bornologies defined in  $R \odot R^{\odot*}$  and in  $K^*$  and has unique extension to a  $R^e$ -morphism  $u_n$  and  $v_n$ , respectively. Check that the diagram (18) is commutative using induction in  $n$ .

$$\begin{aligned} (r_nv_n)^e &= (r_ns_{n-1}v_{n-1}\partial_n)^e = (v_{n-1}\partial_n)^e - (s_{n-2}r_{n-1}v_{n-1}\partial_n)^e \\ &= (v_{n-1}\partial_n)^e - (s_{n-2}v_{n-2}\partial_{n-1}\partial_n)^e = (v_{n-1}\partial_n)^e. \\ (\partial_n u_n)^e &= (\partial_n \sigma_{n-1}u_{n-1}r_n)^e = (\sigma_{n-2}\partial_{n-1}u_{n-1}r_n)^e = (\sigma_{n-2}u_{n-2}r_{n-1}r_n)^e = 0. \end{aligned}$$

**Lemma 11.2** *We have  $vu = 1$  and  $\partial uv = uv\partial = 0$ .*

◀ It follows from the explicit formulae. ▶

**Lemma 11.3** *There exist  $R^e$ -morphisms  $h_n : \text{Ca}_n(R) \rightarrow \text{Ca}_{n+1}(R)$ ,  $n = 0, 1, 2, \dots$  such that for any  $n \geq 1$  the equation holds in  $\text{Ca}_n(R)$ :*

$$1 = u_nv_n + h_{n-1}\partial + \partial h_n, \quad (19)$$

◀ We set  $h_0 = 0$  and argue by induction in  $n$ . Assume that  $1 = u_{n-1}v_{n-1} + h_{n-2}\partial + \partial h_{n-1}$  for some  $n \geq 1$  and consider the morphism  $\xi_n = 1 - u_nv_n - h_{n-1}\partial : R \odot R^{\odot n} \rightarrow R \odot R^{\odot n}$  of left  $R$ -modules. By commutativity of (18), we have

$$\begin{aligned} \partial \xi_n &= \partial - \partial u_nv_n - \partial h_{n-1}\partial = \partial - u_{n-1}r_nv_n - \partial h_{n-1}\partial = \partial - u_{n-1}v_{n-1}\partial - \partial h_{n-1}\partial \\ &= \partial h_{n-1}\partial - \partial h_{n-1}\partial = 0. \end{aligned}$$

The morphism

$$h_n = (\sigma_n \xi_n)^e = (\sigma_n (1 - u_nv_n - h_{n-1}\partial))^e \quad (20)$$

defined on the space  $R \odot R^{\odot n}$  fulfils the equation

$$\partial h_n = \partial \sigma_n \xi_n = (1 - \sigma_n \partial) \xi_n = 1 - u_nv_n - h_{n-1}\partial - \sigma_n \partial \xi_n = 1 - u_nv_n - h_{n-1}\partial$$

This equation holds also on  $\text{Ca}_n(R)$ , since  $h_n$  is a right  $R$ -morphism. This yields (19). ▶

We consider elements of  $\text{Ca}_n(R) = R^{\odot n+2}$  as functions defined in the domain  $U^{\times n+2}$ . Denote by  $z^0, \dots, z^{n+1}$  the coordinate systems on the respective factors, where  $z^i = (z_1^i, \dots, z_N^i)$ ,  $i = 0, \dots, n+1$ . The order  $\text{ord } f$  of a polydifferential operator  $f \in \text{Diff}(R^{\otimes n}, M)$  is defined to be the minimal number  $d$  such that  $(\text{ad } b)^{d+1} f = 0$ , for any  $\beta \in R^{\odot n}$ , where  $(\text{ad } \beta) f = bf - f\beta$ ,  $b(z) = \beta(z, \dots, z)$ .

