

Inverse scattering as nonlinear tomography

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Abstract. The reconstruction problem for the acoustic tomography with non-homogeneous background is discussed. We develop a method of beam-like solutions of a Helmholtz equation which is not sensitive to caustics. For small perturbations of the speed the inverse scattering problem is reduced to inversion of a ray integral transform. An explicit formula is given for beam-like solutions in a homogeneous medium.

Keywords: Helmholtz equation, geodesic, plane wave solution, Debye's method, beam-like solution.

1 Introduction

The method of acoustical tomography is based on reconstruction of an unknown refraction coefficient \mathbf{n} (inverse velocity) from data of scattered solutions of the Helmholtz equation

$$(\Delta + (2\pi\mathbf{k})^2) u(x) = 0, \quad \mathbf{k} = \omega\mathbf{n} \quad (1)$$

in a domain X of an Euclidean space \mathbb{R}^n , $n = 2, 3$ for a given sufficiently large time frequency ω . It is known that a stable reconstruction of a perturbation $\delta(\mathbf{n})$ is possible up to details of the size $\lambda/2$, ($\lambda = 1/\mathbf{k}$ is the local wavelength), see [2],[5]. We outline here a simplified method of reconstruction which works when $\delta(\mathbf{n}) = O(\lambda)$.

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2 Debye's method for plane wave solutions

Consider the wave equation in the space-time $X \times \mathbb{R}$

$$\left(\Delta - \mathbf{n}^2 \frac{\partial^2}{\partial t^2} \right) U = 0, \quad (2)$$

where $\mathbf{c} = \mathbf{n}^{-1}$ is the speed of wave propagation. We suppose that $\mathbf{n} \in C^2(X)$. Let θ be a unit vector in X . If the function \mathbf{n} is constant in a strip $P_\theta(\varepsilon) = \{x \in X; |\langle \theta, x \rangle| < \varepsilon\}$ for some $\varepsilon > 0$, there exists a solution $U_\theta(x, t)$ of (2) that is equal to the plane wave $\delta(t - \mathbf{n} \langle \theta, x \rangle)$ in the time strip $|ct| < \varepsilon$, where δ means the delta-function. For larger $|t|$ the solution has more complicated structure, but for any point $x \in X$ it vanishes for $t < t_x$ for some time value t_x since of finite speed. The inverse Fourier transform

$$u_\theta(x, \omega) = \int_{\mathbb{R}} \exp(2\pi i \omega t) U_\theta(x, t) dt$$

satisfies the Helmholtz equation (1). The solution has the following structure in $P_\theta(\varepsilon)$

$$u_\theta(x, \omega) = \exp(2\pi i \mathbf{k} \langle \theta, x \rangle) + r_\theta(x, \omega), \quad (3)$$

where the main term is the Fourier transform of $\delta(t - \mathbf{n} \langle \theta, x \rangle)$. It is a plane wave solution to the Helmholtz equation with a constant refraction coefficient. The remainder $r_\theta(x, \omega)$ is the diffraction field which is not described in terms of geometrical optics and depends on values of the solution U_θ in the domain $|t| \geq \mathbf{n}\varepsilon$. The solution U_θ is not singular in this domain, hence $r_\theta(x, \omega) = O(\omega^{-N})$ in any compact subset of the strip $P_\theta(\varepsilon)$, where $N \geq 1$ depends on smoothness of the function \mathbf{n} . Let D be a compact domain in X , the function u_θ restricted to the boundary of D is called scattering data of the plane wave (3). The mathematical problem of the acoustic tomography is to recover \mathbf{n} in D from scattering data of a family of plane waves.

Debye's method (ray method), see e.g. [3],[2],[1] gives an extension of the formula (3) to a larger domain in X . This extension is based on solutions of the Hamilton-Jacobi system of ordinary equations

$$\frac{dx}{ds} = -\frac{\xi}{|\xi|}, \quad \frac{dt}{ds} = \mathbf{n}, \quad \frac{d\xi}{ds} = -\nabla \mathbf{n} \tau, \quad \frac{d\tau}{ds} = 0 \quad (4)$$

for the Hamilton function $H(x, t; \xi, \tau) \equiv \mathbf{n}\tau - |\xi|$ with the initial data

$$x(0) = y \in P_\theta, \quad t(0) = 0, \quad \xi(0) = -\theta, \quad \tau(0) = 1, \quad (5)$$

where $P_\theta = \{\langle \theta, x \rangle = 0\}$. The problem (4)-(5) has a C^1 -solution $(x(s, y), \xi(s, y))$ which exists for all values of the independent variables y and s . The curve $\gamma(y)$ given by the equation $x = x(y, s)$, $s \in \mathbb{R}$ (called ray) is for any y a geodesic for the metric $d\sigma^2 = \mathbf{n}^2 dx^2$, where dx^2 is the Euclidean metric.

