Inverse Problems

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IOP Publishing Inverse Problems

Inverse Problems **32** (2016) 035003 (7pp) doi:10.1088/0266-5611/32/3/035003

An analytic method for the inverse problem of MREPT

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Received 19 October 2015, revised 1 December 2015
Accepted for publication 13 January 2016
Published 15 February 2016

CrossMark

Abstract

Magnetic resonance electric properties tomography (MREPT) is a medical imaging modality for visualizing the electrical tissue properties of the human body using radio-frequency magnetic fields. This method consists of reconstructing the admittivity distribution from the positive rotating component of the magnetic field. In the newest paper of Ammari *et al* (2015 *Inverse Problems* **31** 105001) an approximate method of reconstruction of variable admittivity was proposed. In this paper a method for exact reconstruction of the admittivity from data of the positive rotating component of the field is given.

Keywords: admittivity, Bernoulli equation, Riemann–Hilbert problem, Maxwell's equation

1. Introduction

Magnetic resonance electric properties tomography (MREPT) method is a development of electric impedance tomography (EIT) [2] and magnetic resonance EIT (MREIT) techniques [5, 6]. This technique can be applied to obtain high-resolution images of both the conductivity σ and permittivity ε distributions inside the human body. This method uses a time-harmonic magnetic field at the Larmor frequency ω inside an imaging object for determination of the admittivity $\kappa = \sigma + \mathrm{i}\omega\varepsilon$. The positive rotating component H^+ of the magnetic field can be measured by means of the technique called B_1 mapping. It was first suggested in the early nineties by Haacke [1], see later developments in [3, 4]. Determination of κ from data of H^+ was in focus of papers Seo *et al* [6], Song and Seo [7], see also the survey [8]. A method of numerical reconstruction of the variable admittivity was given in the recent paper of Ammari, Kwon, Lee, Kang and Seo [9] based on an optimization algorithm involving solution of a semi-elliptic equation with a small parameter. This method is approximate and time consuming.

We propose a simple analytic method of reconstruction κ from knowledge of H^+. The method is based on reduction to a Riemann–Hilbert problem for holomorphic functions.

2. Governing equation

Consider an object in a three-dimensional domain Ω that lies inside an MRI scanner with a constant magnetic field B_0. Choose an Euclidean coordinate system (x, y, z) such that the external field equals $B_0 = (0, 0, |B_0|)$. The time-harmonic magnetic field $H = (H_x, H_y, H_z)$ in Ω at the Larmor frequency ω relates to the admittivity $\kappa = \sigma + \mathrm{i}\omega\varepsilon$ of the object through the time-harmonic Maxwell's equations so-called 'Helmholtz equation' [6]

$$-\Delta H = \nabla \log \kappa \times (\nabla \times H) - \mathrm{i}\omega\mu_0\kappa H. \tag{1}$$

Here $\mu_0 = 4\pi 10^{-7}\, H\, m^{-1}$ is the magnetic permeability of free space, $\omega/2\pi = 128$ MHz is the Larmor frequency of the $3T$ MRI scanner. The magnetic permeability of the human body is close to μ_0. Equation (1) can be written in the form

$$J \times \nabla\kappa = h, \tag{2}$$

where

$$h(\kappa) \doteqdot \kappa\Delta H - \mathrm{i}\omega\mu_0\kappa^2 H \tag{3}$$

and the current J is can be found from Maxwell's equation

$$\nabla \times H = \kappa E = J.$$

3. Determination of admittivity from the total magnetic field

Suppose that a field H satisfying (1) is known on a domain Ω. The principal part of (2) has the singular matrix $J\times$, since $\langle J, J\times\rangle = 0$ and both sides vanish under scalar multiplication by J. The operator

$$P = -\langle J, J\rangle^{-2} J \times (J \times \cdot)$$

is the projection to the plane orthogonal to J. By (3), the field $p = Ph = -\langle J, J\rangle^{-2} J \times (J \times h)$ is a quadratic function of κ without a free term, that is $p = \kappa p_1 + \kappa^2 p_2$ for some fields p_1, p_2. We have