**Theorem 11.4** *There are  $\mathbb{R}^e$ -morphisms  $u, v$  and  $h$  such that for any cochain  $f \in \text{Hom}(\text{Ca}(\mathbb{R}), \mathbb{R})$  we have  $f = t + \partial^* h^* f + h^* \partial^* f$ , where*

$$h^* g(\alpha) = g(h(\alpha)), t(\alpha) = f(uv(\alpha)), \alpha \in \text{Ca}_*(\mathbb{R})$$

are  $\mathbb{R}^e$ -morphisms. If  $f$  is a polydifferential operator, so are  $t$  and  $g$ ,  $\text{ord } g = \text{ord } f$ , and

$$\begin{aligned} u_n v_n(\alpha)(z^0, z^{n+1}) &= \sum_p (-)^\pi \sum_{1 \leq i(1) < \dots < i(n) \leq \nu} w_{i(1)}^{p(1)} \dots w_{i(n)}^{p(n)} \\ &\times \int \dots \int_{0 \leq t_1 \leq t_2 \leq \dots \leq t_n \leq 1} \frac{\partial^n}{\partial z_{i(1)}^{p(1)} \dots \partial z_{i(n)}^{p(n)}} \alpha(z^0, z(t_1), z(t_2), \dots, z(t_n), z^{n+1}) dt_1 \dots dt_n, \end{aligned} \quad (21)$$

where  $w_i^k = z_i^k - z_i^0$ ,  $\alpha = \alpha(z^0, z^1, \dots, z^n, z^{n+1}) \in \mathbb{R}^{\odot n+2}$ ,  $z(t) = (1-t)z^0 + tz^{n+1}$ , and  $t$  is a skew-symmetric polyderivation:

$$t(\alpha) = \frac{1}{n!} \sum_p (-)^\pi \sum_{1 \leq i(1) < \dots < i(n) \leq \nu} \frac{\partial^n}{\partial z_{i(1)}^1 \dots \partial z_{i(n)}^n} \alpha(z^0, z^0, \dots, z^0) f(w_{i(1)}^{p(1)} \dots w_{i(n)}^{p(n)}),$$

Here  $\pi$  is the signature of the permutation  $p : (1, \dots, n) \mapsto (p(1), \dots, p(n))$ .

◀ **Proof of Theorem.** A direct calculation shows that

$$\partial_1 \partial_2 \dots \partial_n(\alpha) = \varepsilon \sum_{\delta(i)=0,1} (-)^{\delta(1)+\dots+\delta(n)} \alpha(z^{1-\delta(1)}, z^{2-\delta(2)}, \dots, z^{n-\delta(n)}),$$

where  $\varepsilon = (-)^{n(n+1)/2}$ . This yields

$$v_n(\alpha) = \varepsilon s_n \dots s_1 \sum_{\delta(i)=0,1} (-)^{\delta(1)+\dots+\delta(n)} \alpha(z^{1-\delta(1)}, z^{2-\delta(2)}, \dots, z^{n-\delta(n)}),$$

where the operator  $s_i$  acts on the variables  $z^i$ ,  $i = 1, \dots, n$ . Next, we find that

$$\begin{aligned} v_n(\alpha) &= \varepsilon \sum_p \int \dots \int_{0 \leq t_1 \leq t_2 \leq \dots \leq t_n \leq 1} \frac{\partial^n}{\partial z_{i(1)}^{p(1)} \dots \partial z_{i(n)}^{p(n)}} \alpha(z^0, z(t_1), z(t_2), \dots, z(t_n), z^{n+1}) dt_1 \dots dt_n \\ &\times (-)^\pi \mathbf{e}_{i(1)} \wedge \dots \wedge \mathbf{e}_{i(n)}. \end{aligned} \quad (22)$$

The explicit form of  $u_n = (\sigma_{n-1} \dots (\sigma_1 (\sigma_0 r_1)^e r_2)^e \dots r_n)^e$  is

$$u_n(\beta \mathbf{e}_{i_1} \wedge \dots \wedge \mathbf{e}_{i_n}) = \varepsilon \sum (-)^\pi \beta w_{i(1)}^{p(1)} \dots w_{i(n)}^{p(n)}. \quad (23)$$

This yields  $v_n u_n = 1$ . The formula (21) follows from (22) and (23).

