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Hartogs Phenomenon for Systems of Differential Equations

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Abstract A property of extension is studied for solutions of linear elliptic systems of differential equations. We show that the dimension of the characteristic variety of the system plays a key role in the problem.

Keywords Differential module · Characteristic variety · Elliptic module · Determined and overdetermined modules

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1 Introduction

According to the famous theorems of Hartogs and Osgood–Brown, any compact singularity (with no holes) of a holomorphic function of several variables is removable. This fact can be viewed as a property of solutions of the Cauchy–Riemann system of differential equations in a domain of the real space $\mathbb{R}^{2m}$ where $m > 1$. This phenomenon was extended by Ehrenpreis [8], who stated that a compact singularity is always removable for any system of two equations with constant coefficients and relatively prime symbols. It was shown in [15] that the automatic extension property of solutions of a general system M of equations with constant coefficients is governed by the modules $\mathrm{Ext}^k(M, D)$, $k = 1, 2, \dots$. In particular, the equation $\mathrm{Ext}^1(M, D) = 0$ guarantees the absence of a compact singularity. Vanishing of higher Ext implies removing of some noncompact singularities, in particular for singularities supported by a submanifold. Kawai [11] treated a special class of non-elliptic systems with real analytic coefficients. He stated automatic extension of solutions through a submanifold

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of codimension > 1 in the class of hyperfunctions and in the class of real analytic functions.

The theory of monogenic functions of quaternion variables was of permanent interest as a non-commutative version of the theory of holomorphic functions. The Hartogs phenomenon for monogenic functions is a special case of the theory [15] and was addressed in several studies later. It was rediscovered by direct methods by Pertici [17], studied in more detail in [2], proved again by Adams, Berenstein, Loustaunau, Sabadini, Struppa [1], and once again by W. Wang [18, 19]. The Cauchy–Fueter complexes and its variations were under study in the pioneering paper of Baston [4] and later by Colombo, Souček, and Struppa [6] and Bureš, Damiano, and Sabadini [5] with a focus on the purely algebraic side.

The Hartogs phenomenon is not, however, limited by the theory of differential equations with constant coefficients. Moreover, this property and its versions depend rather on the dimension of the support of the corresponding differential module than on a specific form of its resolution. The objective of this paper is to state the phenomenon of compulsory extension for elliptic systems of linear partial differential equations in an open set $X \subset \mathbb{R}^n$ of a general form

$$P(x, \partial_x)u = 0. \tag{1}$$

Here $\partial_x \doteq (\partial/\partial x_1, \dots, \partial/\partial x_n)$, $P = \{p_{ij}\}$ is an arbitrary $s \times r$-matrix differential operator, $u = (u_1, \dots, u_r)$ are unknown functions, and numbers s and r are arbitrary. We show that if the operator is elliptic in a proper meaning and its coefficients are real analytic, then the Hartogs phenomenon has place if the dimension d of the characteristic variety V is strictly less than $n - 1$ (just as for the case of the Cauchy–Riemann system with $m > 1$). An exact statement is: Any solution of (1) defined in $X \setminus U_x(r)$ has compulsory extension to X as a solution, where $U_x(r)$ is a ball with a center $x \in X$, the point $x \in X$ is arbitrary, and the radius $r = r(x)$ is a positive continuous function in X. For smaller d, a stronger statement holds (Theorem 16). In particular, no closed C^1-submanifold S of dimension s can be a support of a non-removable singularity of a solution if $s < n - d - 1$. This is not the case for $s = n - d - 1$.

2 Regularity Conditions for a Differential Matrix

We impose a general condition of regularity on the matrix P which is close to the form proposed by Malgrange [13]. Fix an arbitrary point $x \in X$, and denote by O_x the algebra of germs of analytic functions at the point $x \in \mathbb{R}^n$ and by D_x the algebra of differential operators in O_x. Let $P_i = (p_{i1}, \dots, p_{ir})$, $i = 1, \dots, s$ be rows of the matrix P; consider a linear combination

$$Q(x, \partial_x) = \sum_{i=1}^{s} a_i(x, \partial_x) P_i(x, \partial_x) \in D_x^r \tag{2}$$

with some $a_i \in D_x$, where $Q = (q_1, \dots, q_r)$. We assume that there exist elements $b_i \in D_x, i = 1, \dots, s$ such that

$$Q(x, \partial_x) = \sum b_i(x, \partial_x) P_i(x, \partial_x) \tag{3}$$

and $\operatorname{ord} b_i + \operatorname{ord} p_{ij} \le \operatorname{ord} q_j$ for all $i = 1, \dots, s$, $j = 1, \dots, r$ ($\operatorname{ord} a$ means the order of a differential operator a). In other words, there are no cancellations of higher-order terms in the right-hand side of (3). This condition is not in fact restrictive since it can always be satisfied if the matrix P is supplemented by several lines of the form (2).

Definition Fix some integers $\sigma_1, \dots, \sigma_s$ and $\rho_1, \dots, \rho_t$ (called shifts) such that

$$\deg p_{ij} \le \sigma_i - \rho_j, \quad i = 1, \dots, s;\ j = 1, \dots, r.$$

The principal part of the system is the matrix $\mathrm{P} = \{\mathrm{p}_{ij}\}$, where p_{ij} is the sum of homogeneous terms of p_{ij} of degree $\sigma_i - \rho_j$ ($p_{ij} = 0$ if there are no such terms). Substituting partial derivatives $\partial/\partial x_i$ with independent variables ξ_i, $i = 1, \dots, n$, we obtain homogeneous polynomials $\mathrm{p}_{ij}(x, \xi)$ in $\xi = (\xi_1, \dots, \xi_n)$ with coefficients in O_x. The next condition is essential:
(*) For any point $x \in X$ and polynomials $\mathrm{r}_1, \dots, \mathrm{r}_s \in \mathbb{C}[\xi_1, \dots, \xi_n]$ such that

$$\sum_i \mathrm{r}_i(\xi)\mathrm{P}_i(x, \xi) = 0,$$

where $\mathrm{P}_i(x, \xi)$ denotes the vector $(\mathrm{p}_{i1}(x, \xi), \dots, \mathrm{p}_{ir}(x, \xi))$, there exist functions $\mathrm{R}_1, \dots, \mathrm{R}_s \in O_x[\xi_1, \dots, \xi_n]$ such that

$$\sum_i \mathrm{R}_i(y, \xi)\mathrm{P}_i(y, \xi) = 0$$

for y in a neighborhood of the point x such that $\mathrm{R}_i(x, \xi) = \mathrm{r}_i(\xi)$, $i = 1, \dots, s$. This condition needs to be checked only for a finite number of vectors $(\mathrm{r}_1, \dots, \mathrm{r}_s)$ and it is generic, that is, (*) is always fulfilled in the complement to a nowhere-dense analytic set [16].

Note that in the case $r = s = 1$, the condition (*) means only that the principal part P of P does not vanish at x.

3 Differential Modules and Filtrations

Now we rearrange the above conditions in a more algebraic form. Again let $x \in \mathbb{R}^n$ and D be the algebra of differential operators in $\mathbb{R}^n$ with coefficients in the algebra O of germs at x of analytic functions in $\mathbb{R}^n$ (here and later we omit the subscript x). The algebra D has natural filtration $\{D_k,\ k = 0, 1, \dots\}$, where D_k is the O-module of differential operators $a \in D$ of order $\operatorname{ord} a \le k$ and $D_0 = O$. The associated graded module

$$\mathrm{D} = \operatorname{gr} D = \oplus_{k=0}^{\infty} D_k / D_{k-1}$$

is a commutative O-algebra. Fix a coordinate system $x_1, \dots, x_n$ in $\mathbb{R}^n$. The algebra D is isomorphic to the graded algebra $O[\xi_1, \dots, \xi_n]$, where the generator ξ_i is represented by the operator $\partial/\partial x_i$, $i = 1, \dots, n$. The algebra $\mathrm{D} \otimes_O \mathbb{C}$ is then isomorphic to the graded algebra of homogeneous polynomials in $T_x^*(\mathbb{R}^n)$. Fix a natural r and a vector $\rho = (\rho_1, \dots, \rho_r) \in \mathbb{Z}^r$; the increasing sequence of O-submodules

$$D_k^\rho = \left\{a \in D^r, \operatorname{ord}_\rho \ a \le k\right\}, \quad k \in \mathbb{Z}$$

is called filtration generated by the shift vector ρ, where $\operatorname{ord}_\rho a = \operatorname{ord}_\rho(a_1, \dots, a_r) = \max_i \operatorname{ord} a_i + \rho_i$. The graded vector space

$$\mathrm{D}^\rho = \oplus_k D_k^\rho / D_{k-1}^\rho$$

is a module over the graded commutative algebra D. Let r, s be natural numbers; any morphism of left D-modules $P : D^s \to D^r$ can be written in the form $a \mapsto aP$, where an element $a = (a_1, \dots, a_s) \in D^s$ is thought of as a row and P as an $s \times r$-matrix whose entries p_{ij}, $i = 1, \dots, s$, $j = 1, \dots, r$ are elements of D. Let σ denote the filtration in D^s defined by a shift vector $\sigma = (\sigma_1, \dots, \sigma_s)$. The morphism P agrees with the filtrations, if $\operatorname{ord}_\rho(aP) \le \operatorname{ord}_\sigma a$ for any $a \in D^s$. This condition is equivalent to the inequalities $\operatorname{ord} p_{ij} \le \sigma_i - \rho_j$. Let p_{ij} be the sum of the terms of p_{ij} of order $\sigma_i - \rho_j$. The matrix $\mathrm{P} = \{\mathrm{p}_{ij}\}$ is called the *principal part* of P with respect to the filtrations generated by ρ and σ. The operator P is called *elliptic* at a point x in the sense of Douglis–Nirenberg [7] if $\operatorname{rank} \mathrm{P}(x, \xi) = s$ for any real $\xi \ne 0$.