Because of $|dx/ds| = 1$, the variable s equals the length of the geodesic curve $\gamma(y, x)$ from y to $x(y, s)$ (up to sign). The family of geodesics $x(y, s)$, $y \in P_\theta$ defines a C^1 -map $\pi : P_\theta \times \mathbb{R} \rightarrow X$. The Jacobian function J of the map π is called divergence of this family of geodesics. A zero (s, y) of J is called focal point on the geodesic $\gamma(y)$. We have $J(y, 0) = 1$ for any point $y \in P_\theta$ since $dx/ds|_{s=0} = \theta$ is a unit normal to this hyperplane. Therefore any geodesic $\gamma(y)$ has no focal points in a connected neighborhood of the point $s = 0$. The union of these neighborhoods is an open set U_θ that contains the initial hyperplane $P_\theta \times \{0\}$, the projection $\pi : U_\theta \rightarrow X$ is biunique map onto $X_\theta = \pi(U_\theta)$ and the Jacobian can be considered as a function in X_θ . The function

$$t = \phi(x) = \int_{\gamma(y,x)} \mathbf{n} ds$$

is well defined in X_θ , where $\gamma(y, x)$ is the interval of a geodesic γ between $y \in P_\theta$ and $x \in X_\theta$. It is univalent in X and satisfies the eikonal equation $|\nabla \phi| = \mathbf{n}$. in the domain X_θ . If the coefficient \mathbf{n} is sufficiently smooth in X_θ , the plane wave solution admits the asymptotic representation

$$u_\theta(x, \omega) = \frac{1}{J^{1/2}(x)} \exp(2\pi i \omega \phi(x)) + R_\theta(x, \omega), \quad (6)$$

where the remainder fulfils $R_\theta = O(\omega^{-1})$ on any ray $\gamma(y, x)$, $x \in X_\theta$. Moreover it may be developed in an asymptotic series in powers of ω^{-1} , see details in [1].

3 Perturbed equation

Let $\delta(\mathbf{n})$ be a perturbation of the background refraction coefficient \mathbf{n} . We suppose that $\delta(\mathbf{n})$ is of wavelength magnitude. Let w_θ be the plane wave solution of the perturbed Helmholtz equation

$$\left(\Delta + (2\pi \tilde{\mathbf{k}})^2 \right) w_\theta = 0, \quad \tilde{\mathbf{k}} = \omega \tilde{\mathbf{n}}, \quad \tilde{\mathbf{n}} = \mathbf{n} + \delta(\mathbf{n}). \quad (7)$$

By (6) we get

$$w_\theta(x, \omega) = \frac{1}{J^{1/2}(x)} \exp 2\pi i \omega \left(\phi(x) + \int_{\gamma(y,x)} \delta(\mathbf{n}) ds \right) + \tilde{R}_\theta(x, \omega),$$

where we assume that the integral is equal to $O(\omega^{-1})$ as $\omega \rightarrow \infty$. The perturbations of the ray and of the divergence J are of the rate $O(\omega^{-2})$ and are included in the remainder $\tilde{R}_\theta = O(\omega^{-1})$. Comparing the plane wave solutions of the perturbed and the non-perturbed Helmholtz equations we get

$$\frac{w_\theta(x, \omega)}{u_\theta(x, \omega)} = \exp \left(2\pi i \omega \int_{\gamma(y,x)} \delta(\mathbf{n}) ds \right) + O(\omega^{-1}) \quad (8)$$

for an arbitrary geodesic $\gamma = \gamma(y)$ in X_θ . If the fraction (8) is known, the ray integral of the perturbation $\delta(\mathbf{n})$ can be recovered up to the factor $1 + O(\omega^{-1})$ provided

$$\left| \int_\gamma \delta(\mathbf{n}) ds \right| < \frac{1}{2\omega}. \quad (9)$$

Remark 1. The formulae (6) and (8) remain true in the same sense when the perturbation is a complex-valued function: $\delta(\mathbf{n}) = \delta_{\text{R}}(\mathbf{n}) + i\delta_{\text{I}}(\mathbf{n})$. The imaginary part $\delta_{\text{I}}(\mathbf{n})$ may be considered as the absorption coefficient (in case of weak absorption).