$$\nabla\kappa = p + qJ \tag{4}$$

for some unknown function $q = q(r)$, $r = (x, y)$. Let v be a smooth vector field in Ω orthogonal to J and Γ be a connected integral curve of v. Let $r = r(t)$, $0 \leqslant t \leqslant T$ be a parameterization of Γ such that $\partial r(t)/\partial t = v$. By (4), we obtain the equation has Bernoulli type

$$\frac{\partial\kappa(r)}{\partial t} = \langle \nabla\kappa(r), v\rangle = \langle p(r), v\rangle = \kappa\langle p_1(r), v\rangle + \kappa^2\langle p_2(t), v\rangle. \tag{5}$$

It can be solved for the unknown function $\lambda = \kappa^{-1}$, since $\kappa \neq 0$. By (5)

$$\frac{\partial\lambda}{\partial t} = -\lambda\langle v, p_1\rangle - \langle v, p_2\rangle. \tag{6}$$

The general solution of (6) along Γ is

$$\lambda(r(t)) = -\exp F(r(t))\left[\int_0^t \exp(-F)\langle v, p_2\rangle|_{r=r(s)}\,\mathrm{d}s + C\right],$$

where

$$F(r(t)) = \int_0^t \langle v, p_1 \rangle \mathrm{d}s$$

and C is an arbitrary constant. If the admittivity κ is known at a point $r = r(0)$, the constant C can be determined and the admittivity $\kappa = \lambda^{-1}$ is uniquely reconstructed on Γ. There are many smooth integral curves Γ through any point $r \in \Omega$, if the field J is smooth and does not vanish in Ω.

4. Conditional reconstruction from the positive rotating component

The field $H^+ = H_x + \mathrm{i}H_y$, called positive rotating component of H, can be determined from B_1 mapping method see [3, 4]. (It is not the case for the negative rotating component $H^- = H_x - \mathrm{i}H_y$ at present.) It is shown in [7], this component satisfies

$$\langle V^+, \nabla \log \kappa \rangle = \mathrm{i}\omega\mu_0 H^+ \kappa - \Delta H^+, \tag{7}$$

where $V^+ \doteqdot -(2\partial_\zeta H^+, 2\mathrm{i}\partial_\zeta H^+, \partial_z H^+)$, Δ is the Laplace operator in $\mathbb{R}^3$, and

$$\partial_\zeta = \frac{1}{2}\left(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}\right), \ \zeta = x + \mathrm{i}y, \ \partial_\zeta = \frac{\partial}{\partial \zeta}.$$

Equation (7) can be written in the form

$$\langle V^+, \nabla \log \kappa \rangle = \partial_\zeta H^+ \partial_{\bar\zeta} \log \kappa + \partial_z H^+ \partial_z \log \kappa,$$

where

$$\partial_{\bar\zeta} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\right).$$

Suppose that $\Omega = \Omega_0 \times [0, c]$ for some $c > 0$, where Ω_0 is a bounded connected and simply connected open set with C^1-boundary on $\mathbb{R}^2 \doteqdot \{z = 0\}$, e.g. a disc.

Theorem 1. *Suppose that a solution H^+ of (7) is known on a bounded convex open set $\Omega \subset \mathbb{R}^3$ such that*

$$\partial_\zeta H^+ \neq 0, \quad \partial_z H^+ = 0, \tag{8}$$

and admittivity κ is known on the boundary $\partial\Omega$. Then κ can be found on any region of interest $\Omega(c) = \Omega \cap \{z = c\}$ from solution of a Riemann–Hilbert boundary problem in $\Omega(c)$.

