Remarks on the general Funk transform and thermoacoustic tomography

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Abstract. We discuss properties of a generalized Minkowski-Funk transform defined for a family of hypersurfaces. We prove two-side estimates for the integral operator and show that the range conditions can be written in terms of the reciprocal Funk transform. Some applications to the spherical mean transform are considered.

1 Families of hypersurfaces

Let X and Σ be smooth manifolds of dimension $n > 1$ and F be a closed smooth hypersurface in $X \times \Sigma$. We assume that

(*) *the natural projection* $\pi : F \to \Sigma$ *has rank* n *and the mapping* $g : F \to G^{n-1}T(X)$ *is a local diffeomorphism*, where $g(x, \sigma) = (x, \theta)$, θ is the tangent hyperplane to $F(\sigma)$ at x and $G^{n-1}T(X)$ denotes the variety of $n-1$-subspaces in the tangent bundle $T(X)$ of X.

It follows that the sets $F(\sigma) = \pi^{-1}(\sigma), \sigma \in \Sigma$ are hypersurfaces in X and for any point $x \in X$ and any tangent hyperplane $\theta \subset T_x(X)$ there is a (locally unique) hypersurface $F(\sigma)$ through x tangent to θ. We shall see below that the property (*) is in fact symmetric with respect to X and Σ.

If F is cooriented, we choose a smooth function Φ in $X \times \Sigma$ such that $\Phi = 0$ and $\mathrm{d}\Phi \neq 0$ on F (the *phase* function). If F is not cooriented, one can choose a set of local phase functions $\{\Phi_\alpha\}$ such that $\Phi_\beta = \pm\Phi_\alpha$ in any domain, where both functions Φ_α, Φ_β are defined.

Proposition 1.1 *The condition (*) is equivalent to the inequality* $\det J(\Phi) \neq 0$ *in* F, *where*

$$J(\Phi)=\begin{pmatrix} \frac{\partial^2\Phi}{\partial x_1\partial\sigma_1} & \cdots & \frac{\partial^2\Phi}{\partial x_1\partial\sigma_n} & \frac{\partial\Phi}{\partial x_1} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2\Phi}{\partial x_n\partial\sigma_1} & \cdots & \frac{\partial^2\Phi}{\partial x_n\partial\sigma_n} & \frac{\partial\Phi}{\partial x_n} \\ \frac{\partial\Phi}{\partial\sigma_1} & \cdots & \frac{\partial\Phi}{\partial\sigma_n} & \Phi \end{pmatrix},$$

and $x_1,...,x_n$, $\sigma_1,...,\sigma_n$ *are local coordinates in* X *and in* Σ, *respectively.*

Proof. Suppose that $\det J(\Phi)\neq 0$. Then also $\mathrm{d}_x\Phi\neq 0$ and $\mathrm{d}_\sigma\Phi\neq 0$ which implies that the projections π and $p:F\to X$ have rank n. Choose a point (x_0,σ_0) and take a tangent vector $\theta\neq 0$ to $F(\sigma_0)$ at x_0. The vector $(\theta,0)$ is tangent to F at (x_0,σ_0) and the map q is well defined. Change the coordinates $\sigma_i, i=1,...,n$ in Σ in such a way that $\mathrm{d}_\sigma\Phi(x_0,\sigma_0)=(1,0,...,0)$. This means that the field $\partial/\partial\sigma_1$ moves the point x_0 whereas the fields $\partial/\partial\sigma_2,...,\partial/\partial\sigma_n$ do not move the point but rotate the tangent hyperplane to $F(\sigma_0)$ at x_0. Choose coordinates x_i in such a way that $\mathrm{d}_x\Phi(x_0,\sigma_0)=(1,0,...,0)$. We have then

$$\det J(\Phi)=-\frac{\partial\Phi}{\partial x_1}\frac{\partial\Phi}{\partial\sigma_1}\det J',\ J'\doteq\left\{\frac{\partial^2\Phi}{\partial x_i\partial\sigma_j}\right\}_{i,j=2}^{n}.$$