The equation (6) does not have sense at a focal point since the divergence J vanishes and the plane wave solution has caustics. Therefore the ray integral of $\delta(\mathbf{n})$ can not be properly evaluated. We outline another method which is *not* obstructed by caustics.

4 Beam-like solutions

We focus here on the construction of beam-like solutions [1] (Ch.5). Fix a refraction coefficient $\mathbf{n} \in C^3(X)$ in an Euclidean plane X and a compact interval γ of a geodesic curve of the metric $d\sigma^2 = \mathbf{n}^2 dx^2$ in X given by the equation $x = x(s)$, $0 \leq s \leq S$. Let Γ be an open neighborhood of γ such that for an arbitrary point $x \in \Gamma$ there exists only one closest point $\xi(t) \in \gamma$. The functions $s(x) = s$ and $r(x) = |x - \xi(t)|$ are well defined in Γ and form a smooth coordinate system on either side of γ . The Jacobian matrix of this coordinate system is orthogonal at any point $\xi \in \gamma$. The function

$j = ay_1^2 + 2by_1y_2 + cy_2^2$ plays role of divergence of a beam of geodesics close to γ , where $y_1, y_2 \in C^3(\gamma)$ are solutions of the equation

$$y'' + \mathbf{K}(t)y = 0 \quad (10)$$

such that $y_1y_2' - y_2y_1' = 1$. Here a, b, c are arbitrary real numbers such that $ac - b^2 = 1$, $a > 0$ and

$$\mathbf{K} = \frac{1}{\mathbf{c}} \left(\mathbf{c}_{rr}'' + \frac{1}{2}\mathbf{c}_{tt}'' - \frac{3}{4}\frac{\mathbf{c}_t'^2}{\mathbf{c}} \right).$$

The function j has length dimension and does not vanish. For an arbitrary point $x_0 = \xi(t_0) \in \gamma$ one can choose the above functions in such a way that

$$\varphi(t_0, 0) = \chi(t_0, 0) = 0, \quad j(t_0) = 1, \quad j'(t_0) = 0.$$

For this we choose the solutions of (10) such that $y_1(t_0) = y_2'(t_0) = 1$, $y_1'(t_0) = y_2(t_0) = 0$ and take $a = c = 1$, $b = 0$. The point x_0 is then called waist of the beam. Integrating along γ defines the phase functions

$$\varphi(s) = \int_{s_0}^s \mathbf{n}(s, 0) ds, \quad \chi(s) = - \int_{s_0}^s \frac{ds}{2j(s)}.$$

Introduce the "fast" variable in Γ :

$$q(x) \doteq \left(\frac{\mathbf{n}(s, 0)\omega}{j} \right)^{1/2} r(x)$$

and consider the function

$$U_\gamma(x) = \frac{1}{j^{1/4}(s)} \exp i \left(\chi(s) + 2\pi\omega\varphi(s) + \frac{\pi r^2}{2}\omega \frac{d}{ds} \frac{\mathbf{n}(s)}{j(s)} \right) \exp(-\pi q^2) \quad (11)$$

It is concentrated near γ and is called asymptotic beam solution. In the case of constant speed U_γ is also called Gaussian beam. Each asymptotic solution U_γ is the main term of a true solution:

Theorem 4.1 *Let $\mathbf{n} \in C^5(X)$ and γ be an arbitrary compact interval of a geodesic. There exists a solution of the Helmholtz equation in Γ of the form*

$$u_\gamma(x, \omega) = U_\gamma(x, \omega) + R_\gamma(x, \omega), \quad (12)$$

where the remainder fulfils the inequality $|R_\gamma(x, \omega)| \leq C\omega^{-1/2}$.