Now we take a polydifferential operator  $f$  and check that so is  $g$ , where  $g[\alpha] = f(h_n[\alpha])$  for arbitrary  $[\alpha] = [a_1|\dots|a_{n-1}] \in \mathbb{R}^{\otimes(n-1)}$ . By (20) we have  $g[\alpha] = \phi[\alpha] - \tau[\alpha] - \phi(h_{n-1}[\alpha])$ , where  $\phi[\alpha] = f(\sigma_{n-1}[\alpha])$ ,  $\tau[\alpha] = t(\sigma_{n-1}[\alpha])$  and  $\sigma_{n-1}[\alpha] = [\alpha|1]$ . Obviously  $\phi$  and  $\tau$  are polydifferential operators and  $\text{ord } \phi \leq \text{ord } f$ ,  $\text{ord } \tau \leq \text{ord } t \leq \text{ord } f$ . The same is true for the term  $\phi(h_{n-1}[\alpha])$  if we use induction on  $n$ . The case  $n = 0$  is trivial, which completes the proof of Theorem 11.4.  $\blacktriangleright$

**Remark.** Suppose that  $U = V \times W$ , where  $V$  and  $W$  are sets in some coordinate subspaces of  $\mathbb{C}^N$ , then there is injective homomorphism  $\mathbb{R}(V) \rightarrow \mathbb{R}(U)$ . The construction of the homotopy operator  $\eta$  is hereditary in the sense, that if a cochain  $f$  vanishes on products  $[a_1|\dots|a_n]$  for any  $a_1, \dots, a_n \in \mathbb{R}(V)$ , then so is the cochain  $g = \eta^* f$ . This follows from the above proof. The latter works also for non convex domains  $U$ , but the formula (21) is more complicated.

## 12 Splitting

Let  $\mathcal{R}$  be a resolution of  $X$  on a covering  $\mathbb{P}$ . Consider the functor  $\mathcal{O} : A \rightsquigarrow \mathcal{O}(A) \doteq \mathcal{O}|_{X_A}$  from the category  $\mathbb{P}$  to the category of sheaves. We have the natural transformation of functors  $\pi : \mathcal{R} \implies \mathcal{O}$ ,  $\mathcal{R}(A) \rightarrow \mathcal{O}(A)$  which induces isomorphism in cohomology. Define the functor  $\mathcal{O} : A \rightarrow \mathcal{O}(A) \doteq \Gamma(X_A, \mathcal{O}(A))$  on  $\mathbb{P}$  with values in the category of  $\mathbb{C}$ -algebras. This functor is a  $\mathbb{R}$ -module and the bicomplex  $\text{Diff}(\text{Ca}_*(\mathbb{R}), \mathcal{O})$  is well defined with the differentials  $f \mapsto fd$  and  $f \mapsto [\partial, f]$ . Denote by  $\text{Tot}(\mathcal{R}, \mathcal{O})$  the total complex with the differential  $D^* f = (-)^{\text{Deg} f} fd + [\partial, f]$ . The natural map

$$\pi : \text{Tot}(\mathcal{R}) \rightarrow \text{Tot}(\mathcal{R}, \mathcal{O})$$

is a morphism of complexes.

**Lemma 12.1** *The morphism  $\pi$  is a surjection and generates the isomorphism of cohomology*

$$\text{QT}^*(X) = H^*(\text{Tot}(\mathcal{R})) \cong H^*(\text{Tot}(\mathcal{R}, \mathcal{O}))$$

$\blacktriangleleft$  A proof is based on the fact that the homomorphism  $\pi(A) : \mathbb{R}(A) \rightarrow \mathcal{O}(A)$  is a resolution. We omit details.  $\blacktriangleright$

The grading  $|\cdot|$  in the complex  $\text{Tot}(\mathcal{R}, \mathcal{O})$  is non-negative and any homogeneous cochain  $p \in \text{Tot}^n(\mathcal{R}, \mathcal{O})$  has a unique representation in the form  $p = p_0 + p_1 + \dots + p_n$ , where the term  $p_k$  has bidegree  $(k, n - k)$ ,  $k = 0, \dots, n$ . We say that the cochain  $p$  is *normal*, if  $p_0$  is a derivation and  $p_n(1 - e_{n+1}) = 0$ , for  $n \geq 1$ , where  $e_{n+1} \in \mathbb{Q}[\Sigma_{n+1}]$  is Barr's shuffle-projector in  $\mathbb{R}^{\otimes n+1}$ , and  $\Sigma_{n+1}$  denotes the permutation group acting in  $\mathbb{R}^{\otimes n+1}$ , see [1]. The action of the group  $\Sigma_{n+1}$  is modified taking in account the

grading in  $\mathbb{R}^{\odot n+1}$ . In particular, the term  $p_1$  of a normal cochain is skew-symmetric, that is  $q_1(b|a) = -(-)^{|a||b|} q_1(a|b)$ .

