PAPER

Reconstruction from cone integral transforms

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Reconstruction from cone integral transforms

Victor Palamodov

Tel Aviv University, Tel Aviv-Yafo, Israel

E-mail: palamodo@post.tau.ac.il

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Abstract
The paper contains new reconstruction formulas for a function on 3D space from data of its cone integrals with fixed opening and integrable weight. In the case of cone integrals with the (non integrable) weight modelling photometric law, a reconstruction is obtained for the non redundant data of cones with the apex running on a curve.

Keywords: convolution, nongeodesic Funk transform, singular ray transform

(Some figures may appear in colour only in the online journal)

1. Introduction

One of the methods of the Compton tomography is based on the count of single-scattered photons on electrons with specific incident and outcome energies. The scattering angle θ is characterized by the Compton formula

$$\lambda_{\mathrm{sc}} - \lambda_{\mathrm{in}} = \lambda_{\mathrm{el}} (1 - \cos\theta),$$

where $\lambda_{\mathrm{in}} = \mathrm{hc}/\mathrm{E}_{\mathrm{in}}$, $\lambda_{\mathrm{sc}} = \mathrm{hc}/\mathrm{E}_{\mathrm{sc}}$ are the wavelengths of a photon before and after scattering, and $\lambda_{\mathrm{el}} = \mathrm{h}/\mathrm{m}_{\mathrm{el}}\mathrm{c}$ is the Compton wavelength of an electron in rest. The Compton shift $\lambda_{\mathrm{sc}} - \lambda_{\mathrm{in}}$ is measured by the electronically collimated Compton camera which provides the number of photons scattered at a specific angle θ. This data is modeled by the integral of the gamma source distribution over the cone of rotation whose half-opening is θ.

This method provides multiple views of the object which can be considered as a Radon-type cone transform of the photon source distribution. Cree and Bones [7] proposed reconstruction formulae from data of the regular cone transform on $3D$ space with apices restricted to a plane orthogonal to the axis and all openings. Analytic reconstructions from the cone transform with restricted apex were obtained by Nguyen and Truong [9], Smith [10],Nguyen, Truong, Grangeat [11], Maxim *et al* [12], Maxim [13], Haltmeier [14], Terzioglu [17], Moon [19], Jung and Moon [18] gave inversion formulae for cone transform on the space of arbitrary dimension n. In [18] a scheme was proposed for collecting non redundant data from a line of detectors and rotating axis. Basko *et al* [8] proposed a numerical method based on developing the unknown function into spherical harmonics from cone integrals with swinging axis and only one opening. Gouia-Zarrad and Ambartsoumian [15, 16] found the reconstruction formula for the regular cone transform on a half-space with free apex and one opening.

This paper contains new reconstruction formulas for reconstruction of an unknown density function on the half-space from its weighted integrals over cones with constant axis and apex running the half-space. In the regular case (the weight density is integrable over the cone) the cone transform is the convolution with a distribution supported by the cone. Our reconstruction is given in terms of the cone transform with another weight. We note a connection of the regular cone transforms with the fundamental solution of the wave operator. This approach does not work when the weight density is not integrable. Then the cone integral data can not be collected for all positions of the apex. A more complicated reconstruction method is proposed from data of cone integrals with apices running a 1D set. Here we apply the inversion formula for the nongeodesic Funk transform.

2. Cone integral transforms

The cone of rotation in an Euclidean space E^n can be written in the form

$$C(\lambda) = \{x \in E^n : \lambda x_1 = r\}, \ \lambda > 0, \ r = \sqrt{x_2^2 + \ldots + x_n^2}$$

in the appropriate coordinate system. The line $r = 0$ is the axis and $\lambda = \tan\theta$, $0 < \theta < \pi/2$ is the half-opening of the cone. The integral operator

$$g(y) = \int_{x \in C(\lambda)} f(y + x)\,\omega(x)\mathrm{d}S, \ y \in E^n \tag{1}$$

is discussed in recent publications under the name of *cone Radon* or *cone* transform ($\omega = 1$). Here dS is the Euclidean hypersurface element. If $\omega(x) = |x|^{-k}$ we call this transform *regular* in the case $k < n - 1$ and *singular* in the case $k = n - 1$. The realistic model of point spread function for single-scattering optical 3D tomography is based on the photometric law of scattered radiation modelled by the singular cone transform. On figure 2, one can see that the number of scattering events close to the point $x = 0$ caused by a small portion of incident photons at a point $x \neq 0$ moving inside an angular area ω is proportional to $|x|^{-2}$. See more details in [9].

We focus on analytic inversion of regular and singular cone transforms in 3D.

3. Inversion of regular cone transforms

Setting $\omega = |x|^{-k}$, $k < 2$ and changing x to $-x$ we can write the cone transform (1) in the convolution form

$$g_k = \mathrm{D}_k * f_k, \tag{2}$$

where D_k is the functional

$$\mathrm{D}_k(\varphi) \doteqdot \int_{\mathbf{C}(\lambda)} |x|^{-k} \varphi(x)\mathrm{d}S \tag{3}$$

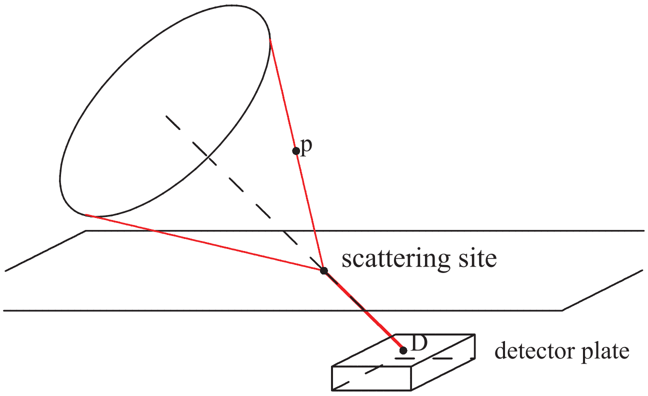

Figure 1. Scheme of the Compton camera.