Proof. By (8) the basic equation (7) is reduced to *the equation of Bernoulli type*

$$\partial_{\bar\zeta}\kappa = \kappa^2 h_2 + \kappa h_1, \quad h_1 = -\frac{\Delta_3 H^+}{2\partial_\zeta H^+}, \quad h_2 = \frac{\mathrm{i}\omega\mu_0 H^+}{2\partial_\zeta H^+}. \tag{9}$$

The function $\lambda = \kappa^{-1}$ fulfils the equation

$$\partial_{\bar\zeta}\lambda = -\lambda h_1 - h_2, \tag{10}$$

which is defined on any transverse plane $\{z = \text{const}\}$. The general solution is

$$\lambda(x, y, z) = -\exp(F)\left(\frac{1}{\pi\zeta} * \exp(-F)h_2 + f\right),$$

where f is a holomorphic function on $\Omega(z)$ and

$$F(x, y, z) = \frac{1}{\pi\zeta} * h_1 = \frac{1}{\pi}\int\int \frac{h_1(x', y', z)\mathrm{d}x'\mathrm{d}y'}{x - x' + \mathrm{i}(y - y')}.$$

Note that the kernel of the convolution satisfies the classical equation

$$\partial_{\bar{\zeta}}\left(\frac{1}{\pi\zeta}\right) = \delta_0,$$

where δ_0 is the delta-function at the origin in $\mathbb{R}^2$. It follows from the classical formula

$$\delta_0 = \Delta_2\left(\frac{1}{4\pi}\log(x^2 + y^2)\right) = \frac{1}{\pi}\partial_{\bar{\zeta}}\partial_{\zeta}(\log\bar{\zeta} + \log\zeta) = \partial_{\bar{\zeta}}\frac{1}{\pi\zeta},$$

since $\Delta_2 = 4\partial_{\bar{\zeta}}\partial_{\zeta}$ on the complex plane. We have

$$\partial_{\bar{\zeta}}F = \partial_{\bar{\zeta}}\left(\frac{1}{\pi\zeta}\right) * h_1 = \delta_0 * h_1 = h_1,$$

$$\partial_{\bar{\zeta}}\left(\frac{1}{\pi\zeta} * \exp(-F)h_2\right) = \exp(-F)h_2$$

and (10) follows.

By (8), we have $\Delta_3 H^+ = \Delta_2 H^+ = 4\partial_{\bar{\zeta}}\partial_{\zeta}H^+$ and

$$\begin{aligned} F &= -\frac{2}{\pi\zeta} * \frac{\partial_{\bar{\zeta}}\partial_{\zeta}H^+}{\partial_{\zeta}H^+} = -\frac{2}{\pi\zeta} * \partial_{\bar{\zeta}}\log\partial_{\zeta}H^+ \\ &= -2\partial_{\bar{\zeta}}\left(\frac{1}{\pi\zeta}\right) * \log\partial_{\zeta}H^+ = -2\log\partial_{\zeta}H^+. \end{aligned}$$

Because of (8), the function $\log\partial_{\zeta}H^+$ is well defined on the convex set $\Omega(c)$ for any c. This implies $\exp(-F) = (\partial_{\zeta}H^+)^2$ and by (9)

$$\exp(-F)h_2 = \frac{1}{2}\mathrm{i}\omega\mu_0 H^+\partial_{\zeta}H^+ = \frac{\mathrm{i}\omega\mu_0}{4}\partial_{\zeta}(H^+)^2,$$

which yields

$$\lambda = -\frac{\mathrm{i}\omega\mu_0}{4}\frac{1}{\partial_{\zeta}(H^+)^2}\left(\frac{1}{\pi\zeta} * \partial_{\zeta}(H^+)^2 + f\right). \tag{11}$$

Suppose that the function $\mathrm{Re}\,\lambda = \mathrm{Re}\,\kappa^{-1}$ is known on the boundary $\partial\Omega$. The unknown function f satisfies the boundary condition

$$\mathrm{Re}(f\exp F)|_{\partial\Omega(c)} = -\mathrm{Re}\,\lambda + \mathrm{Re}\exp(F)\left(\exp(-F)h_2 * \frac{1}{\pi\zeta}\right)\Bigg|_{\partial\Omega(c)}, \tag{12}$$

where $c \in \mathbb{R}$ is the parameter. This is the Riemann–Hilbert type problem in $\Omega(c)$. According to the general theory [10] section 39, if $\mathrm{k} \geqslant -1$, problem (10)–(12) can be solved in an explicit form and is unique up to a linear combination of $\mathrm{k} + 1$ linearly independent solutions of the homogeneous problem. The number k is called the index of the problem and is equal to

$$\mathrm{k} = -\frac{1}{\pi}[\arg F]_{\partial\Omega(c)},$$

where $[\varphi]_L$ denotes the increment of a function φ along the curve L. According to this formula, the index vanishes, since the function F is regular and has no zeros on the convex set $\Omega(c)$. It follows that the solution of (12) is unique up to a constant function $f = \mathrm{i}c$, where c is real constant. The solution can be found by an explicit formula see addendum.