Let M be a left D-module; suppose that M has an increasing filtration by O-submodules M_k, $k \in \mathbb{Z}$, such that $\cup M_k = M$ and $D_i M_k \subset M_{k+i}$ for any i and k. Then we call M a filtered D-module. For a filtered module M, the direct sum

$$\operatorname{gr} M = \oplus_{k=-\infty}^{\infty} M_k / M_{k-1}$$

has a natural structure of D-module.

Let M and N be filtered left (or right) D-modules. We say that a D-morphism $\alpha : M \to N$ *agrees* with the filtrations, if $\alpha(M_k) \subset N_k$ for $k = 0, 1, 2, \dots$. If α agrees with filtrations, it generates a morphism of graded modules $\operatorname{gr} \alpha : \operatorname{gr} M \to \operatorname{gr} N$, and the correspondence $\alpha \mapsto \operatorname{gr} \alpha$ is a functor.

4 Complexes and Symbols

Let

$$\dots \to D^t \xrightarrow{Q} D^s \xrightarrow{P} D^r \tag{4}$$

be an exact sequence of left D-modules. The morphisms $P, Q, \dots$ in (13) act by right multiplication as above. Suppose that the modules in (4) are supplied with filtrations generated by some shift vectors $\rho, \sigma, \tau, \dots$, and denote these modules by $\dots, D^\tau, D^\sigma, D^\rho$. We assume that the morphisms $\dots, Q, P$ agree with these filtrations, which means $\dots, \operatorname{ord} q_{jk} \le \tau_j - \sigma_k, \operatorname{ord} p_{ij} \le \sigma_i - \rho_j$ for entries of these matrices. The sequence of graded D-modules is then well defined,

$$\dots \to \mathrm{D}^\tau \xrightarrow{\mathrm{Q}} \mathrm{D}^\sigma \xrightarrow{\mathrm{P}} \mathrm{D}^\rho.$$

Taking a tensor product over the algebra O, we get the complex

$$\cdots \to \mathrm{D}^{\tau} \otimes \mathbb{C} \overset{\mathrm{Q}\otimes\mathbb{C}}{\to} \mathrm{D}^{\sigma} \otimes \mathbb{C} \overset{\mathrm{P}\otimes\mathbb{C}}{\to} \mathrm{D}^{\rho} \otimes \mathbb{C}$$

of free graded modules over the commutative algebra $\mathrm{A} = \mathrm{D} \otimes \mathbb{C} \cong \mathbb{C}[\xi_1, \dots, \xi_n]$. The set of maximal ideals in the algebra A is isomorphic to $\mathbb{C}^n$. For a maximal ideal $\mathfrak{m}$ in A we take a tensor product with the quotient algebra. This yields a complex of $\mathbb{C}$-vector spaces

$$\cdots \to \big(\mathrm{D}^{\tau} \otimes \mathbb{C}\big)\otimes_{\mathrm{A}}\mathrm{A}/\mathfrak{m} \overset{\mathrm{Q}_{\mathfrak{m}}}{\to} \big(\mathrm{D}^{\sigma} \otimes \mathbb{C}\big)\otimes_{\mathrm{A}}\mathrm{A}/\mathfrak{m} \overset{\mathrm{P}_{\mathfrak{m}}}{\to} \big(\mathrm{D}^{\rho} \otimes \mathbb{C}\big) \otimes_{\mathrm{A}} \mathrm{A}/\mathfrak{m},$$

where $\dots, \mathrm{P}_{\mathfrak{m}} = \mathrm{P} \otimes \mathbb{C}\otimes \mathrm{A}/\mathfrak{m}$. Because $\mathrm{A}/\mathfrak{m} \cong \mathbb{C}$, this complex can be written in a simple form,

$$\cdots \to \mathbb{C}^{t} \overset{\mathrm{Q}(x,\xi)}{\to} \mathbb{C}^{s} \overset{\mathrm{P}(x,\xi)}{\to} \mathbb{C}^{r}, \tag{5}$$

where ξ is the point in $\mathbb{C}^n$ corresponding to the ideal $\mathfrak{m}$. Here, $\dots, \mathrm{Q}(x,\xi), \mathrm{P}(x,\xi)$ are matrices whose entries are analytic functions of x and polynomial functions of ξ.

Definition We call (5) the *principal symbol* of (4). The complex (4) is called *elliptic* in the sense of Douglis–Nirenberg if the symbol is exact at any real point $\xi \neq 0$.

5 Local Solvability

Let $x \in \mathbb{R}^n$, $D = D_x$, and (4) be an exact sequence of left D-modules. Denote by E the space of germs at x of C^{∞}-functions defined in $\mathbb{R}^n$. This space has a natural structure of left D-module. Applying the functor $\mathrm{Hom}_D(\cdot, E)$ to (4) we obtain a complex of vector spaces:

$$E^r \overset{P}{\to} E^s \overset{Q}{\to} E^t \to \cdots, \tag{6}$$

where the matrices $P, Q, \dots$ act by left multiplication as in (1).

Theorem 1 *If* (4) *is exact and elliptic, then the sequence* (6) *is exact.*

The case of arbitrary operator P with constant coefficients is considered in Malgrange [12] and in [15]. For the case of analytic coefficients, this statement is essentially due to Malgrange [14] who proved it for Newlander–Nirenberg's operator by reduction to the case of germs of analytic functions (the method coming from the Hodge theory). A proof in the general case was done by Andreotti–Nacinovich [3] by the same method. For a special case, a quite different method was used by Hörmander [10].

We state here a quantitative version of this theorem. Let E be the sheaf of germs of C^{∞}-functions in $\mathbb{R}^n$. The space $E(U) = \Gamma(U, E)$ for any open $U \subset \mathbb{R}^n$ has a natural Fréchet topology. Any $s \times r$ matrix P as in (1) defined in X generates for any open

$U \subset X$ a linear continuous operator $P : E^r(U) \to E^s(U)$. We denote its kernel by $E_P(U)$.

Fix a Euclidean structure in $\mathbb{R}^n$; for a point $x \in \mathbb{R}^n$ and a number $r > 0$ the notation $U_x(r)$ means the r-neighborhood of x.

Theorem 2 *Let $D = D_X$ be the sheaf of germs of analytic differential operators in an open set $X \subset \mathbb{R}^n$ and (4) be an elliptic complex of filtered D-modules, then*

A. *There exists a continuous function a_x in X such that for an arbitrary point $x \in X$, arbitrary $0 < r \le 1$, and arbitrary $g \in E_Q(U_x(r))$ there exists $f \in E^r(U_x(a_x r))$ such that*

$$Pf = g \tag{7}$$

in $U_x(a_x r)$.

B. *There exists a linear continuous operator $\mathrm{s}_{x,r} : E_Q(U_x(r)) \to E^r(U_x(a_x r))$ that provides a solution to (7).*

Proof **1.** We will construct a Laplace-like operator Ω for (4) and reduce the statement to the case when $\Omega g = 0$. Because the morphisms agree with the filtrations, the inequalities

$$\operatorname{ord} q_{ij} \le \tau_i - \sigma_j, \qquad \operatorname{ord} p_{ij} \le \sigma_i - \rho_j$$

are fulfilled, where $Q = \{q_{ij}\}$, $P = \{p_{ij}\}$ are the entries of matrices Q and P. For any differential operator A in $X \subset \mathbb{R}^n$ with analytic coefficients, the formal adjoint operator A^* acts on smooth densities and on distributions with compact support in X:

$$\int_X A^*(v)u = \int_X vA(u).$$

Identifying a function u with the density $u\mathrm{d}x$, where $\mathrm{d}x$ is the Euclidean volume form in $\mathbb{R}^n$, we make the adjoint operator a^* acting on functions. It also has analytic coefficients and $(AB)^* = B^*A^*$. The Laplace operator Δ in $\mathbb{R}^n$ is self-adjoint; we denote $\Lambda = -\Delta$. For a natural k and a vector $\omega = (\omega_1, \dots, \omega_k)$ with natural coordinates, we denote by Λ^ω the diagonal $k \times k$ matrix $(\Lambda^{\omega_1}, \dots, \Lambda^{\omega_k})$. Set $t = \max(\tau_1, \dots, \tau_t)$ and denote $t + \rho = (t + \rho_1, \dots, t + \rho_r)$, $t - \tau = \cdots$. The differential operator

$$\Omega = P\Lambda^{t+\rho}P^* + \Lambda^\sigma Q^* \Lambda^{t-\tau} Q \Lambda^\sigma$$

is well defined in the sheaf E^s.