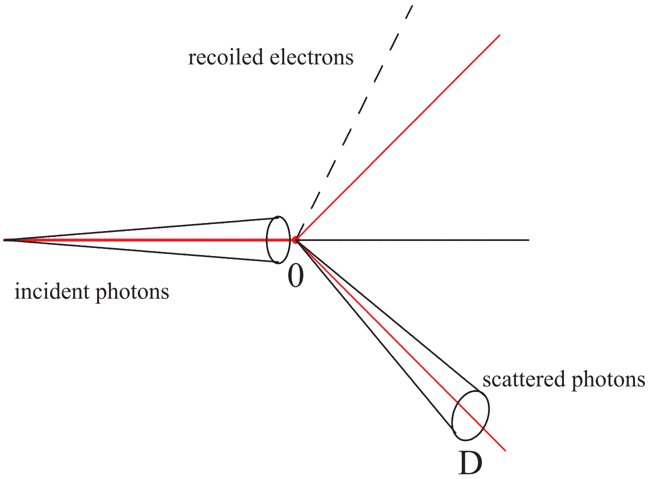

Figure 2. Photometric law.

supported on $C(\lambda)$. This is a tempered distribution on E^n and the convolution (2) is well defined for an arbitrary distribution or generalized function f_k that decreases sufficiently fast as $x_1 \to -\infty$. From now on we assume that any function f under consideration fulfils this condition. If f_k is supported on the half-space $H_+ = \{x_1 \geqslant 0\}$, then the same is true for g_k since supp $\mathrm{D}_k * f \subset$ supp $f + C(\lambda)$. It is easy to check that the solution of (2) is unique under this condition .

The operator

$$\Box_n = \frac{\partial^2}{\partial x_1^2} - \lambda^2 \left(\frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_n^2} \right)$$

can be considered as the wave operator in E^n with time variable x_1. The product $\Theta(x) \doteqdot h(x_1)\,\delta_0(x_2, \dots, x_n)\,\mathrm{d}x$ is a distribution in E^n supported on the ray $\{x = (t, 0, \dots, 0),\ t \geqslant 0\}$ where δ_0 denotes the delta function, $h(t) = 1$ for $t \geqslant 0$ and $h(t) = 0$ for $t < 0$. It product satisfies $\partial\Theta/\partial x_1 = \delta_0$.

Theorem 1. *If $n=3$ and $k=0$ the solution of (2) can be found in the form*

$$
\begin{aligned}
f_0(x) &= z\Box^2 \mathrm{D}_1 * \Theta * g_0 \\
&= z\Box^2 \int\int \left(\int_{x_1-\lambda s}^{\infty} g_0\left(y, x_2 - t_2, x_3 - t_3\right) \mathrm{d}y \right) \frac{\mathrm{d}t_2 \mathrm{d}t_3}{s},
\end{aligned} \tag{4}
$$

where $s = \left(t_2^2 + t_3^2\right)^{1/2}$ and

$$
z = \frac{\cos\theta}{\mathrm{j}^2 \tan^2\theta}, \ \mathrm{j} = 2\pi i.
$$

If $k=1$ the solution of (2) reads

$$
\begin{aligned}
f_1(x) &= z\Box^2 \mathrm{D}_0 * \Theta * g_1 \\
&= z\Box^2 \int\int \left(\int_{x_1-\lambda s}^{\infty} g_1\left(y, x_2 - t_2, x_3 - t_3\right) \mathrm{d}y \right) \mathrm{d}t_2 \mathrm{d}t_3.
\end{aligned} \tag{5}
$$

Inversion of (2) for $k=0$ was given first in [16] in terms of Fourier transform in the variables x_2, x_3; see also [15] for reconstructions in arbitrary dimensions.

Corollary 2. *For any distribution f, we have*

$$
\mathrm{supp} f \subset \mathrm{supp}\ \mathrm{D}_k * f + V(\lambda), \ k = 0, 1
$$

where $V(\lambda)$ means the convex hull of $C(\lambda)$.

Let $\mathrm{d}x$ be the volume form on E^n. The Fourier transform of a tempered distribution u is defined by

$$
F(u)(p) = u\left(\exp\left(-\mathrm{j}\left\langle p, x\right\rangle\right)\right) \doteq \lim_{m\to\infty} u\left(\mathrm{a}\left(x/m\right)\exp\left(-\mathrm{j}\left\langle p, x\right\rangle\right)\right), \ \mathrm{j} \doteq 2\pi i
$$

where a is an arbitrary test function on E^n such that $\mathrm{a}(0) = 1$. For a tempered distribution g on the dual space E_p^n, the inverse Fourier transform $F^{-1}(g)$ is defined in the similar way with $-$ replaced by $+$ in the exponent.