5. The general case

Without the assumption $\partial_z H^+ = 0$, equations (9) and (10) can be written in the form

$$\partial_{\bar\zeta}\kappa = h_1\kappa + h_2\kappa^2 + h_3\partial_z\kappa,\ h_3 = \frac{\partial_z H^+}{2\partial_\zeta H^+}.$$

Theorem 2. *Suppose that data of positive rotating components of two magnetic fields H and $\tilde H$ are available such that*

$$\frac{\partial_z \tilde H^+}{\partial_\zeta \tilde H^+} \neq \frac{\partial_z H^+}{\partial_\zeta H^+} \tag{13}$$

on a cylinder $\Omega = \Omega_0 \times [0, c]$. Then λ can be determined in Ω from solution of a Riemann–Hilbert problem in Ω_0.

Proof. Let

$$\partial_{\bar\zeta}\lambda = -h_1\lambda - h_2 + h_3\partial_z\lambda, \tag{14}$$

$$\partial_{\bar\zeta}\lambda = -\tilde h_2\lambda - \tilde h_2 + \tilde h_3\partial_z\lambda \tag{15}$$

be the basic equations for $\lambda = \kappa^{-1}$ obtained from data of H^+ and $\tilde H^+$. By subtracting, we get

$$0 = (\tilde h_1 - h_1)\lambda + (\tilde h_2 - h_2) + (h_3 - \tilde h_3)\partial_z\lambda,$$

$$\partial_z\lambda = \frac{h_1 - \tilde h_1}{h_3 - \tilde h_3}\lambda + \frac{h_2 - \tilde h_2}{h_3 - \tilde h_3},$$

where $h_3 - \tilde h_3 \neq 0$ according to (13). Solving the last equation yields

$$\lambda(r, z) = \exp(A(r, z))\int_0^z \exp(-A(r, t))\frac{h_2(r, t) - \tilde h_2(r, t)}{h_3(r, t) - \tilde h_3(r, t)}\mathrm{d}t + \lambda(r, 0),$$

where $r = (x, y)$ and

$$A(x, y, z) = \int_0^z \frac{h_1(r, t) - \tilde h_1(r, t)}{h_3(r, t) - \tilde h_3(r, t)}\mathrm{d}t.$$

It follows from (14) and (15)

$$\partial_{\bar\zeta}\lambda = \frac{\tilde h_3 h_1 - h_3\tilde h_1}{h_3 - \tilde h_3}\lambda + \frac{\tilde h_3 h_2 - h_3\tilde h_2}{h_3 - \tilde h_3}.$$

We set $z = 0$ and follow the method of section 4

$$\lambda(r, 0) = \exp\gamma(r)\left(\exp(-\gamma(r))g_2(r, 0) * \frac{1}{\pi\zeta} + f(x + \mathrm{i}y)\right),$$

where f is an arbitrary holomorphic function in Ω_0 and

$$G(r) = \frac{1}{\pi\zeta} * g_1(r, 0),\ g_1 = \frac{\tilde{h}_3 h_1 - h_3\tilde{h}_1}{h_3 - \tilde{h}_3},\ g_2 = \frac{\tilde{h}_2 h_3 - h_2\tilde{h}_3}{h_3 - \tilde{h}_3}.$$