◀ This fact follows from [5], Proposition 6.1, where the construction of the Leontovich-Fock solution coincides with (11) up to the factor $\omega^{-1/4}$. ▶

5 Perturbation of a beam solution

Take perturbation $\delta(\mathbf{n}) = O(\omega^{-1})$ in (12) and omit all the terms of the rate $O(\omega^{-2})$. We obtain the formula for a beam solution of the perturbed Helmholtz equation in Γ :

$$w_\gamma(x, \omega) = \frac{1}{j^{1/4}(s)} \exp i \left(\chi(s) + 2\pi\omega\tilde{\varphi}(s) + \frac{\pi r^2}{2} \omega \frac{d\mathbf{n}(s)}{ds} j(s) \right) \\ \times \exp(-\pi q^2) + O(\omega^{-1/2}),$$

where $s = s(x)$, $\mathbf{n}(s) = \mathbf{n}(x(s), 0)$ and

$$\tilde{\varphi}(s) = \varphi(s) + \int_{s_0}^s \delta(\mathbf{n}) ds$$

is the perturbed phase function. The ray γ and the divergence j are the same modulo $O(\omega^{-2})$. The perturbation of the product $q^2 j$ in Γ is equal to

$$\delta(q^2 j) = 2\omega r^2 \delta(\mathbf{n}^2) = O(\omega^{-1})$$

since $r = O(\omega^{-1/2})$ in Γ and $\delta(\mathbf{n}) = O(\omega^{-1})$.

Conclusion 5.1 *Comparing beam solutions of the perturbed and the non-perturbed Helmholtz equations we get*

$$\frac{w_\gamma(x, \omega)}{u_\gamma(x, \omega)} = \exp \left(2\pi i \omega \int_{s_0}^s \delta(\mathbf{n}) ds \right) + O(\omega^{-1/2}) \quad (13)$$

Remark 2. This equation looks similarly to the equation (8) for the plane wave solutions, but it is now valid for any compact geodesic interval γ . Let now $D \subset X$ be a compact set and γ be a compact interval of a geodesic curve in D with both end points $x(s_0), x(s_1)$ on ∂D . Assuming that the quotient (13) is known on the boundary of the set D and the condition (9) is fulfilled, we can evaluate the ray integral

$$\int_\gamma \delta(\mathbf{n}) ds = \int_{s_0}^{s_1} \delta(\mathbf{n}) ds.$$

If many integrals of this type are known, the perturbation $\delta(\mathbf{n})$ can be evaluated by methods of integral geometry.

The condition (9) appeared (in equivalent form) in [4] (1.3), where quite different methods for numerical reconstruction were applied. Nevertheless the heuristics of [4] agrees with our conclusion.

6 Construction of beam solutions

The above method depends on a construction of beam solutions of non-perturbed and perturbed equations for many rays. An explicit construction can be done in the case of constant speed in a n -dimensional Euclidean space X . We denote $\langle x, \xi \rangle = x_1 \xi_1 + \dots + x_n \xi_n$, $\xi \in \mathbb{R}^n$, $|\xi| = (\sum \xi^2)^{1/2}$; $d\Sigma$ means the angular measure on a sphere. The space frequency is denoted by $\mathbf{k} = \omega \mathbf{n}$.

Theorem 6.1 *Let \mathbf{e} be an arbitrary unit vector in X , then the function*

$$u_{\mathbf{e}}(x, \omega) = \int_{|\xi|=1} \exp(2\pi \mathbf{k} (\langle \mathbf{e}, \xi \rangle - 1)) \exp(2\pi i \mathbf{k} \langle x, \xi \rangle) d\Sigma \quad (14)$$

is a beam solution of the Helmholtz equation in X for the ray $\gamma = \{x = s\mathbf{e}, s \in \mathbb{R}\}$ with waist at the origin. (The quantity 1 in the exponent has space dimension).