The inequality $\det J'\neq 0$ implies that the forms $\partial\mathrm{d}_x\Phi(x_0,\sigma)/\partial\sigma_2,...,\partial\mathrm{d}_x\Phi(x_0,\sigma)/\partial\sigma_n$ are independent. This yields (*). Inverting the arguments we check that (*) implies $\det J(\Phi)\neq 0$. ►

Corollary 1.2 *The condition (*) implies that the projection* $p:F\to X$ *has rank* n *and the mapping* $\gamma:F\to G^{n-1}T(\Sigma)$ *is a local diffeomorphism, where* $\gamma(x,\sigma)=(t,\sigma)$, t *denotes the tangent hyperplane to* $F(x)$ *at* σ.

This follows from the symmetry of the condition $\det J(\Phi)\neq 0$.

2 The Funk transform

Let F be a hypersurface in $X\times\Sigma$ that fulfils (*) and Φ be a phase function of F. We define the Funk transform M for a density f in X with compact support by the integral

$$M\mathrm{f}(\sigma)\doteq\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{|\Phi(\cdot,\sigma)|\leq\varepsilon}\mathrm{f},\ \sigma\in\Sigma. \tag{1}$$

If X is oriented, then the limit is equal to the hypersurface integral

$$M\mathrm{f}(\sigma)=\int_{F(\sigma)}\frac{\mathrm{f}}{\mathrm{d}_x\Phi},\ \sigma\in\Sigma,$$

where the orientation of $F(\sigma)$ is determined by the phase function and $\omega = \mathrm{f}/\mathrm{d}_x\Phi$ is a differential $n-1$-form such that $\mathrm{d}_x\Phi \wedge \omega = \mathrm{f}$. The form ω is defined up to a term $\mathrm{d}_x\Phi \wedge \chi$, where χ is a $n-2$-form. Therefore the restriction of ω to $F(\sigma)$ is well-defined. In the general case, when X is not oriented and/or F is not cooriented, then this formula holds at least locally. The function $M\mathrm{f}$ is also continuous if f is so.

The Funk transform $M = M_\Phi$ does not depend essentially on the phase function: for any function Λ on $X \times \Sigma$ that does not vanish on F the equation $M_{\Lambda\Phi}(\Lambda \mathrm{f}) = M_\Phi(\mathrm{f})$ holds.

Example 1. Let $n > 1$ and $X = S^n$ and $\Sigma = S^n$ be unit spheres in Euclidean $n+1$-spaces and F be the hypersurface defined by the phase function $\Phi(x,\sigma) = x_0\sigma_0 + ... + x_n\sigma_n$. Check the condition (*). In the chart where $x_0 = \sigma_0 = 1$ we have

$$J = x_1\sigma_1 + ... + x_n\sigma_n = -x_0\sigma_0 = -1.$$

The same equation $J = -1$ holds in any chart $x_i = \sigma_j = 1, 0 \le i, j \le n$. The operator M is the classical Minkowski-Funk transform [4]. The same phase function Φ is defined up to the factor ± 1 in the product of the projective spaces $X = S^n/\mathbb{Z}_2, \Sigma = S^n/\mathbb{Z}_2$ and defines the hypersurface $P \subset X \times \Sigma$. The fibres $P(x) \subset \Sigma$ and $P(\sigma) \subset X$ of P are projective hyperplanes which are not coorientable. If n is odd they are not orientable either.