To compute f, we consider the Riemann–Hilbert problem

$$\mathrm{Re}\,(f\exp G|_{\partial\Omega_0}) = \mathrm{Re}\,\lambda(r, 0)|_{r\in\partial\Omega_0} - \mathrm{Re}\exp G\left((\exp(-G)g_2) * \frac{1}{\pi\zeta}\right)\bigg|_{r\in\partial\Omega_0},$$

where data of λ on the skeleton $\{r \in \partial\Omega_0, z = 0\}$ of the domain Ω are supposed known. The index equals

$$k = -\frac{1}{\pi}[\arg G]_{\partial\Omega_0}.$$

6. Conclusion

It is shown that the governing equation (7) has solution (11) depending only on the positive rotating component of a magnetic field if the vertical derivative of the field may be neglected. The holomorphic function f is defined on each horizontal plane from the Riemann–Hilbert problem (12). In the general case, our method is based on data of two magnetic fields H^+, $\tilde{H}^+$ that are in the 'general position'. Instead of the family of Riemann–Hilbert problems, only one problem at $z = 0$ need to be solved. The explicit solution of the Riemann–Hilbert problem with zero index is given below.

7. Addendum: Riemann–Hilbert problem

Theorem 3. *Let ω be a bounded simply connected open set the complex plane with smooth boundary, γ and b be Hölder continuous functions on $\partial\omega$. If $[\arg\gamma]_{\partial\omega} = 0$, then a solution of the Riemann–Hilbert problem*

$$\mathrm{Re}\,(\gamma(\zeta)f(\zeta)) = b(\zeta),\ \zeta \in \partial\omega \tag{16}$$

can be found in an explicitly form. The solution is unique up to a pure imaginary constant.,

For the general case see [10] (be causious of misprints).

Proof. For simplicity, we assume that ω is the unit disc. Let g be an arbitrary Hölder continuous function on $\partial\omega$ such that $[\arg g] = 0$. The function $\log g$ is well defined on the boundary and the Cauchy integral

$$\varphi_\pm(z) = \frac{1}{2\pi\mathrm{i}}\int_{\partial\omega}\frac{\log g(\zeta)\mathrm{d}\zeta}{\zeta - z}$$

is well defined for $z \in \mathbb{C}\backslash\partial\omega$, where the curve $\partial\omega$ is oriented counter clockwise. We denote it by $\varphi_+(z)$ for $z \in \omega$ and by $\varphi_-(z)$ for $z \in \mathbb{C}\backslash\bar{\omega}$. By the Plemelj theorem [10], both functions have continuous limits on $\partial\omega$ and fulfils $\varphi_+ - \varphi_- = \log g$. This yields

$$\exp(\varphi_+) = g \exp(\varphi_-). \tag{17}$$

In the similar way, we define the functions

$$A_\pm(z) = \frac{1}{2\pi \mathrm{i}} \int_{\partial\omega} \frac{a(\zeta)\mathrm{d}\zeta}{(\zeta - z)\exp\varphi_+(\zeta)},$$

where $\varphi_+(\zeta)$ means the limit of φ_+ from ω, and set $\psi_\pm = \exp(\varphi_\pm)A_\pm$. By (17) we have on $\partial\omega$

$$\begin{aligned} \psi_+ - g\psi_- &= \exp(\varphi_+)A_+ - g\exp(\varphi_-)A_-, \\ &= \exp(\varphi_+)(A_+ - A_-) = a, \end{aligned} \tag{18}$$

where the equalion

$$A_+ - A_- = \frac{a}{\exp(\varphi_+)}$$

follows from the Plemelj theorem.

Set now $g = -\gamma^{-1}\bar{\gamma}$, $a = \gamma^{-1}b$ and have $[\arg g] = 0$. Define

$$f(z) = \psi_+(z) + \bar{\psi}_-(z^*),$$

where $\psi^*(z) = \psi_-(z^*)$, $z^* = 1/\bar{z}$. This function is holomorphic in ω and by (18)

$$\begin{aligned} 2\,\mathrm{Re}(\gamma f) &= \gamma f + \bar{\gamma}\bar{f} = \gamma(f + \gamma^{-1}\bar{\gamma}\bar{f}) = \gamma(f - g\bar{f}), \\ &= \gamma(\psi_+ - g\psi_- + \bar{\psi}_- - \bar{g}\bar{\psi}_+) = \gamma(a - \bar{g}\bar{a}), \\ &= \gamma a + \bar{\gamma}\bar{a} = 2\,\mathrm{Re}\,\gamma a = 2b \end{aligned}$$

on the boundary of ω, which yields (16). □

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