Proposition 3. *For $n=3$, we have*

$$
F\left(\mathrm{D}_0\right)(p) = -\frac{\tan\theta}{2\pi\cos\theta} p_1 \left(\left(p_1 - \mathrm{i}0\right)^2 - \lambda^2\left(p_2^2 + p_3^2\right)\right)^{-3/2}, \tag{6}
$$

$$
F\left(\mathrm{D}_1\right)(p) = -\,\mathrm{i}\tan\theta\left(\left(p_1 - \mathrm{i}0\right)^2 - \lambda^2\left(p_2^2 + p_3^2\right)\right)^{-1/2}. \tag{7}
$$

Both functions have analytical continuation at $\mathbb{H}_- = \left\{p \in \mathbb{C}^3 : \mathrm{Im} p_1 \leqslant 0\right\}$.

A proof will be given in the next section.

Proof of theorem 1. Equations (6) and (7) immediately yield

$$
-\frac{\cos\theta}{\mathrm{j}\tan^2\theta}\left(p_1^2 - \lambda^2\left(p_2^2 + p_3^2\right)\right)^2 F\left(\mathrm{D}_0\right) F\left(\mathrm{D}_1\right) = p_1. \tag{8}
$$

The inverse Fourier transform gives

$$
z\Box^2 \mathrm{D}_1 * \mathrm{D}_0 = \frac{\partial}{\partial x_1}\delta_0, \tag{9}
$$

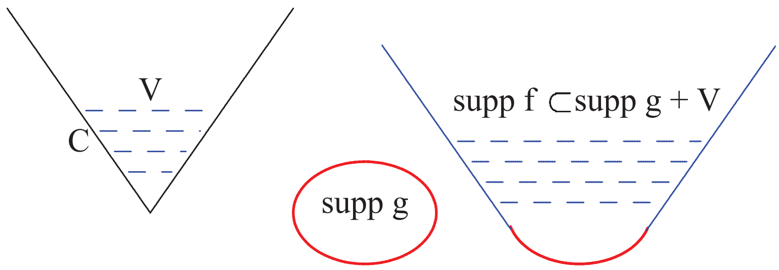

Figure 3. Support of $g = D_k * f$ and support of f.

since

$$F(\square) = -4\pi^2 \left(p_1^2 - \lambda^2 \left(p_2^2 + p_3^2\right)\right)$$

and $F^{-1}(p_1) = -\mathrm{j}^{-1}\partial/\partial x_1$. Take convolution with Θ and get

$$z\square^2 \mathrm{D}_1 * \mathrm{D}_0 * \Theta = \frac{\partial}{\partial x_1}\delta_0 * \Theta = \delta_0$$

since the distributions Θ, D_0 and D_1 are supported on H_+ and commute. Applying (9) to f_0 gives

$$f_0 = z\square^2 \mathrm{D}_1 * \Theta * \mathrm{D}_0 * f_0 = z\square^2 \mathrm{D}_1 * \Theta * g_0$$

which is equivalent to (4). Commuting factors in (9) yields

$$f_1 = z\mathrm{D}_0 * \Theta * \mathrm{D}_1 * \square^2 f_1 = z\mathrm{D}_0 * \Theta * \square^2 g_1$$

and (5) follows. ▶ □

Remark 1. Solution of (2) could be done in the form $F(f) = F(g)/F(\mathrm{D}_k)$ on the frequency space. Implementation of this method assumes cutting out the 'plumes' of g as in figure 3 which cause the artifacts of the reconstruction.

Remark 2. Constant attenuation can be included in this method. It is sufficient to replace $p_1 - \mathrm{i}0$ by $p_1 - \mathrm{i}\varepsilon$ in (8).

4. Fourier transform of homogeneous distributions

In this section we check (6) and (7). Let u be a homogeneous distribution of degree $\alpha \neq 0, -1, -2, \ldots$ on the Euclidean space E^n. This means that

$$\mathrm{d}\left(\mathbf{e} \llcorner u\right) = \alpha u$$

where $\mathbf{e} = \sum x_i \partial/\partial x_i$ is the Euler field and $t \llcorner a$ denotes the inner product of a field t and a differential form a. The Fourier transform $F(u)$ is a homogeneous generalized function of degree $-\alpha$ which can be found by

$$F(u)(p) = \Gamma(\alpha)\mathrm{j}^{-\alpha} \int_\Sigma \left(\langle p, x\rangle - \mathrm{i}0\right)^{-\alpha} \mathbf{e} \llcorner u, \qquad (10)$$

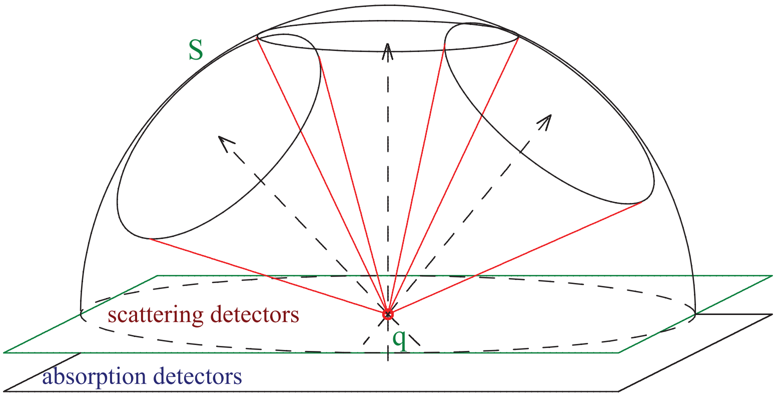

Figure 4. Cones of constant opening.