◀ The integral (14) obviously satisfies (2). We may assume that $\mathbf{e} = (1, 0, \dots, 0)$. Then $\langle x, \xi \rangle = s\xi_1 + \langle y, \eta \rangle$, $y = (x_2, \dots, x_n)$, $\eta = (\xi_2, \dots, \xi_n)$ and (14) looks as follows

$$u_{\mathbf{e}}(x, \omega) = \int_{|\xi|=1} \exp(2\pi \mathbf{k} ((1 + \imath s) \xi_1 - 1 + \imath \langle y, \eta \rangle)) d\Sigma.$$

We have $\xi_1 = (1 - |\eta|^2)^{1/2} = 1 - |\eta|^2 / 2 + \rho$, where $\rho = O(|\eta|^4)$ on the sphere $\{|\xi| = 1\}$, hence

$$u_{\mathbf{e}}(x, \omega) = \exp(2\pi i \mathbf{k} s) \int_{|\xi|=1} \exp 2\pi \mathbf{k} \left[(1 + \imath s) \left(\rho - \frac{|\eta|^2}{2} \right) + \imath \langle y, \eta \rangle \right] d\Sigma.$$

We remove the term with ρ in the phase function by means of a change of the variables η in a small neighborhood V of the points $\xi = \pm \mathbf{e}$. Then we perform integration in V and neglect the complement to V , where the integrand is small. This yields

$$u_{\mathbf{e}}(x, \omega) = \frac{2 \exp(2\pi i \mathbf{k} s)}{[\mathbf{k}(1 + \imath s)]^{(n-1)/2}} \exp\left(-\frac{\pi \mathbf{k} |y|^2}{1 + \imath s}\right) (1 + O(\omega^{-1})). \quad (15)$$

The main term of (15) is obviously small when the product $\mathbf{k}|y|$ is big. In the case $n = 2$ it coincides with (11) up to factor $2\mathbf{k}^{-1/2}$, since

$$r = |y|, \quad j = 1 + s^2, \quad \frac{\exp(i\chi)}{j^{1/4}} = \frac{1}{\sqrt{1 + is}},$$

$$-q^2 + iq^2 \frac{j}{2} \frac{\partial}{\partial s} \log j = \omega r^2 \left(-\frac{1}{j} + i \frac{j'}{2j} \right) = -\omega |y|^2 \frac{1}{1 + is}. \blacktriangleright$$

7 Beam solutions to the wave equation

Proposition 7.1 *For a constant speed \mathbf{c} and any unit vector $\mathbf{e} \in \mathbb{R}^n$ the function*

$$u(x, t) = \mathbf{c} \int_{\mathbb{R}^n} \exp(2\pi(\langle \mathbf{e}, \xi \rangle - |\xi| + i\langle x, \xi \rangle - it\mathbf{c}|\xi|)) d|\xi| d\Sigma$$

satisfies the wave equation (2) and we have

$$\int u(x, t) \exp(2\pi i\omega t) dt = u_{\mathbf{e}}(x, \omega),$$

where $u_{\mathbf{e}}$ is the beam solution as above.

◀ By straightforward integration, we get

$$\int_{\mathbb{R}} \exp(-2\pi i\mathbf{c}t|\xi|) \exp(2\pi i\omega t) dt = \int \exp(2\pi it(\omega - \mathbf{c}|\xi|)) dt = \delta(\omega - \mathbf{c}|\xi|),$$

where δ means delta-function. This yields

$$\begin{aligned} \int u(x, t) \exp(2\pi i\omega t) dt &= \mathbf{c} \int_{\mathbb{R}^n} \delta(\omega - \mathbf{c}|\xi|) \exp(2\pi(\langle \mathbf{e}, \xi \rangle - |\xi| + i\langle x, \xi \rangle)) d|\xi| d\Sigma \\ &= \int_{|\xi|=\mathbf{k}} \exp(2\pi(\langle \mathbf{e}, \xi \rangle - \mathbf{k} + i\langle x, \xi \rangle)) d\Sigma \\ &= \int_{|\eta|=1} \exp(2\pi\mathbf{k}(\langle \mathbf{e}, \eta \rangle - 1 + i\langle x, \eta \rangle)) d\Sigma \\ &= u_{\mathbf{e}}(x, \omega), \end{aligned}$$

where we change the variables $\eta = \mathbf{k}^{-1}\xi$. ▶

Proposition 7.2 For an arbitrary unit vector $\mathbf{e} \in \mathbb{R}^2$ we have for $n = 2$

$$u(x, 0) = \frac{\mathbf{c}}{(x^2 - 2i \langle \mathbf{e}, x \rangle)^{1/2}}, \quad u'_t(x, 0) = \frac{\mathbf{c}^2}{2\pi i (x^2 - 2i \langle \mathbf{e}, x \rangle)^{3/2}}, \quad (16)$$

where $(x^2 - 2i \langle \mathbf{e}, x \rangle)^\lambda$ is the boundary value on \mathbb{R}^2 of the holomorphic function

$$(z^2 - 2i \langle \mathbf{e}, z \rangle)^\lambda, \quad z^2 \doteq z_1^2 + z_2^2$$

defined in the tube $U = \{z \in \mathbb{C}^2; |\operatorname{Im} z - \mathbf{e}| < 1\}$, where the branch of $\zeta^{1/2}$ is positive for $\zeta > 0$.