Lemma 3 *Ω is an elliptic operator in the sense of Douglis–Nirenberg in X with the shift vector equal to $2t + 2\sigma = (2t + 2\sigma_1, \dots, 2t + 2\sigma_s)$.*

We postpone a proof of this Lemma. Because Ω is elliptic, there exists a countable family of local fundamental solutions E defined in open sets D_E such that $X = \cup D_E$. Take a fundamental solution E (Lemma 4), an arbitrary point $x \in D_E$, and a number $r > 0$ such that $U_x(r) \subset D_E$. Choose a smooth cut function e with support in $U_x(r)$

that is equal to 1 in $U_x(r/2)$. Suppose that a function $g \in E^s(U_x(r))$ fulfils $Qg = 0$, and set

$$g_e = eg, \qquad h = Q^* \Lambda^{t-\tau} Q \Lambda^\sigma E g_e, \qquad f = \Lambda^{t+\rho} P^* E g_e.$$

We have

$$Pf = P\Lambda^{t+\rho} P^* E g_e = \Omega E g_e - \Lambda^\sigma Q^* \Lambda^{t-\tau} Q \Lambda^\sigma E g_e = g_e - \Lambda^\sigma h,$$

and see that the function f is a solution of (7) modulo a function $\Lambda^\sigma h$. On the other hand,

$$\begin{aligned}\Omega h &= \Lambda^\sigma Q^* \Lambda^{t-\tau} Q \Lambda^\sigma Q^* \Lambda^{t-\tau} Q \Lambda^\sigma E g_e \\ &= \Lambda^\sigma Q^* \Lambda^{t-\tau} Q \Omega E g_e = \Lambda^\sigma Q^* \Lambda^{t-\tau} Q g = 0\end{aligned}$$

in $U_x(r/2)$ since $P^* Q^* = 0$ and $QP = 0$. It follows that the function h is analytic in $U_x(r/2)$ since Ω is elliptic and has analytic coefficients.

Proof **2.** We show that h has holomorphic continuation in a quantified complex neighborhood Z_x of the point x.

Lemma 4 *Let A be an $m \times m$ matrix differential operator with analytic coefficients in an open set $X \subset \mathbb{R}^n$ that is elliptic in the sense of Douglis–Nirenberg. Then there exist positive continuous functions $r = r(q)$, $s = s(q)$ in X and for any $q \in X$ a fundamental solution $E = E(x, u)$ defined in $U_q(r) \times U_q(r)$ that admits a holomorphic extension $E(z, w)$ in the domain*

$$Z = \Big\{ z = x + \imath y, w = u + \imath v,\ s|y - v| \le |x - u| < r \Big\}. \tag{8}$$

Moreover, ϕE defines a bounded operator in $L_2(U_q(r)) \to L_2(\mathbb{R}^n)$ for any test function ϕ with support in $U_q(r)$.

Proof We apply the method of E. Levi. According to the assumption, A defines a map $A : D^m \to D^m$ that agrees with filtrations D^σ and D^ρ in D^m generated by some shift vectors ρ and σ. Let $\mathrm{A}(x, D)$ be the principal part of this operator, that is, $\mathrm{A} = \{\mathrm{a}_{ij}\}$, where a_{ij} is the homogeneous part of a_{ij} of order $\sigma_i - \rho_j$. The principal symbol $\mathrm{A}(x, \xi)$ is elliptic for any $x \in X$, which implies that the scalar operator $\det \mathrm{A}(x, \xi)$ is elliptic. The elliptic operator $A_q(\partial_x) = A(q, \partial_x)$ with constant coefficients possesses a fundamental solution $E_q(x, y) = E_q(x - y)$ in $\mathbb{R}^n$, and the function $E_q(x)$ has a holomorphic extension $E_q(z)$ to a neighborhood of $\mathbb{R}^n \backslash \{0\}$ of the form $\{z \in \mathbb{C}^n; |\mathrm{Im}\, z| < s_A |\mathrm{Re}\, z|\}$, where the constant s_A is determined from the condition $\det \mathrm{A}_q(\xi + \imath\eta) \ne 0$ for $|\eta| < s_A|\xi|$, where $A_q(\zeta)$ is the symbol of A_q. Such a fundamental solution can be written as the Fourier–Laplace integral of $A_q^{-1}(\zeta)$ taken over an n-cycle in $\mathbb{C}^n$ that coincides with $\mathbb{R}^n$ up to a compact subset. Choose a number r such that $U_q(2r) \subset X$ and take a test function ϕ in U that is equal to 1 in $U_q(r)$. Consider an operator series

$$E = E_q \sum_{k=0}^{\infty} F^k, \qquad F = \phi(A - A_q) E_q, \tag{9}$$

where we set $F = 0$ in $\mathbb{R}^n \backslash U_q(r)$. Let e_{ij} be the $m \times m$ matrix whose entry equals 1 in the ij-place and 0 otherwise, and let b be a differential operator in $\mathbb{R}^n$ with constant coefficients such that $\operatorname{ord} b \le \sigma_i - \rho_j$. The operator be_{ij} defines a map $D^\sigma \to D^\rho$ which agrees with filtrations. This yields that the composition $be_{ij}E_q$ is a bounded operator in $L_2(\mathbb{R}^n)^s$. We have

$$A - A_q = \sum_{i,j=1}^{r} \beta_{ij}(x) b_{ij} e_{ij},$$

where for all i, j, $\operatorname{ord} b_{ij} \le \sigma_i - \rho_j$ and $\beta_{ij}(z)$ are analytic functions that vanish for $x = q$. The norm $N = \|\phi(A - A_q)E_q\|$ can be made smaller than $1/2$ if we take $r = r(q)$ sufficiently small, and the series (9) converges as an operator in $L_2(U_q(r))^m \to L_2(U_q(2r))^m$. Moreover, we have

$$\|\psi E\| < 2\|\psi E_q\| < \infty$$

for any test function ψ since the kernel E_q has weak singularity. It is easy to check that $AE = \mathrm{id}$ in $U_q(r)$. The kernel $E(x,u)$ is real analytic in $U_q(r) \times U_q(r)$ out of the diagonal since the operator A is elliptic. Moreover, it has a holomorphic extension to the domain (8) with $s(q) = s_{A_q}/2$ due to Hörmander [9], Theorem 5.3.3.

Lemma 5 *For an arbitrary point $q \in X$ and arbitrary $r \le r(q)$, any solution f of the equation $Af = 0$ in the ball $U_q(r)$ has a unique holomorphic extension to the ball $Z_q(\rho) \subset \mathbb{C}^n$ with the center q and radius $\rho = s(q)r$, where $r(q)$ and $s(q)$ are the functions as in the previous lemma.*

Proof Choose a test function e supported in $U_q(r)$ that is equal to 1 in $U_q(r/2)$, and evaluate the solution f in $U_q(r/2)$ by means of the integral

$$f(x) = \int_{\mathbb{R}^n} A^*(u, \partial_u) e(u) f(u) E(x,u) \mathrm{d}u.$$

We can now move the point x to an arbitrary point $z = x + \imath y$ such that $|y| < s(q)r/2$. The function $E(z,u)$ is holomorphic, since the support of the integrand is contained in $U_q(r) \backslash U_q(r/2)$, hence $|x - u| > r/2$. This gives a holomorphic extension of f to the ball $Z_q(s(q)r/2)$.

Thus the function h as above has holomorphic extension to the ball $Z_x(b_x r)$, where $b_x = s(x)/2$. This construction has the property **B** with any $r < \operatorname{dist}(x, \partial D_E)$ and a positive continuous function $b_x = b_{E,x} \le 1$ defined in the domain D_E. Take the maximum $a_x = \max\{b_{E,x}\delta_E(x), x \in D_E\}$, where $\delta_E(x) \doteq \min\{\operatorname{dist}(x, \partial D_E), 1\}$ for $x \in D_E$ and $\delta_E(x) = 0$ for $x \in X \backslash D_E$, over the family of all fundamental solutions E constructed by means of Lemma 4. The function a_x is continuous in X, and $a_x \le b_{E,x}$ for any $x \in X$ and some E, hence the function a_x fulfils **A** and **B**.