**Lemma 12.2** *If  $p \in \text{Tot}(\mathcal{R}, \mathcal{O})$  and the cochain  $D^*p$  is normal, then there exists a finite cochain  $r \in \text{Tot}(\mathcal{R}, \mathcal{O})$  such that  $p' \doteq p - D^*r$  is normal.*

◀ Fix a simplex  $A \in \mathbb{N}$  and consider the subcomplex  $\text{Harr}(A)$  in  $\text{Diff}(\text{Ca}_*(\mathbb{R}(A)), \mathcal{O}(A))$  of cochains  $f$  such that  $fe_{n+1} = 0$ ,  $n = \text{deg } f$ . This equation is equivalent to vanishing of  $f$  on all shuffle products. The cohomology of the complex  $\text{Harr}(A)$  is trivial in degrees  $m > 0$  according to Theorem 11.1. For any cochain  $f$  the cochain  $g \doteq f(1 - e_{n+1})$  is the projection of  $f$  to  $\text{Harr}(A)$ . We 'kill' successively the projections to  $\text{Harr}(A)$  of  $p_n, p_{n-1}, \dots$ . The cochain  $p_n$  is  $\partial^*$ -closed, and so is the cochain  $p_n(1 - e_{n+1})$ . We can find a solution  $r_{n-1}(A) \in \text{Harr}(A)$  of the equation  $\partial^*r_{n-1}(A) = p_n(A)(1 - e_{n+1})$ . We use induction on  $\dim A$  and hereditary of the homotopy operator of Theorem 11.1 to ensure functoriality  $A \mapsto r_{n-1}(A)$ . Consider the cocycle  $p' = p - D^*r_{n-1}$  and have  $p'_n(1 - e_{n+1}) = \partial^*r_{n-1} - \partial^*r_{n-1}(1 - e_n) = 0$ , since  $(\partial q)e_{n+1} = \partial(qe_n)$ , which is a property of Barr's projectors. On the other hand,  $\partial^*p'_{n-1}(1 - e_n) + p'_nd(1 - e_n) = 0$ , since the cochain  $D^*p' = D^*p$  is normal. We have  $p'_nd(1 - e_{n+1}) = (p'_n(1 - e_{n+1}))d = 0$ , since  $d$  is a derivation, which implies  $\partial^*(p'_{n-1}(1 - e_n)) = (\partial^*p'_{n-1})(1 - e_{n+1}) = 0$ , that is  $p'_{n-1}(1 - e_n)$  is a cocycle in the complex  $\text{Harr}(A)$ . We apply again Theorem 11.1 and so on. Finally, we find an equivalent cocycle  $\tilde{p} = \tilde{p}_0 + \tilde{p}_1 + \dots + \tilde{p}_n$  such that all the terms  $\tilde{p}_k$ ,  $k = n, n-1, \dots, 1$  are normalized. The equation  $\partial^*\tilde{p}_0 = -[d, \tilde{p}_1]$  implies that  $\partial^*\tilde{p}_0 = 0$ , since the left-hand side is symmetric and the right-hand side is skew-symmetric. ▶

◀ Proof of Theorem 6.1. Take a class  $\tilde{p} \in H^n(\text{Tot}(\mathcal{R}, \mathcal{O})) = \text{QT}^n(X)$  and choose a normal cocycle  $p \in \tilde{p}$  by means of Lemma 12.2. By definition we have  $p = p_0 + p_1 + \dots + p_n$  and the term  $p_0$  is a derivation, that is  $p_0 \in \text{Der}(\mathcal{R}, \mathcal{O})$ . This term defines a class  $\tilde{p}_0 \in \text{T}^*(X)$ . Check that the map  $t : \tilde{p} \rightarrow \tilde{p}_0$  is well defined. If  $p'$  is another finite normal cocycle in the class  $\tilde{p}$ , then  $p' - p = D^*r$ . According to Lemma 12.2, there exists a cochain  $s \in \text{Tot}$  such that  $r' \doteq r - D^*s$  is normal, which implies  $r'_0 \in \text{Der}(\mathcal{R})$ . The equation  $p' - p = D^*r'$  yields  $p'_0 - p_0 = r'_0d$  and the equation  $\text{cl}(p'_0) = \tilde{p}_0$  follows. ▶

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