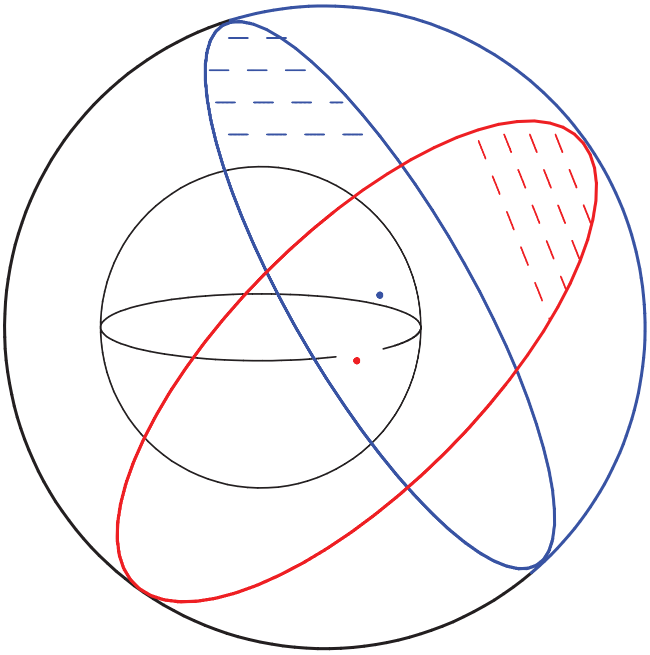

Figure 5. Nongeodesic circles on a sphere.

where Σ is an arbitrary smooth hypersurface in E^3 that meets each central ray transversely [20]. We apply (10) to $u = \mathrm{D}_0$ and choose $\Sigma = \{x_1 = 1\}$. This choice is admissible since Σ can be deformed outside of supp $\mathrm{D}_0 = C(\lambda)$ to meet the transversality condition.

Remind that for a smooth function a in E^n such that $\nabla a(x) \neq 0$, the delta function $\delta_0(a)$ is defined by

$$\delta_0(a)\,(\rho) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_{|a| \leqslant \varepsilon} \rho = \int_{a=0} \frac{\rho}{|\nabla a|}$$

where ρ is an arbitrary test density on E^n. By (3) we have

$$\mathrm{D}_0(\varphi) = \frac{1}{\cos\theta}\delta(\lambda x_1 - r)\,\mathrm{d}x,$$

where $\mathrm{d}x$ is the volume form on E^3. This is a homogeneous distribution of degree 2 since the function $\delta_0(a)$ has degree -1 for any linear function a and $\deg\ \mathrm{d}x = 3$. Consider the delta-like sequence of functions on a line $v_k(t) = kv_1(kt)$, $k = 1, 2, \ldots$ where $v_1(t) = 1$ for $|t| \leqslant 1/2$ and $v(t) = 0$ otherwise. The sequence of functions $v_k(\lambda x_1 - r)$ tends to $\delta(\lambda x_1 - r)$ and $\mathbf{e} \llcorner \mathrm{d}x = \mathrm{d}x_2\mathrm{d}x_3$ on Σ, which yields for any test function ψ in Σ,

$$\begin{aligned}\cos\theta\int_\Sigma \mathbf{e}\llcorner \mathrm{D}_0(\psi) &= \lim_k \int_\Sigma v_k(\lambda - r)\,\psi\mathbf{e}\llcorner \mathrm{d}x = \lim_{k\to\infty} k\int_{|\lambda - r|\leqslant 1/2k} \psi \mathrm{d}x_2\mathrm{d}x_3 \\ &= \int_{r=\lambda} \psi\frac{\mathrm{d}x_2\mathrm{d}x_3}{\mathrm{d}r} = \lambda\int_{r=\lambda}\psi\mathrm{d}\varphi, \end{aligned} \tag{11}$$

where $\mathrm{d}\varphi$ is the angular measure on $\{r = \lambda\} \subset \Sigma$. Substitute (11) to (10) and calculate the integral for the case $p_1 > \lambda q$ where $q \doteqdot (p_2^2 + p_3^2)^{1/2}$:

$$F(\mathrm{D}_0)(p) = \frac{\tan\theta}{\mathrm{j}^2\cos\theta}\int_{r=\lambda}(\langle\xi, x\rangle - \mathrm{i}0)^{-2}\,\mathrm{d}\varphi = \frac{\tan\theta}{\mathrm{j}^2\cos\theta}\int_{r=\lambda}\frac{\mathrm{d}\varphi}{(p_1 + p_2x_2 + p_3x_3)^2}.$$

We can write $p_1^{-1}(p_2x_2 + p_3x_3) = a\cos\varphi$, where $0 \leqslant a = \lambda q/p_1 < 1$. Then by means of [6], 3.613.2 we get

$$\begin{aligned}\int_{r=\lambda}\frac{\mathrm{d}\varphi}{(p_1 + p_2x_2 + p_3x_3)^2} &= \frac{2}{p_1^2}\int_0^\pi \frac{\mathrm{d}\varphi}{(1 + a\cos\varphi)^2} = \frac{2}{p_1^2}\frac{\pi}{(1 - a^2)^{3/2}} \\ &= \frac{2\pi p_1}{\left((p_1 - \mathrm{i}0)^2 - \lambda^2q^2\right)^{3/2}}\end{aligned}$$

which implies

$$F(\mathrm{D}_0)(p) = -\frac{\tan\theta}{2\pi\cos\theta}\frac{p_1}{\left((p_1 - \mathrm{i}0)^2 - \lambda^2q^2\right)^{3/2}}.$$

This proves (6) for $p_1 > \lambda q$.