Proof **3.** Now it is sufficient to prove the statements of Theorem 2 for the sheaf H of germs of analytic functions in $\mathbb{R}^n$. A construction of a solution to (7) in the space of germs of analytic functions can be done by the method of [16] that guarantees the properties **A**, **B** in terms of balls $Z_x(r)$ in $\mathbb{C}^n$. □

Proof **4.** Proof of Lemma 3. We have

$$\begin{aligned}\operatorname{ord}\big(P\Lambda^{t+\rho}P^*\big)_{ij} &\le \max_k\big(\operatorname{ord} p_{ik} + \operatorname{ord}\Lambda^{t+\rho_k} + \operatorname{ord} p_{jk}\big)\\ &\le \max_k(\sigma_i - \rho_k + 2\rho_k + 2t + \sigma_j - \rho_k) = \sigma_i + \sigma_j + 2t.\end{aligned}$$

The same inequality holds for the matrix $\Lambda^\sigma Q^*\Lambda^{t-\tau}Q\Lambda^\sigma$ and for Ω. The principal symbol $\Omega(z,\xi)$ of Ω with respect to the shift vector $2t+2\sigma$ is equal to

$$\Omega = \mathrm{P}\mathrm{R}^{t+\rho}\mathrm{P}^* + \mathrm{R}^\sigma\mathrm{Q}^*\mathrm{R}^{t-\tau}\mathrm{Q}\mathrm{R}^\sigma,$$

where R^ω means the symbol of the operator Λ^ω. We will check that $\det\Omega(z,\xi)\neq 0$ as $\xi\in\mathbb{R}^n\backslash\{0\}$. If it is not the case for a point ξ, then there exists a non-zero vector $v\in\mathbb{R}^s$ such that $\Omega(z,\xi)v=0$. Define the coordinate scalar product $\langle,\rangle$ in $\mathbb{R}^s$ and write

$$\begin{aligned}0 &= \big\langle\Omega(x,\xi)v,v\big\rangle = \big\langle\mathrm{P}\mathrm{R}^{t+\rho}\mathrm{P}^*v,v\big\rangle + \big\langle\mathrm{R}^\sigma\mathrm{Q}^*\mathrm{R}^{t-\tau}\mathrm{Q}\mathrm{R}^\sigma v,v\big\rangle\\ &= \big\langle\mathrm{R}^{(t+\rho)/2}\mathrm{P}^*v,\mathrm{R}^{(t+\rho)/2}\mathrm{P}^*v\big\rangle + \big\langle\mathrm{R}^{(t-\tau)/2+\sigma}\mathrm{Q}v,\mathrm{R}^{(t-\tau)/2+\sigma}\mathrm{Q}v\big\rangle,\end{aligned}$$

where $\mathrm{R}^{\omega/2}$ means a diagonal matrix with the positive diagonal terms $\sqrt{\mathrm{R}^{\omega_i}}$, $i=1,\dots,k$. Both terms in the right-hand side are non-negative, hence vanish. This yields

$$\mathrm{P}^*(x,\xi)v=0,\qquad \mathrm{Q}(x,\xi)v=0. \tag{10}$$

By Proposition 12, the sequence of symbols (5) is exact at any real point (x,ξ), $\xi\neq 0$. Therefore, the first equation (10) implies that $v=\mathrm{Q}^*(x,\xi)w$ for some vector $w\in\mathbb{R}^t$. By the second equation (10) we find $0=\langle\mathrm{Q}(x,\xi)v,w\rangle=\langle v,v\rangle$, that is, $v=0$. This contradicts the assumption and completes the proof. □

Corollary 6 *For any $x\in X$ and $r\le 1$ there exist linear continuous operators $\mathrm{R}_r : E^s(U_x(r))\to E^r(U_x(a_x^3 r))$ and $\Sigma_r : E^t(U_x(r))\to E^s(U_x(a_x^3 r))$ such that*

$$(P\mathrm{R}_r + \Sigma_r Q)g = g \tag{11}$$

for any $g\in E^s(U_x(r))$.

Proof Write (4) with two more terms,

$$\cdots\to D^v\xrightarrow{S}D^u\xrightarrow{R}D^t\xrightarrow{Q}D^s\xrightarrow{P}D^r$$

and apply the functor $\mathrm{Hom}_D(\cdot,E)$:

$$E^r\xrightarrow{P}E^s\xrightarrow{Q}E^t\xrightarrow{R}E^u\xrightarrow{S}E^v\to\cdots.$$

Here $P, Q, R, S, \ldots$ are linear operators as in (6). By Theorem 2 applied to these terms, there exist linear continuous operators

$$\rho_r : E_Q\big(U_x(r)\big) \to E^r\big(U_x(a_x r)\big), \qquad \sigma_r : E_R\big(U_x(r)\big) \to E^s\big(U_x(a_x r)\big)$$
$$\tau_r : E_S\big(U_x(r)\big) \to E^t\big(U_x(a_x r)\big)$$

with the properties $P\rho_r f = f$, $Q\sigma_r g = g$, $R\tau_r h = h$ in $U_x(a_x r)$. We have $Q(g - \sigma_r Qg) = 0$ for any $g \in E^s(U_x(r))$. Therefore, we can set $\mathrm{R}_r g = \rho_{a^2 r}(g - \sigma_{ar} Qg)$ and similarly $\Sigma_r h = \sigma_{ar}(h - \tau_r Rh)$. We now have for any $g \in E^s(U_x(r))$,

$$\begin{aligned}(P\mathrm{R}_r + \Sigma_r Q)g &= P\rho_{a^2 r}(g - \sigma_{ar} Qg) + \sigma_{ar}(Qg - \tau_r RQg)\\ &= g - \sigma_{ar} Qg + \sigma_{ar} Qg = g\end{aligned}$$

and (11) follows. □

6 Solutions with Compact Support

Let X be an open set in $\mathbb{R}^n$; the topological dual space $E^*(X)$ to $E(X)$ is identified with the space of distributions in $\mathbb{R}^n$ with compact support contained in X. An arbitrary differential $s \times r$ matrix P in U with analytic coefficients defines a continuous operator $P : E^r(X) \to E^s(X)$ and the adjoint operator $P^* : E^*(X)^s \to E^*(X)^r$ which acts by

$$P^*\phi = \psi,\ \psi(u) = \phi(Pu), \quad \phi \in E^*(X)^s,\ u \in E^s(X).$$

For any complex (4) of left D_X-modules defined in an open set $X \subset \mathbb{R}^n$ and any open set $U \subset X$, the sequence

$$\cdots \to E^*(U)^t \xrightarrow{Q^*} E^*(U)^s \xrightarrow{P^*} E^*(U)^r$$

is a complex of vector spaces.

Theorem 7 *If* (4) *is an elliptic complex in an open set $X \subset \mathbb{R}^n$, then for any point $x \in X$ and any r, $0 < r \le 1$,*

- **C**. *The kernel of $P^* : E^*(U_x(c_x r))^s \to E^*(U_x(c_x r))^r$ is contained in the image of $Q^* : E^*(U_x(r))^t \to E^*(U_x(r))^s$.*
- **D**. *A function $\alpha \in E^*(U_x(c_x r))^r$ is equal to $P^*\beta$ for some $\beta \in E^*(U_x(r))^s$ if and only if $\alpha(u) = 0$ for any $u \in E_P(U_x(c_x r))$, where $c_x = a_x^3$ and a_x is the function as in Theorem 2.*

Proof Dualizing (11), we get

$$\mathrm{R}_r^* P^*\alpha + Q^*\Sigma_r^*\alpha = \alpha$$

for an arbitrary $\alpha \in E^*(U_x(cr))^s$. If $\alpha P = 0$, this equation yields $\alpha = Q\beta$, where $\beta = \Sigma_r^*\alpha \in E^*(U_x(r))^r$. This proves statement **C**.

Check **D**. If $\alpha = P^*\beta$, then $u(\alpha) = Pu(\beta) = 0$. Vice versa, let $u(\alpha) = 0$ for any $u \in E_P(U_x(cr))$. The distribution $\beta = \alpha - P^*\mathrm{R}_r^*\alpha$ fulfils $v(\beta) = w(\alpha)$, where $w = v - \mathrm{R}_r Pv$ for an arbitrary $v \in E^s(U_x(r))$. We have $w \in E^s(U_x(cr))$ and $Pw = (P - P\mathrm{R}_{cr}P)v = 0$ because of (11). Therefore, $w(\alpha) = 0$; hence, $v(\beta) = 0$, which yields $\beta = 0$ and $\alpha = P^*\gamma$, $\gamma = \mathrm{R}_r^*\alpha$. □

7 Resolutions

Now we take a more invariant point of view on systems like (1).

Definition [13] Let $D = D_x$ for some $x \in \mathbb{R}^n$, M be a filtered left D-module, and

$$\operatorname{gr} M = \oplus_{k\in\mathbb{Z}} M_k/M_{k-1}$$

the corresponding graded D-module. We assume that

(i) the D-module $\operatorname{gr} M$ is finitely generated,
(ii) the O-module $\operatorname{gr} M$ is free.

Definition Let M be a left D-module satisfying (i). The product $\operatorname{gr} M \otimes_O \mathbb{C}$ is a module of finite type over the polynomial algebra $\mathrm{A} = \mathrm{D} \otimes \mathbb{C} \cong \mathbb{C}[\xi_1, \dots, \xi_n]$. The *characteristic* set of M is by definition the support of $V = V(M)$ in the support of the A-module $\operatorname{gr} M \otimes_O \mathbb{C}$. The set V is an algebraic cone in the set $\mathbb{C}^n$ of maximal ideals of the algebra A. Any point $\xi \in V$ generates a multiplicative functional μ : $\operatorname{gr} M \otimes_O \mathbb{C} \to \mathbb{C}$ such that $\mu(am) = a(\xi)\mu(m)$ for arbitrary $a \in \mathrm{A}$, $m \in \operatorname{gr} M \otimes_O \mathbb{C}$ (and vice versa).

We call M *elliptic* if the characteristic variety $V(M)$ contains no real point $\xi \neq 0$.

Remark It is easy to check that the condition (**ii**) for $M = \operatorname{Cok} P : D^\sigma \to D^\rho$ is equivalent to (*) for P. The characteristic set V of this module coincides with the set $\{\xi \in \mathbb{C}^n;\ \operatorname{rank} P(x,\xi) < s\}$.

Definition Let $\alpha : E \to F$ be a morphism of filtered D-modules. It is called *strict* if it agrees with the filtrations and $\alpha(E_k) = \alpha(E) \cap F_k,\ k \in \mathbb{Z}$.