Remark 4. It follows that any beam solution of the Helmholtz equation can be obtained by solving the Cauchy problem for the wave equation with the initial data (16). Note that the data are real analytic except for the origin, where they have rather simple singularity.

◀ We have $u(x, 0) = \mathbf{c}v(x - i\mathbf{e})$, where

$$v(x) = \int_{\mathbb{R}^2} \exp(-2\pi|\xi| + 2\pi i x \xi) d\xi = \int_0^\infty d\rho \exp(-2\pi\rho) \int_{|\xi|=\rho} \exp(2\pi i x \xi) d\varphi.$$

The inner integral is equal to

$$\int_{|\eta|=1} \exp(2\pi i r \rho \cos \varphi) d\varphi = 2\pi J_0(2\pi r \rho),$$

where $r = 2\pi|x|$. This yields

$$v(x) = 2\pi \int_0^\infty d\rho \exp(-2\pi\rho) J_0(2\pi r \rho) = \frac{1}{(r^2 + 1)^{1/2}}.$$

This function has holomorphic continuation

$$v(z) = \frac{1}{(z^2 + 1)^{1/2}}$$

in the ball $\{z \in \mathbb{C}^2; |\operatorname{Im} z| < 1\}$, where $z^2 = \sum z_i^2$. Therefore the function

$$v(z - i\mathbf{e}) = \frac{1}{(z^2 - 2i \langle \mathbf{e}, z \rangle)^{1/2}},$$

is well defined in the tube U . It has moderate growth as $y \rightarrow 0$, $\langle \mathbf{e}, y \rangle > 0$, hence its boundary value for $y = 0$ exists as a generalized function of moderate growth. Further we have $u'_t(x, 0) = -i\mathbf{c}^2 w(x - i\mathbf{e})$, where

$$\begin{aligned} w(x) &= \int \exp(-2\pi |\xi| + 2\pi i x \xi) d\xi \\ &= 2\pi \int_0^\infty d\rho \exp(-2\pi\rho) \rho J_0(r\rho) = \frac{1}{2\pi (r^2 + 1)^{3/2}}. \end{aligned}$$

and the second equation (16) follows. \blacktriangleright

8 Variable speed

Note that the integral in (14) is a superposition of plane waves solutions $\exp(2\pi i \mathbf{k} \langle x, \xi \rangle)$ of the Helmholtz equation. The factor $\exp(2\pi \mathbf{k} (\langle \mathbf{e}, \xi \rangle - 1))$ is equal to 1 for $\xi = \mathbf{e}$ and exponentially small for another unit directions.

A beam solution u_γ for a variable coefficient \mathbf{n} can be constructed as follows. Suppose that $\mathbf{n} = \text{const}$ in $X \setminus D$, where D is a compact convex subset of X . Choose a point $x_0 \in X \setminus D$, a unit vector \mathbf{e} and consider the family of plane wave solutions $u_\theta(x, \omega)$ as in Sec.2 defined in X for all unit θ in a neighborhood Ω of the vector \mathbf{e} and such that the hyperplanes $P_\theta + x_0$ are contained in $X \setminus D$. Take a smooth function ψ on the unit sphere that $\psi(\mathbf{e}) = 1$ supported by Ω and consider the superposition

$$U_\gamma(x) = \int \exp(2\pi \mathbf{k} (\langle \mathbf{e}, \theta \rangle - 1)) u_\theta(x - x_0, \omega) \psi(\theta) d\Sigma(\theta).$$

This is a reasonable approximation to the beam solution u_γ for the ray γ that has direction \mathbf{e} at the point x_0 , since the omitted integral with factor $\exp(2\pi \mathbf{k} (\langle \mathbf{e}, \theta \rangle - 1))$ is exponentially small as $\omega \rightarrow \infty$.

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