The case $k = 1$ is treated in the similar way. We have $|x|^{-1} = \cos\theta$ on Σ and

$$\begin{aligned}F(\mathrm{D}_1)(p) = \mathrm{j}^{-1}\int_\Sigma(\langle p, x\rangle - \mathrm{i}0)^{-1}\,\mathbf{e}\llcorner \mathrm{D}_1 &= -\frac{2\tan\theta}{\mathrm{j}(p_1 - \mathrm{i}0)}\int_0^\pi \frac{\mathrm{d}\varphi}{1 + a\cos\varphi} \\ &= \frac{2\tan\theta}{\mathrm{j}(p_1 - \mathrm{i}0)}\frac{\pi}{(1 - a^2)^{1/2}} = \frac{\tan\theta}{i}\left((p_1 - \mathrm{i}0)^2 - \lambda^2q^2\right)^{-1/2}\end{aligned}$$

where we use again [6], 3.613.2. This implies (7) for $p_1 > \lambda q$. Note the right hand sides of (6,7) have analytic continuation at $\mathbb{H}_-$. The Fourier transforms of D_0 and of D_1 have also analytic continuations at $\mathbb{H}_-$ since the distributions are supported by H_+ hence the above equations hold for all p. ►

5. Cone transform and the wave equation

The forward propagator for the wave operator in E_n with the the time variable x_1 is the distribution P_n supported on the $H_+ = \{x : x_1 \geqslant 0\}$ satisfying $\Box \mathrm{P}_n = \delta_0$. The forward propagator is unique and can be found by the formula

$$\mathrm{P}_n(x) = F\left(\mathrm{j}^2\left(p_1^2 - \lambda^2 q^2\right)\right) = F^{-1}\left(\mathrm{j}^2\left(p_1^2 - \lambda^2 q^2\right)\right) \tag{12}$$

where $q = \left(p_2^2 + ... + p_n^2\right)^{1/2}$. In particular for $n = 3$,

$$\mathrm{P}_3(x) = \frac{1}{2\pi\lambda}\frac{\theta\left(\lambda x_1 - r\right)}{\left(\lambda^2 x_1^2 - r^2\right)^{1/2}}\mathrm{d}x$$

where $r = \left(x_2^2 + x_3^2\right)^{1/2}$. The support of P_n is contained in the cone $V(\lambda) = \{\lambda x_1 \geqslant |y|\}$ and coincides with this cone for any odd n. The convolution $u = \mathrm{P}_n * f$ solves the equation $\Box_n u = f$.

Let K be a convex cone in H_+. All distributions supported by K form the algebra A_K with respect to the convolution since $u * v \in A_K$ if $u, v \in A_K$. In this section, we consider the case $n = 3$ only.

Theorem 4. *The square root of the forward propagator* P_3 *is defined in the convolution algebra* $A_{C(\lambda)}$:

$$\mathrm{P}_3 = \frac{1}{2\pi\tan\theta}\mathrm{D}_1 * \frac{1}{2\pi\tan\theta}\mathrm{D}_1. \tag{13}$$

Proof. By (7) we have

$$F\left(\mathrm{D}_1\right)^2(p) = -\tan^2\theta\frac{1}{\left(p_1 - \mathrm{i}0\right)^2 - \lambda^2 q^2}$$

By the inverse Fourier transform this implies

$$\mathrm{D}_1 * \mathrm{D}_1 = -\tan^2\theta F^{-1}\left(\frac{1}{\left(p_1 - \mathrm{i}0\right)^2 - \lambda^2 q^2}\right).$$

Therefore

$$\begin{aligned}\Box\left(\mathrm{D}_1 * \mathrm{D}_1\right) &= -\tan^2\theta\Box F^{-1}\left(\frac{1}{\left(p_1 - \mathrm{i}0\right)^2 - \lambda^2 q^2}\right)\\ &= 4\pi^2\tan^2\theta F^{-1}(1) = 4\pi^2\tan^2\theta\delta_0\end{aligned}$$

which yields

$$\left(2\pi\tan\theta\right)^{-2}\Box\left(\mathrm{D}_1 * \mathrm{D}_1\right) = \delta_0.$$

This implies equation (13) since both sides are supported by H_+. ▶

It is easy to check that the square root $\sqrt{\mathrm{P}_3}$ is unique up to the factor ± 1.

Remark 3. The forward propagator P_4 for the wave operator om E^4 vanishes on the set $E^4 \backslash V$ which is called the exterior lacuna. The distribution P_4 vanishes also on the open cone $\mathrm{int}V = \{\lambda x_1 > |y|\}$; this domain is called the interior lacuna. The same is true for propagators P_n on E^n for any even $n > 2$. For odd n, the forward propagator P_n have the exterior but not the interior lacuna whereas the kernel $\sqrt{\mathrm{P}_3}$ has both lacunae.