Proposition 8 *Let*

$$E \xrightarrow{\alpha} F \xrightarrow{\beta} G \tag{12}$$

be a complex of morphisms of filtered vector spaces. If $\operatorname{Ker}\operatorname{gr}\beta = \operatorname{Im}\operatorname{gr}\alpha$, *the complex* (12) *is exact and* α *is strict.*

Proof Let $\beta(f) = 0$ for an element $f \in F$. We have $f \in F_k$ for some k and $\operatorname{gr}\beta(f) = 0$. By the condition there is an element $e_k \in E_k$ such that $\operatorname{gr}\alpha(e_k) = \operatorname{gr} f$, that is, $f - \alpha(e_k) \in F_{k-1}$. The element $g \doteq f - \alpha(e_0)$ is contained in $\operatorname{Ker}\beta$, we repeat the above arguments with k replaced by $k - 1$ and obtain an element

$e_{k-1} \in E_{k-1}$ such that $f' - \alpha(e_{k-1}) \in F_{k-2}$ and so on. Finally we get $f = \alpha(e)$, where $\alpha = \alpha_k + \alpha_{k-1} + \cdots \in E_k$. □

Proposition 9 *If the complex* (12) *is exact*, α *and* β *are strict*, *then* $\operatorname{Ker} \operatorname{gr} \beta = \operatorname{Im} \operatorname{gr} \alpha$.

Proof We have $\operatorname{gr} \beta \operatorname{gr} \alpha = 0$. Show that $\operatorname{Ker} \operatorname{gr} \beta \subset \operatorname{Im} \operatorname{gr} \alpha$. Take an element $\mathrm{f} \in \mathrm{F}_k \doteq F_k / F_{k-1}$ such that $\operatorname{gr} \beta(\mathrm{f}) = 0$. Let $f \in F_k$ be an element of the class f. We have $\beta(f) \in G_{k-1}$ and $\beta(f) \in \beta(F_{k-1})$ since β is strict, that is $\beta(f - g) = 0$ for an element $g \in F_{k-1}$. We have $f - g = \alpha(e)$, $e \in E$, since (12) is exact and $\alpha(e) = \alpha(\tilde{e})$ for some $\tilde{e} \in E_k$, since α is strict. This yields $\mathrm{f} = \operatorname{gr} \alpha(\mathrm{e})$, where e is the class of $\tilde{e}$. □

Let M be a filtered left D-module and

$$\cdots \to D^{\tau} \stackrel{Q}{\to} D^{\sigma} \stackrel{P}{\to} D^{\rho} \stackrel{\pi}{\to} M \to 0 \tag{13}$$

be a strict exact sequence of filtered left D-modules. The complex of D-modules

$$\cdots \to \mathrm{D}^{\tau} \stackrel{\mathrm{Q}}{\to} \mathrm{D}^{\sigma} \stackrel{\mathrm{P}}{\to} \mathrm{D}^{\rho} \stackrel{\pi}{\to} \operatorname{gr} M \to 0 \tag{14}$$

is then well defined where all morphisms have degree 0. We call (14) the *principal part* of (13).

Proposition 10 *If a left D-module M fulfils* (***i***), *then for any point $x \in X$ there exist a neighborhood U of x and a resolution of the graded module* $\operatorname{gr} M$.

Proof The product $\operatorname{gr} M \otimes_{O} \mathbb{C}$ is a module over the polynomial algebra $\mathrm{A} \doteq \mathrm{D} \otimes \mathbb{C}$. Construct a strict resolution of this module of the form

$$\cdots \to \mathrm{D}^{\tau} \otimes \mathbb{C} \stackrel{\mathrm{Q_A}}{\to} \mathrm{D}^{\sigma} \otimes \mathbb{C} \stackrel{\mathrm{P_A}}{\to} \mathrm{D}^{\rho} \otimes \mathbb{C} \stackrel{\pi_{\mathrm{A}}}{\to} \operatorname{gr} M \otimes \mathbb{C} \to 0. \tag{15}$$

By (**i**) there exists a surjective morphism $\pi : \mathrm{D}^{r_0} \to \operatorname{gr} M$. We choose a shift vector $\rho = (\rho_1, \ldots, \rho_{r_0})$, where $\rho_i \doteq \operatorname{ord} \pi_{\mathrm{A}}(e_i)$, $i = 1, \ldots, r_0$ for the standard generators $e_1, \ldots, e_{r_0}$ of the module D^{r_0} and introduce the filtration D^{ρ} in this module. The morphism $\pi : \mathrm{D}^{\rho} \to \operatorname{gr} M$ has degree 0 and generates A-morphism $\pi_{\mathrm{A}} : \mathrm{A}^{\rho} \to \operatorname{gr} M \otimes \mathbb{C}$. Because the algebra A is Noetherian, the submodule $\operatorname{Ker} \pi_{\mathrm{A}}$ is generated by some homogeneous elements $\mathrm{p}_1, \ldots, \mathrm{p}_{r_1}$. Let $\mathrm{P_A} : \mathrm{A}^{r_1} \to \mathrm{A}^{\rho}$ be the morphism such that $\mathrm{P_A}(e'_i) = \mathrm{p}_i$, $i = 1, \ldots, r_1$ for the standard generators $e'_1, \ldots, e'_{r_1}$ of A^{r_1}. Set $\sigma = (\sigma_1, \ldots, \sigma_{r_1})$, where $\sigma_i \doteq \operatorname{ord}_{\rho} \mathrm{p}_i$, and introduce the corresponding filtration D^{σ} in D^{r_1}. The morphism $\mathrm{P_A}$ is homogeneous of degree 0 and $\operatorname{Im} \mathrm{P_A} = \operatorname{Ker} \pi_{\mathrm{A}}$. We can apply the same arguments to $\operatorname{Ker} \mathrm{P_A}$ and choose morphism $\mathrm{Q_A}$, and so on.

By (**ii**) all the n morphisms $\mathrm{P_A}, \mathrm{Q_A}, \ldots$ have extensions to some D-morphisms $\mathrm{P}, \mathrm{Q}, \ldots$ such that the sequence (14) is a complex and $\mathrm{P} \otimes \mathbb{C} = \mathrm{P_A}, \mathrm{Q} \otimes \mathbb{C} = \mathrm{Q_A}, \ldots$. It can be shown by standard homological arguments since the O-modules $\operatorname{gr} M, \mathrm{D}^{\rho}, \mathrm{D}^{\sigma}, \ldots$ are flat. By Nakayama's lemma, exactness of (15) implies exactness of (14). □

Proposition 11 *For any free graded resolution* (14) *there exists a free strict resolution* (13) *of M such that* (14) *is the principal part of* (13).

Proof For any $i = 1, \ldots, r_0$, choose an element $m_i \in M_{\rho_i}$ whose image in $M_{\rho_i}/M_{\rho_i - 1}$ is equal to $\pi(e_i)$, and define a D-morphism $\pi : D^{r_0} \to M$ such that $\pi(e_i) = m_i$, $i = 1, \ldots, r_0$. This morphism agrees with the ρ-filtration in D^{r_0} and filtration in M; it is surjective, because so is π. Next we lift P to a D-morphism $P_0 : D^\sigma \to D^\rho$. For any standard generator e'_k of D^σ, the row $\mathrm{p}_k \doteq \mathrm{P}(e'_k) \in \mathrm{D}^\rho$ satisfies $\pi \mathrm{p}_k = 0$, which means $\pi \mathrm{p}_k \in M_{\rho_k - 1}$, $k = 1, \ldots, r_1$. Because of the exactness of (14), there exists an element $q_k \in D^{r_0}$ such that $\operatorname{ord} q_k = \rho_k - 1$ and $\pi(q_k) = \pi \mathrm{p}_k$. We have $\pi(\mathrm{p}_k - q_k) \in M_{\rho_k - 2}$ and so on up to filtration -1. Finally, we collect the lines $\mathrm{p}_k - q_k - q'_k - \cdots$, $k = 1, \ldots, r_1$ in a matrix P of size $r_0 \times r_1$ and have $\pi P = 0$. The principal part of the line $P(e'_k)$ is equal to p_k, that is, the principal part of P is P. By Proposition 8, P is strict and $\operatorname{Im} P = \operatorname{Ker} \pi$.

The image of the composition $P\mathrm{Q} : D^{r_2} \to D^{r_0}$ is contained in $\operatorname{Ker} \pi$ and $\operatorname{ord}_\rho P\mathrm{Q}(e'') < \operatorname{ord}_\tau e''$ for each standard generator e'' of D^{r_2}. Because $\operatorname{Ker} \pi = \operatorname{Im} \mathrm{P}$, there exists an element $q_1 \in D^{r_1}$ such that $\operatorname{ord}_\sigma q_1 = \operatorname{ord}_\rho P\mathrm{Q}(e'')$ and $P\mathrm{Q}(e'') = \mathrm{P} q_1$ up to a term of filtration $< \operatorname{ord}_\sigma q_1$. We make a matrix $Q_1 : D^{r_2} \to D^{r_1}$ from the lines $Q_1(e'') = q_1$, where e'' runs over the set of generators of D^{r_2}. Consider the composition $P(\mathrm{Q} - Q_1) : D^{r_2} \to D^{r_0}$. We now have $\operatorname{ord}_\rho P(\mathrm{Q} - Q_1)(e'') < \operatorname{ord}_\rho P\mathrm{Q}(e'')$ and can find an element $q_2 \in D^{r_1}$ such that $\operatorname{ord}_\sigma q_2 = \operatorname{ord}_\rho P(\mathrm{Q} - Q_1)(e'')$ up to a term of filtration $< \operatorname{ord} q_2$. Define a matrix Q_2 by $Q_2(e'') = q_2$ for the set of standard generators e'', then consider the matrix $\mathrm{Q} - Q_1 - Q_2$, and so on. This series is finite since $\cdots < \operatorname{ord}_\sigma q_2 < \operatorname{ord}_\sigma q_1 < \operatorname{ord}_\tau e''$. We set $Q = \mathrm{Q} - Q_1 - Q_2 - \cdots$. By Proposition 8, P_1 is strict and $\operatorname{Im} Q = \operatorname{Ker} P$. We construct a matrix R such that $\operatorname{Im} R = \operatorname{Ker} Q$ in a similar way, and so on. □

Proposition 12 *If M is an elliptic module, then any strict resolution* (13) *of M is elliptic.*

Proof Consider $\mathrm{A} \doteq \mathrm{D} \otimes \mathbb{C}$ as a non-graded algebra; it is a polynomial with the spectrum $\mathbb{C}^n$. Take an arbitrary real point $\xi \neq 0$ of the spectrum; let $\mathfrak{m}$ be the corresponding maximal ideal in A. All the terms of (15) except for the right one are free over A and

$$\operatorname{Tor}^*(\operatorname{gr} M \otimes \mathbb{C}, \mathrm{A}/\mathfrak{m}) = 0,$$

since $(\operatorname{gr} M \otimes \mathbb{C}) \otimes_{\mathrm{A}} \mathrm{A}/\mathfrak{m} = 0$ because ξ does not belong to the characteristic set of $\operatorname{gr} M \otimes \mathbb{C}$. Therefore, tensoring (15) by A-module $\mathrm{A}/\mathfrak{m}$, we get the exact sequence

$$\cdots \to \mathrm{D}^\tau \otimes \mathbb{C} \otimes_{\mathrm{A}} \mathrm{A}/\mathfrak{m} \overset{\mathrm{Q}(x,\xi)}{\to} \mathrm{D}^\sigma \otimes \mathbb{C} \otimes_{\mathrm{A}} \mathrm{A}/\mathfrak{m} \overset{\mathrm{P}(x,\xi)}{\to} \mathrm{D}^r \otimes \mathbb{C} \otimes_{\mathrm{A}} \mathrm{A}/\mathfrak{m} \to 0,$$

which proves ellipticity of (13). □

8 Key Lemma

Let M be a filtered D-module. The set $\operatorname{Hom}_D(M, D)$ of D-morphisms $h : M \to D$ has a natural structure of two-side D-module since D has such a structure. It possesses the dual filtration ρ^* such that $\mathrm{ord}_{\rho^*}(h) = k$ if $h(M_i) \subset D_{i+k}$ for any i. In particular, $\mathrm{Hom}(D^\rho, D) \cong D^{-\rho}$, where $D^{-\rho}$ is a free D-module of the same rank as D^ρ with the shift vector $-\rho$. Any morphism of left D-modules $P : D^\sigma \to D^\rho$ generates the *dual* morphism

$$P' \doteq \mathrm{Hom}(P, D) : D^{-\rho} \to D^{-\sigma},\ h \mapsto Ph,$$

where we interpret an element $h \in D^{-\rho}$ as a column. The map P' is a morphism of *right* D-modules.

Fix a point $x \in \mathbb{R}^n$. Let

$$R : \cdots \to D^{\rho_2} \overset{P_1}{\to} D^{\rho_1} \overset{P_0}{\to} D^{\rho_0} \to 0 \tag{16}$$

be a strict resolution of a left D-module M, where $\rho_0, \rho_1, \rho_2, \ldots$ are some shift vectors. The complex $\mathrm{Hom}_D(R, D)$ looks like

$$0 \to D^{-\rho_0} \overset{P_0'}{\to} D^{-\rho_1} \overset{P_1'}{\to} D^{-\rho_2} \to \cdots \to D^{-\rho_{k-1}} \overset{P_{k-1}'}{\to} D^{-\rho_k} \to \cdots, \tag{17}$$

where $D^{-\rho_i}$ is a free right D-module of the same rank r_i as D^{ρ_i} and all morphisms agree with the filtrations and P' means left multiplication of a column by a matrix P. It is a complex of right D-modules.

Lemma 13 *If a left D-module M satisfies* (***i***, ***ii***), *then the sequence* (17) *is exact at the terms $D^{-\rho_k}$ with $k = 0, \ldots, m-1$, where $m = n - \dim_{\mathbb{C}} V(M)$.*

Proof The principal part of (17) is the complex of modules over the graded commutative algebra D:

$$0 \to \mathrm{D}^{-\rho_0} \overset{\mathrm{P}_0'}{\to} \mathrm{D}^{-\rho_1} \to \cdots \to \mathrm{D}^{-\rho_{k-1}} \overset{\mathrm{P}_{k-1}'}{\to} \mathrm{D}^{-\rho_k} \to \cdots,$$

which is equal to $\mathrm{Hom}_{\mathrm{D}}(\mathrm{R}, \mathrm{D})$, where R is the principal part of (16). Let Π be the trunk of this complex up to the morphism P_{m-1}'. We are going to show that Π is exact. By Proposition 9, R is a resolution of $\mathrm{gr}\, M$. By condition (**ii**), the complex $\mathrm{R} \otimes \mathbb{C}$ is a resolution of $\mathrm{gr}\, M \otimes \mathbb{C}$ over the polynomial algebra $\mathrm{A} \doteq \mathrm{D} \otimes \mathbb{C}$, where $\otimes = \otimes_O$. This yields

$$H^k(\Pi \otimes \mathbb{C}) = H^k\big(\mathrm{Hom}_{\mathrm{A}}(\mathrm{R} \otimes \mathbb{C}, \mathrm{A})\big) \cong \mathrm{Ext}^k(\mathrm{gr}\, M \otimes \mathbb{C}, \mathrm{A}), \quad k < m. \tag{18}$$

The right-hand side of (18) vanishes for $k < m$, by virtue of [15, Corollary 1, §13], which means that the complex $\Pi \otimes \mathbb{C}$ is acyclic. By definition, for any shift vector ω we have $\mathrm{D}^\omega = \oplus_i \mathrm{D}_i^\omega$, where D_i^ω is an O-module of homogeneous elements of grading i and $\mathrm{D}^\omega \otimes \mathbb{C} = \oplus_i \mathrm{D}_i^\omega \otimes \mathbb{C}$, where $\mathrm{D}_i^\omega \otimes \mathbb{C}$ is a finite dimensional space. Therefore,

$$\Pi = \oplus \Pi_i, \qquad \Pi \otimes \mathbb{C} = \oplus \Pi_i \otimes \mathbb{C},$$

where Π_i is a complex of free O-modules of finite type. As we know, the complex $\Pi_i \otimes \mathbb{C}$ is acyclic. By Nakayama's lemma, the complex Π_i is also acyclic and the same true for the complex Π. By Proposition 8 the complex (17) is acyclic in degrees $k < m$ and the morphisms $P'_0, \ldots, P'_{m-1}$ are strict. □

Corollary 14 *The complex* (17) *is elliptic in any degree* $k < m$.

Proof Take an arbitrary real point $\xi \neq 0$ and check that the complex $\Pi \otimes \mathrm{A}/\mathfrak{m}$ is acyclic, where $\mathfrak{m}$ is the corresponding maximal ideal of the algebra A. We have $H^*(\Pi \otimes \mathrm{A}/\mathfrak{m}) \cong H^*(\Pi) \otimes \mathrm{A}/\mathfrak{m}$, since A-module Π is flat. This implies the first statement, since $H^*(\Pi) = 0$ by Lemma 13. □

9 Extension of Solutions of Overdetermined Systems

Definition Let M be a left D-module with good filtration that fulfills the condition (**i**). The characteristic set $V = V(\mathrm{gr}M)$ is an algebraic cone in $\mathbb{C}^n$. We say that M is *underdetermined* if $V = \mathbb{C}$, *determined* if $\dim_{\mathbb{C}} V < n$, and *overdetermined* if $\dim_{\mathbb{C}} V < n - 1$.

Now let $\mathcal{D} = \mathcal{D}_X$ be the sheaf of germs of analytic differential operators in an open set X and $\mathcal{M}$ be a filtered left $\mathcal{D}$-module. We say that $\mathcal{M}$ fulfills the conditions (**i**, **ii**) is called elliptic, overdetermined, etc., if so are $\mathcal{M}_x$ in any point $x \in X$. Suppose that $\mathcal{M}$ can be included in a strict exact sequence of filtered left $\mathcal{D}$-modules

$$\mathcal{D}^\sigma \xrightarrow{P} \mathcal{D}^\rho \xrightarrow{\pi} \mathcal{M} \to 0, \tag{19}$$

where $\mathcal{D}^\sigma, \mathcal{D}^\rho$ denote some filtrations in free left $\mathcal{D}$-modules defined as in Section 3 and the filtration in $\mathcal{M}$ is the image of the filtration in $\mathcal{D}^\rho : \mathcal{M}_k = \pi(\mathcal{D}^\rho_k)$, $k \in Z$. Here P is a matrix differential operator as in (1) with analytic coefficients defined in an open set $X \subset \mathbb{R}^n$. It acts as a morphism of left $\mathcal{D}$-modules: $a \mapsto aP$.