6. Regular cone transforms in 4D

We calculate the inversions to the regular cone transforms $\mathrm{D}_k,\ k = 0, 1, 2$. We have $\deg \mathrm{D}_k = 3 - k$ hence $F(\mathrm{D}_k)$ is homogeneous function of degree $k - 3$ on the impulse space E^4_p. For $k = 1$, we have

$$F(\mathrm{D}_1)(p) = \mathrm{j}^{-2} \int_\Sigma \frac{\mathbf{e} \llcorner \mathrm{D}_1}{(p_1 + p_2 x_2 + p_3 x_3 + p_4 x_4)^2} \tag{14}$$

where $\mathbf{e} = \sum x_i \partial/\partial x_i$ and $\Sigma \doteqdot \{x_1 = 1\}$ By (3) we find

$$\mathrm{D}_1 = \frac{1}{\cos\theta} \delta(\lambda x_1 - r) |x|^{-1} \, \mathrm{d}x$$

where $r = \left(x_2^2 + x_3^2 + x_4^2\right)^{1/2}$, and by the arguments of section 4

$$\mathbf{e} \llcorner \mathrm{D}_1|_\Sigma = \delta(\lambda - r) \, \mathrm{d}x_2 \mathrm{d}x_3 \mathrm{d}x_4$$

since $\cos\theta\, |x| = x_1 = 1$. Suppose that $p_1 > \lambda q$ and have

$$p_2 x_2 + p_3 x_3 + p_4 x_4 = p_1 a \cos\varphi$$

for $a = \lambda q / p_1 < 1$ and some angle $\varphi,\ 0 \leqslant \varphi \leqslant \pi$. We have $|\nabla r| = 1$ and $\mathrm{d}S = \lambda^2 \sin\varphi \mathrm{d}\varphi \mathrm{d}\theta$ in (14).

$$\begin{aligned} F(\mathrm{D}_1)(p) &= \mathrm{j}^{-2} \int_\Sigma \frac{\delta(\lambda - r) \, \mathrm{d}x_2 \mathrm{d}x_3 \mathrm{d}x_4}{(p_1 + p_2 x_2 + p_3 x_3 + p_4 x_4)^2} = \frac{1}{\mathrm{j}^2 p_1^2} \int_{r=\lambda} \frac{\mathrm{d}S}{(1 + a\cos\varphi)^2} \\ &= -\frac{\lambda^2}{2\pi p_1^2} \int_0^\pi \frac{\sin\varphi \mathrm{d}\varphi}{(1 + a\cos\varphi)^2} \end{aligned}$$

where $\mathrm{d}S = \lambda^2 \sin\varphi \mathrm{d}\varphi \mathrm{d}\psi$ is the Euclidean measure on the sphere $r = \lambda$; the factor 2π appears by integrating against $\mathrm{d}\psi$. Change the variable $t = a\cos\varphi$ in the last integral and get

$$\int_0^\pi \frac{\sin\varphi \mathrm{d}\varphi}{(1 + a\cos\varphi)^2} = \int \frac{1}{a(1+t)} \bigg|_a^{-a} = \frac{2}{1 - a^2} = \frac{2p_1^2}{(p_1 - \mathrm{i}0)^2 - \lambda^2 q^2}.$$

Finally

$$F(\mathrm{D}_1)(p) = -\frac{\tan^2\theta}{\pi\left((p_1 - \mathrm{i}0)^2 - \lambda^2 q^2\right)} = 4\pi \tan^2\theta F(\mathrm{P}_4).$$

By (12) this yields

$$\mathrm{D}_1 = 4\pi \tan^2\theta \mathrm{P}_4.$$

For $k = 0$, we have

$$\mathrm{D}_0 = |x| \, \mathrm{D}_1 = \frac{1}{\cos\theta} x_1 \mathrm{D}_1 = 4\pi \frac{\tan^2\theta}{\cos\theta} x_1 \mathrm{P}_4.$$

By the Leibniz formula

$$\Box_4 x_1 \mathrm{P}_4 = x_1 \Box_4 \mathrm{P}_4 + \frac{\partial}{\partial x_1} x_1 \cdot 2 \frac{\partial}{\partial x_1} \mathrm{P}_4 = 2 \frac{\partial}{\partial x_1} \mathrm{P}_4$$

since $x_1 \delta_0 = 0$. The inversion can be given by

$$\frac{\cos\theta}{8\pi \tan^2\theta} \Theta * \Box^2 \mathrm{D}_0 = \delta_0.$$

For $k = 2$, we have

$$x_1 \mathrm{D}_2 = \cos\theta \mathrm{D}_1 = 4\pi \frac{\sin^2\theta}{\cos\theta} \mathrm{P}_4,$$

hence the inversion of D_2 can be given in the form

$$\frac{\cos\theta}{4\pi \sin^2\theta} \Box (x_1 \mathrm{D}_2) = \delta_0.$$

7. Inversion of the singular cone transforms with swinging axe

Consider the singular transform on E^3

$$G(q,\sigma) = \int_{C(\sigma,\lambda)} f(q+x) \frac{\mathrm{d}S}{|x|^2},\ \sigma \in \mathrm{S}^2,\ q \in E^3, \tag{15}$$

where $C(\sigma,\lambda)$ stands for the cone of rotation with apex $x = 0$, axis $\sigma \in \mathrm{S}^2$ and opening $\lambda = \tan\theta$. The integral is well defined if f has compact support and $f(q+x) = O(|x|)$ for small $|x|$.

Theorem. *For any $\lambda > 0$ and any set $Q \subset E^3$, an arbitrary function $f \in C^2(E^3)$ with compact support can be recovered from data of integrals (15) for $q \in Q$ provided:*

(i) any plane H which meets supp$f$ *has a common point with Q,*
(ii) for any point $q \in Q$, there exists a unit vector $\sigma(q)$ such that supp $f \subset q + C(\sigma(q),\lambda)$.