Proposition 15 *For any compact set $K \subset X$, the sequence* (19) *can be extended to a strict exact complex of $\mathcal{D}$-sheaves*

$$\cdots \to \mathcal{D}^\tau \xrightarrow{Q} \mathcal{D}^\sigma \xrightarrow{P} \mathcal{D}^\rho \xrightarrow{\pi} \mathcal{M} \to 0 \tag{20}$$

defined in a neighborhood of K where $\ldots, \mathcal{D}^\tau$ are filtered free $\mathcal{D}$-sheaves of the same type.

Proof Let D_X be the sheaf in X whose stalks are the algebras D and D^ω_X be the graded D_X-sheaf where ω is an arbitrary shift vector. Consider the sequence of graded D_X-modules

$$\mathrm{D}^\sigma_X \xrightarrow{\mathrm{P}} \mathrm{D}^\rho_X \xrightarrow{\pi} \mathrm{gr}\,\mathcal{M} \to 0$$

generated by (19). It is exact since of Proposition 9. For $k = 0, 1, 2, \ldots$ we consider $\mathcal{O}$-sheaf $(\mathrm{Ker}\,\mathrm{P})_k : (\mathrm{D}^\sigma_X)_k \to (\mathrm{D}^\rho_X)_k$. It is a coherent analytic sheaf in the real domain X. Let L be a compact set in X such that $K \Subset L$. By the classical theory of coherent sheaves the sheaf $(\mathrm{Ker\,P})_k$ is generated in each point $x \in L$ by a finite set S_k of its sections. The total set $\mathrm{S} \doteq \cup_k \mathrm{S}_k$ generates D_x-sheaf $\mathrm{Ker\,P}_x : \mathrm{D}_x^\sigma \to \mathrm{D}_x^\rho$ at each point $x \in L$. On the other hand, for any point x there is a finite subset $\mathrm{q}_x \subset \mathrm{S}$ that generates the stalk $(\mathrm{Ker\,P})_x$ since the algebra D_x is Noetherian. Obviously the set q_x generates the sheaf $\mathrm{Ker\,P}$ also in a neighborhood of x. Therefore there is a finite set $F \subset L$ such that the union $\mathrm{q}_L = \cup\{\mathrm{q}_x, x \in F\}$ generates the D-sheaf $\mathrm{Ker\,P}$ at each point $x \in L$. Let D_X^t be a free D-sheaf with generators $e_1, \dots, e_t$. Consider a D_X-morphism $\mathrm{Q} : \mathrm{D}_X^t \to \mathrm{D}_X^\sigma$ such that $\mathrm{q}_j = \mathrm{Q}(e_j),\ \ j = 1, \dots, t$ are all elements of the set q_L. Define a filtration D_X^τ in D_X^t by means of a shift vector $\tau = (\tau_1, \dots, \tau_t)$, where $\tau_j = \deg \mathrm{q}_j$, $j = 1, \dots, t$. The morphism $\mathrm{Q} : \mathrm{D}^\tau \to \mathrm{D}^\rho$ agrees with the filtrations and $\mathrm{Im\,Q} = \mathrm{Ker\,P}$. Next we consider the restriction Q_L of Q to L and repeat these arguments for the D-sheaf $\mathrm{Ker\,Q}_L$ and so on. We obtain in this way an exact sequence of D_Y-sheaves

$$\dots \xrightarrow{\mathrm{R}_x} \mathrm{D}_Y^\tau \xrightarrow{\mathrm{Q}_x} \mathrm{D}_Y^\sigma \xrightarrow{\mathrm{P}} \mathrm{D}_Y^\rho \xrightarrow{\pi} \mathrm{gr}\mathcal{M}_Y \to 0$$

defined in a neighborhood Y of K. Then we construct a strict exact sequence (20) by means of arguments of Propositions 10. $\square$

Let $\mathcal{M}$ be a filtered left $\mathcal{D}$-module as in (20) such that the stalk $\mathcal{M}_x$ fulfils the conditions (**i**, **ii**). We set

$$\dim_X \mathcal{M} \doteq \max_X \dim_{\mathbb{C}} V(\mathrm{gr}\,\mathcal{M}_x \otimes \mathbb{C}).$$

Note that the function $x \mapsto \dim_{\mathbb{C}} V(\mathrm{gr}\,\mathcal{M}_x \otimes \mathbb{C})$ is locally constant in X. This follows from (**ii**).

Theorem 16 *Let $\mathcal{M}$ be a left $\mathcal{D}$-module as in* (20) *that is elliptic and overdetermined* (*which implies* $\dim_X \mathcal{M} \leq n-2$). *Let Y be a relatively compact subset of X and S be a closed C^1-submanifold of Y of dimension $d = n - 2 - \dim_X \mathcal{M}$. There exists an open neighborhood V of S such that any solution u of* (1) *in $Y \backslash \bar{V}$ has a unique extension in X as a solution.*

Example 1 Let $d = 0$, then the statement tells that for any point $x \in X$ there exists a compact neighborhood $K_x \subset X$ such that arbitrary solution defined in $X \setminus K$ is uniquely extended to a solution in X. If the module $\mathcal{M}$ is not overdetermined, then solutions of (1) in $X \setminus K$ may have non-removable singularity in K, as e.g. a fundamental solution of a scalar operator of P.

Proof of Theorem Introduce an Euclidean structure in $\mathbb{R}^n$.

Lemma 17 *There exist positive constants $b \leq c < 1$ that depends only on K such that for an arbitrary subspace Z in $\mathbb{R}^n$ of dimension d and arbitrary open balls $Y(r)$ and $Z(s)$ of radius r in $Y = Z^\perp$, respectively in Z of radii $r, s \leq 1,\ cr \leq s$ such that $Y(r) \times Z(s) \subset K$ an arbitrary solution u of $\mathcal{M}$ defined in the set $Y(r) \backslash Y(br) \times Z(s)$ has a unique extension to $Y(r) \times Z(s)$ as a solution of M.*

Here and later we denote by $Y(r')$ the ball with the same center as $Y(r)$; notation $Z(s')$ has a similar meaning. To prove the Theorem we take for an arbitrary point $x_0 \in S$ the tangent subspace Z to S at x_0 and set $Y = Z^{\perp}$. In the case $d = 0$ we take $Z = 0, Y = \mathbb{R}^n$. Choose a positive number r such that $Y(r)\backslash Y(br) \times Z(cr) \subset X\backslash S$. This choice is possible since S is contained in $o(r)$-neighborhood of Z. By Lemma 17 any solution u can be extended to the set $Y(r) \times Z(cr)$. This set contains a neighborhood of x_0. We take for V the union of these neighborhoods for all $x_0 \in S$ and complete the proof of Theorem. □

Proof of Lemma 17 Choose some positive numbers $r_0, s_0 \le 1$ such that $Y(r_0) \times Z(s_0) \Subset X$; we may assume that $r_0 = 1, s_0 = c$ by coordinate rescaling. Set $b = c^{d+1}$, $c = \inf_U c_x/4$, where c_x is the function as in Lemma 14. Choose a coordinate system (y, z) in $Y \times Z$ such that the centers of $Y(r)$ and $Z(r)$ are in the origins.

Take a smooth function e in Y with support in $Y(2b)$ such that $e = 1$ in $Y(b + \varepsilon)$ and set $v_0(x) = P_0(e(y)u(x))$, the function v_0 is extended by zero to $Y(b) \times Z(c)$. Take a convex polytope $\Pi \subset Z(c)\backslash Z(c/2)$; let $F_\alpha, \alpha \in N$ be its faces. Let N_k be the subset of N of faces F_α of dimension $k = 0, 1, \dots, \dim Z$; the face Π is the only one of dimension $d \doteq \dim Z$. The notation α_k always will mean that $\alpha_k \in N_k$. We suppose that each face F_{α_k} of dimension $k < d$ is a simplex and the inequality holds

$$2b \le \operatorname{diam} F_{\alpha_1} \le 3b \tag{21}$$

for each 1-face. We call *k-flag* any sequence $A = (\alpha_k, \alpha_{k+1}, \dots, \alpha_{d-1})$ such that $F_{\alpha_k} \subset F_{\alpha_{k+1}} \subset \cdots \subset F_{\alpha_{d-1}}$. For a set $G \subset Z$ and a positive ε we denote by $(G)_\varepsilon$ the open ε-neighborhood of G.

Take a smooth function f_0 in Z with compact support in $(\Pi)_b$ such that $f_0 = 1$ in Π. For an arbitrary $k < d$ and $\alpha_k \in N_k$ we choose a smooth function f_{α_k} that fulfils

I. $\operatorname{supp} f_{\alpha_k} \subset (F_{\alpha_k})_{b/c}$ and
II. $\sum_{\alpha_k} f_{\alpha_k} = 1$ in $(\cup_{\alpha_k} F_{\alpha_k})_b$.

Take an arbitrary k-flag $A = (\alpha_k, \alpha_{k+1}, \dots)$ and define the function

$$v_A = P_{d-k+1}\big(f_{\alpha_k} \cdots P_2\big(f_{\alpha_{d-1}} P_1(f_0 v_0)\big) \cdots\big), \tag{22}$$

where $P_1, \dots, P_{d+1}$ are differential operators as in (17) (strokes are omitted).