Proof. The singular ray transform

$$Zf(q,\xi) = \int_0^\infty f(q+r\xi) \frac{\mathrm{d}r}{r},\ \xi \in \mathrm{S}^2,\ q \in Q$$

is well defined since f is smooth and vanishes on Q because of (ii). By Fubini's theorem

$$G(q,\sigma) \doteqdot \int_{\mathrm{S}(\sigma,\lambda)} \int_0^\infty f(q+\xi(s)r) \frac{\mathrm{d}r}{r} \mathrm{d}s = \int_{\mathrm{S}(\sigma,\lambda)} Zf(q,\xi(s))\, \mathrm{d}s,$$

where $\xi(s)$ runs over the circle $\mathrm{S}(\sigma,\lambda) \doteqdot C(\sigma,\lambda) \cap \mathrm{S}^2$ of radius $\rho = \sin\theta$ and the center $\cos\theta\sigma$. The planes containing these circles are tangent to the central ball B of radius $\rho = \cos\theta$.

8. Nongeodesic Funk transform

Theorem 5. *For arbitrary ρ, $0 \leqslant \rho < 1, \alpha \in E^3, |\alpha| \leqslant 1$, an arbitrary function $g \in C^2\left(\mathrm{S}^2\right)$ can be reconstructed from data of integrals*

$$\gamma(\sigma) = \int_{\langle \xi - \alpha, \sigma \rangle = \rho} g(\xi) \frac{\mathrm{d}\Omega}{\langle \sigma, \mathrm{d}\xi \rangle}, \ \sigma \in \mathrm{S}^2$$

by

$$g(\xi) = -\frac{|\xi - \alpha|}{2\pi^2} \int_{\mathrm{S}^2} \frac{\gamma(\sigma)}{\left(\langle \xi - \alpha, \sigma \rangle - \rho\right)^2} \mathrm{d}\Sigma \tag{16}$$

provided there exists a vector $\sigma_0 \in S^2$ such that $\langle \xi - \alpha, \sigma_0 \rangle \geqslant \rho$ on supp *g. Here $\mathrm{d}\Sigma$ is the euclidean measure on the sphere $\mathrm{S}^2 = \{|\sigma| = 1\}$.*

The singular integral in (16) is regularized as follows. Let φ be the zenith angle on S^2 with the pole $\eta \doteqdot (\xi - \alpha)|\xi - \alpha|^{-1}$ and θ is the azimuth angle. Write $\sigma = \sigma(\varphi, \theta) \in \mathrm{S}^2$ and consider the tangent field on S^2

$$\tau_\xi \doteqdot \frac{\langle \xi - \alpha, \sigma^\perp \rangle}{|\xi - \alpha, \sigma^\perp|^2} \left\langle \sigma^\perp, \frac{\partial}{\partial \sigma} \right\rangle = |\xi - \alpha| \sin^{-1} \varphi \frac{\partial}{\partial \varphi} \tag{17}$$

where $\sigma^\perp$ is any unit vector in the plane spanned by $\xi - \alpha$ and σ orthogonal to σ. We have

$$\tau_\xi\left(\langle \xi - \alpha, \sigma \rangle - \rho\right) = 1.$$

For an arbitrary $f \in C^2\left(\mathrm{S}^2\right)$, we have

$$\begin{aligned}\int_{\mathrm{S}^2} \tau_\xi f \mathrm{d}S &= \int\int \tau_\xi f \sin\varphi \mathrm{d}\varphi \mathrm{d}\theta = |\xi - \alpha|^{-1} \int_0^{2\pi} \int_0^\pi \frac{\partial f(\varphi, \theta)}{\partial \varphi} \mathrm{d}\varphi \mathrm{d}\theta \\ &= 2\pi |\xi - \alpha|^{-1} \left(f(\eta) - f(-\eta)\right).\end{aligned}$$

This yields

$$\begin{aligned}\int_{\mathrm{S}^2} \frac{\gamma(\sigma)\mathrm{d}S}{\left(\langle \xi - \alpha, \sigma \rangle - \rho\right)^2} &= -\int \tau_\xi \left(\frac{1}{\langle \xi - \alpha, \sigma \rangle - \rho}\right) \gamma(\sigma) \mathrm{d}S = \int \frac{\tau_\xi \gamma(\sigma) \mathrm{d}S}{\langle \xi - \alpha, \sigma \rangle - \rho} \\ &\quad - \frac{2\pi}{|\xi - \alpha|} \left(\frac{\gamma(\eta)}{|\xi - \alpha| - \rho} + \frac{\gamma(-\eta)}{|\xi - \alpha| + \rho}\right).\end{aligned}$$

Finally we can define the regularization of (16) as follows

$$\begin{aligned}g(\xi) = &-\frac{|\xi - \alpha|^2}{2\pi^2} \int_0^{2\pi} \mathrm{d}\theta \int_0^\pi \frac{\partial \gamma(\sigma)}{\partial \varphi} \frac{\mathrm{d}\varphi}{\langle \xi - \alpha, \sigma \rangle - \rho} \\ &+ \frac{1}{\pi} \left(\frac{\gamma(\eta)}{|\xi - \alpha| - \rho} + \frac{\gamma(-\eta)}{|\xi - \alpha| + \rho}\right)\end{aligned}$$

where the principal value of the inner integral is taken.

9. End of the proof

By (i) formula (16) can be applied to $\gamma(q,\sigma)$ for $\alpha = 0$, $\rho = \left(1+\lambda^2\right)^{-1/2} = \cos\theta$ and arbitrary $q \in Q$. This provides the reconstruction of $g(q,\xi) = Zf(q,\xi)$ for all $q \in Q$ and $\xi \in \mathrm{S}^2$. For any $x \in E^3$ and any unit orthogonal vectors ω, ξ, we have

$$\langle \omega, \nabla_\xi \rangle^2 f(q + r\xi) = r^2 \langle \omega, \nabla_q \rangle^2 f(q + r\xi)$$

which yields for any q,

$$\int_{\langle \omega,\xi\rangle = 0} \langle \omega, \nabla_\xi \rangle^2 Zf(q,\xi)\,\mathrm{d}\varphi = \int_0^{2\pi} \int_0^\infty \langle \omega, \nabla_\xi \rangle^2 f(q + r\xi)\,\frac{\mathrm{d}r}{r}\mathrm{d}\varphi$$

where $\mathrm{d}\varphi$ is the angular measure on the plane $\{\langle \omega, \xi\rangle = 0\}$. We have

$$\langle \omega, \nabla_\xi \rangle^2 f(q + r\xi) = r^2 \langle \omega, \nabla_q \rangle^2 f(q + r\xi)$$

and by Grangeat's trick [4] we get

$$\begin{aligned}\int_{\langle \omega,\xi\rangle = 0} \langle \omega, \nabla_\xi \rangle^2 Zf(q,\xi)\,\mathrm{d}\varphi &= \langle \omega, \nabla_q \rangle^2 \int_0^{2\pi} \int_0^\infty f(q + r\xi)\, r\mathrm{d}r\mathrm{d}\varphi \\ &= \frac{\partial^2}{\partial p^2} \int_{\langle \omega, q\rangle = p} f(q)\mathrm{d}S,\end{aligned}$$

where $\mathrm{d}S$ is the Euclidean area element. For an arbitrary $x \in \operatorname{supp} f$, we apply the Lorentz–Radon formula (see e.g. [3])

$$\begin{aligned} f(x) &= -\frac{1}{8\pi^2} \int_{\omega \in \mathrm{S}^2} \frac{\partial^2}{\partial p^2} \int_{\langle \omega, q - x\rangle = 0} f(q)\mathrm{d}q\mathrm{d}S \\ &= -\frac{1}{8\pi^2} \int_{\omega \in \mathrm{S}^2} \mathrm{d}S \int_{\langle \omega,\xi\rangle = 0} \langle \omega, \nabla_\xi \rangle^2 Zf(q(\omega),\xi)\,\mathrm{d}\varphi(\xi)\,. \end{aligned}$$

Here for any $\omega \in \mathrm{S}^2$, we choose a point $q = q(\omega) \in Q$ such that $\langle q(\omega) - x, \omega\rangle = 0$. It is possible since of (II). This completes the reconstruction. ►

Example. Suppose for simplicity that supp$f$ is contained in the ball B of radius 1 with the center at the origin. If S is the concentric sphere of radius $r = \left(1+\lambda^2\right)^{1/2}/\lambda$ and $\sigma(q) = -q/|q|$ for a point $q \subset S$, then $B \subset q + C(\sigma(q), \lambda)$ that is condition (II) of theorem 7 is fulfilled for this point. If a set $Q \subset S$ meets any plane H such that $\operatorname{dist}(H, 0) \leqslant 1$, then condition (I) is also satisfied and theorem 7 can be applied to Q.

10. Singular cone transform on E^4

The reconstruction in 4-space can be done as in sections 7–9. The only difference that the inversion of the nongeodesic transform γ on S^3 reads

Theorem 6. *If ρ, $0 \leqslant \rho < 1$, $\alpha \in E^4$, $|\alpha| \leqslant 1$, then an arbitrary function $g \in C^2\left(\mathrm{S}^3\right)$ can be reconstructed from data of integrals*

$$\gamma(\sigma) = \int_{\langle \xi - \alpha, \sigma\rangle = \rho} g(\xi)\,\frac{\mathrm{d}S}{\langle \sigma, \mathrm{d}\xi\rangle},\ \sigma \in \mathrm{S}^3$$

by

$$g(\xi) = -\frac{|\xi-\alpha|^2}{8\pi^2\left(|\xi-\alpha|^2-\rho^2\right)^{1/2}}\int_{\mathrm{S}^3}\delta''\left(\langle\xi-\alpha,\sigma\rangle-\rho\right)\gamma(\sigma)\,\mathrm{d}\Sigma$$

provided there exists a vector $\sigma_0 \in \mathrm{S}^3$ *such that* $\langle\xi-\alpha,\sigma_0\rangle \geqslant \rho$ *on* supp g.

For any smooth function $\gamma \in C^2\left(\mathrm{S}^3\right)$, the integral can be calculated as follows

$$\int_{\mathrm{S}^3}\delta''\left(\langle\xi-\alpha,\sigma\rangle-\rho\right)\gamma(\sigma)\,\mathrm{d}S = \int_{S_\xi}\tau_\xi^2\gamma(\sigma)\,\mathrm{d}S_\xi$$

where τ_ξ is the vector field on S_ξ as in (17). ►

Remark 4. The analytic reconstruction of an even function on S^2 from known great circle integrals was given by Paul Funk's [1] after uniqueness result of Hermann Minkowski 1905. Funk's reconstruction was generalized for higher dimensions by Helgason in 1959–2006 [5] and by Semyanistyi in 1961 [2]. The general result for arbitrary dimension containing theorems 5 and 6 was published in 2016 [20], p 76. Salman [21] proved the particular case $n = 2$, $\rho = 0$, $|\alpha| < 1$ see also Quellmalz [22].

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