Lemma 18 ***III.*** *The function v_A is supported by $(F_{\alpha_k})_b$.*
IV. *For any $k+1$-flag B we have*

$$\sum_{\alpha_k} v_{\alpha_k, B} = 0,$$

where the sum is taken for all k-flags that contain B.

Proof of Lemma Statement **III** follows from **I** and equation **IV** follows from **II**:

$$\sum_{\alpha_k} v_{\alpha_k, B} = P_{d-k+1} \sum_{\alpha_k} f_{\alpha_k} v_B = P_{d-k+1} v_B = 0.$$

□

For any 1-flag $A=(\alpha_1,\dots)$ we have $v_A = v_{\alpha_0,A}+v_{\beta_0,A}$, where $\alpha_0,\beta_0\in N_0$ are the vertices of the face F_{α_1} hence (α_0, A) and (β_0, A) are 0-flags. By **III** we have $\operatorname{supp} v_{\alpha_0,A}\Subset (F_{\alpha_0})_b$ and similarly for $v_{\beta_0,A}$. the left inequality (21) implies that the supports of the distributions $v_{\alpha_0,A}$ and $v_{\beta_0,A}$ are disjoint. The formula (22) yields $P_{d+1}v_{\alpha_0,A}=P_{d+1}v_{\beta_0,A}=0$ hence by Lemma 14 there exist solutions to the equations

$$v_{\alpha_0,A}=P_d w_{\alpha_0,A},\qquad v_{\beta_0,A}=P_d w_{\beta_0,A} \tag{23}$$

with compact supports $\operatorname{supp} w_{\alpha_0,A}\Subset (F_{\alpha_0})_{b/2c}$, $\operatorname{supp} w_{\beta_0,A}\Subset (F_{\beta_0})_{b/2c}$. Set $w_A = w_{\alpha_0,A}+w_{\beta_0,A}$ and have $P_d w_A = v_A$. By (21) for any α_0, $\operatorname{supp} w_{\alpha_0,A}\Subset (F_{\alpha_1})_{b/2c}\subset (F)_{3b+b/2c}\subset (F)_{b/c}$ since $3b+b/2c\le b/c$. By **IV** we have

$$\sum_{\alpha_0,\alpha_1} v_{\alpha_0,\alpha_1,B}=\sum_{\alpha_1} v_{\alpha_1,B}=0,$$

where the sum is taken over all flags that contain the 2-flag $B=(\alpha_2,\alpha_3,\dots)$. Therefore we can assume that also

$$\sum_{\alpha_0,\alpha_1} w_{\alpha_0,\alpha_1,B}=\sum_{\alpha_1} w_{\alpha_1,B}=0. \tag{24}$$

Define $v'_A = v_A-\sum w_{\beta,A}$ for any 1-flag A. By (21) we have $\operatorname{supp} v'_A\Subset (F_{\alpha_0})_{b/c}$ for any 1-flag A and an arbitrary vertex F_{α_0} of the face F_{α_1}. Due to (23) we have $P_d v'_A=0$, hence by Lemma 14 there exists a solution w_A to $P_{d-1}w_A=v'_A$ with compact support in $(F_{\alpha_0})_{b/2c^2}$. Set for any 2-flag B

$$v'_B = v_B-\sum_{\alpha_1} w_{\alpha_1,B},$$

where the sum is taken over all 1-flags that contains the flag B. We have $\operatorname{supp} v'_B\Subset (F_{\alpha_0})_{b/c^2}$. By (24) and **II** we have

$$P_{d-1}v'_B = P_{d-1}\sum_{\alpha_1}(f_{\alpha_1}v_B - w_{\alpha_1,B}) = \sum_{B\subset A}(v_A - P_{d-1}w_A)=\sum_{\alpha_1} w_{\alpha_1,B}=0.$$

By Lemma 14 we can solve the equation $P_{d-2}w_B = v'_B$ for a function w'_B with compact support in $(F_{\alpha_0})_{b/2c^3}$ under the condition

$$\sum_{\alpha_2} w_{a_2,C}=0$$

for any 3-flag C. Set

$$v'_C = v_C - \sum w_{\alpha_2,C}$$

and have $P_{d-2}v'_C=0$ for any 3-flag C. Continuing arguing in this way $d-1$ time, we get the function

$$v'_0 = v_0-\sum_{\alpha_{d-1}} w_{\alpha_{d-1}},$$

where $\operatorname{supp} w_{\alpha_{d-1}} \Subset (F_{\alpha_0})_{b/c^{d-1}}$ and $P_1 v_0' = 0$. We have $\operatorname{supp} v_0' \Subset Y(2c) \times Z(c)$ and $v_0' = v_0$ in $Y \times Z(c)$. We apply again Lemma 14 and find a solution to the equation $P_0 w_0 = v_0'$ with compact support in $Y(1/2) \times Z(c/2)$. We have $P_0 w_0 = f_0 P_0(eu)$ in $Y(1/2) \times Z(c/2)$. Because $f_0 = 1$ in $Y \times \Pi$ we have $P_0(eu - w_0) = 0$, hence $P_0((1-e)u + w_0) = 0$ in $Y \times \Pi$. The function $U \doteq (1-e)u + w_0$ fulfils the equation $P_0 U = 0$ and coincides with u in $Y(1)\backslash Y(1/2) \times Z(c/2)$. By uniqueness of analytic continuation we have $U = u$ in $Y(1)\backslash Y(c) \times Z(c)$. □

Example 2 The statement of Theorem 16 does not hold in general for $\dim S = \dim_X \mathcal{M} + 1$. Let $\mathbb{R}^n = Y \oplus Z$, where Z is spanned by the coordinates $x_1, \dots, x_d$. Consider the D-module $M = D/(p_0, p_1, \dots, p_d)$, where

$$p_0 = p_0(\partial_{x_{d+1}}, \dots, \partial_{x_n}), \quad p_i = \partial_{x_i}, \ i = 1, \dots, d,$$

where p_0 is an elliptic operator with constant coefficients in Y. It is an elliptic module and $V(\operatorname{gr} \mathcal{M}_x) = \{(x, \xi); \xi_1 = \cdots = \xi_d = \mathrm{p}_0(\xi_{d+1}, \dots, \xi_n) = 0\}$ for any $x \in \mathbb{R}^n$. Dimension of this characteristic manifold is equal to $n - d - 1$ however there is no compulsory extension for solutions of M from $\mathbb{R}^n \backslash Z$ on $\mathbb{R}^n$ since any fundamental solution E of p_0 considered as a function in $\mathbb{R}^n$ has singularity in Z.

10 Moments Condition for Extension

If $\mathcal{M}$ is not a overdetermined module, then a solution u of M in a domain $U\backslash K$ may have non removable singularity on K (Example 1). A necessary condition for a solution u to have an extension to U as a solution is vanishing of some momenta. Fix a smooth density ϕ with support in an open set $V \subset U$ such that $\phi = \mathrm{d}x$ in a neighborhood of K, take an arbitrary solution v of $P^* v = 0$ defined in a neighborhood V of K such that $\operatorname{supp} \nabla\phi \Subset V\backslash K$ and consider the integral

$$\int_{U\backslash K} u P^*(\phi v). \tag{25}$$

Note that if u has extension to U as solution of M, we can integrate in (25) by parts and get the equation $\int \phi v P(u) = 0$. We state an inverse implication:

Theorem 19 *Let $\mathcal{M}$ be an elliptic $\mathcal{D}$-module in X and c_x be the function in X as in Theorem 7. Let $x \in X$, $0 < r \le c_x$, $U_x = U_x(r)$, $V_x = U_x(c_x r)$ and $K \subset V_x$ be a compact set without holes. Then an arbitrary solution of $Pu = 0$ defined in $U_x\backslash K$ has a unique extension to U_x as a solution provided the integral (25) vanishes for any smooth solution v of $P^* v = 0$ in V_x.*

Proof We may assume that $\operatorname{supp} \phi \subset V_x$ and $\phi(x) = \mathrm{d}x$ in a neighborhood W of K. Take a smooth function e in $\mathbb{R}^n$ supported in W that is equal to 1 in a neighborhood W_0 of K. The function $P(eu)$ is supported in W and vanishes in W_0. Set

$\alpha = P(eu)\mathrm{d}x$ in $V_x \backslash K$ and $\alpha = 0$ in K. We have $\alpha \in E^*(V_x)^s$ and for any solution w of the equation $P'w = 0$ in V_x

$$\alpha(w) = \int_{V_x \backslash K} P(eu) w \mathrm{d}x = \int eu P^*(w \mathrm{d}x) = \int u P^*(\phi w)$$

since the distribution $P^*(\phi w)$ is supported in $V_x \backslash W$. By the assumption the right-hand side vanishes for any w. By Theorem 7 **D** there exists a distribution $\beta \in E^*(U_x)^r$ such that $P\beta = \alpha$, that is $Pu' = 0$ in U_x, where $u' \doteq eu - \beta$. The functions u and u' coincide in $U_x \backslash \operatorname{supp} e \cup \operatorname{supp} \beta$ and are analytic in $U_x \backslash K$, hence $u' = u$ in K since K has no holes. □

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