Example 2. The Radon transform R in $X = \mathbb{R}^n$ is another parameterization of the Minkowski-Funk transform. We set $\Sigma = S^{n-1} \times \mathbb{R}_+$ and take the phase function $\Phi(x; p, \omega) = \langle \omega, x \rangle - p$. Then

$$R(\mathrm{f})(p,\omega) = \left(1 + p^2\right)^{-1/2} M(\mathrm{g})(\sigma),$$

where $\mathrm{f} = f\mathrm{d}x,\ \mathrm{g} = g\mathrm{d}S \in H^0(S^n, \Omega)$,

$$g\left(\frac{1}{1+|x|^2}, \frac{x}{1+|x|^2}\right) = \left(1 + |x|^2\right)^{n/2} f(x),\ \sigma = \left(\frac{1}{(1+p^2)^{1/2}}, \frac{\omega p}{(1+p^2)^{1/2}}\right).$$

Example 3. Let $\mathbf{E}$ be an Euclidean space, X be an open set and Y be a hypersurface in $\mathbf{E}$. Take $\Sigma = Y \times \mathbb{R}_+$ and $\Phi(x; y, r) = |x - y| - r$ for $x \in X,\ y \in Y, r > 0$. The hypersurface $F = \{\Phi = 0\}$ is a family of spheres $F(y,r)$ in X and the Funk transform M coincides with the spherical integral transform:

$$M\mathrm{f}(y,r) = \int_{F(y,r)} \frac{\mathrm{f}}{\mathrm{d}_x\Phi} = \int_{F(y,r)} f\mathrm{d}S,\ \mathrm{f} \doteq f\mathrm{d}x.$$

Proposition 2.1 *The condition (*) is fulfilled for the family F of spheres in X, if and only if for any point $(x; y, r) \in F$ the vector $x - y$ is not parallel to the tangent space T_yY.*

Proof. Let $\sigma_1, ..., \sigma_{n-1}$ be local coordinates in Y and $\sigma_n = r$. We have in F

$$J = \left(-\frac{2}{r}\right)^{n+1} \begin{pmatrix} \frac{\partial y_1}{\partial \sigma_1} & ... & \frac{\partial y_1}{\partial \sigma_{n-1}} & 0 & x_1 - y_1 \\ . & ... & . & . & . \\ \\ . & ... & . & . & . \\ \frac{\partial y_n}{\partial \sigma_1} & ... & \frac{\partial y_n}{\partial \sigma_{n-1}} & 0 & x_n - y_n \\ \\ \sum (x_j - y_j) \frac{\partial y_j}{\partial \sigma_1} & ... & \sum (x_j - y_j) \frac{\partial y_j}{\partial \sigma_{n-1}} & r & 0 \end{pmatrix}$$

$$= 2^{n+1} r^{-n} \begin{pmatrix} \frac{\partial y_1}{\partial \sigma_1} & ... & \frac{\partial y_1}{\partial \sigma_{n-1}} & x_1 - y_1 \\ . & ... & . & . \\ \\ . & ... & . & . \\ \frac{\partial y_n}{\partial \sigma_1} & ... & \frac{\partial y_n}{\partial \sigma_{n-1}} & x_n - y_n \end{pmatrix}.$$

The right-hand side is different from zero, if and only if the vectors $\partial y / \partial \sigma_1, ..., \partial y / \partial \sigma_{n-1}, (x - y)$ are linearly independent. ▶

3 Estimate from above

Let X be a smooth manifold and α, r be some real numbers, $0 \leq r \leq 1$. We denote by $H^\alpha (X, \Omega^r)$ the Sobolev space of r-densities on X, which locally belong to the Sobolev class of order α. For a compact set $K \subset X$ the subspace $H_K^\alpha (X, \Omega^r)$ of r-densities supported by K form a Hilbert space with respect to the Sobolev-type norm $\|\cdot\|^\alpha$. Define the direct limit of Hilbert spaces

$$H_{\text{comp}}^\alpha (X, \Omega^r) \doteq \lim_K H_K^\alpha (X, \Omega^r).$$

The space $H^\alpha (X, \Omega^r)$ has the natural Fréchet topology. The natural bilinear form

$$(\mathrm{f}, \mathrm{g}) \mapsto \int_X \mathrm{fg}$$

generates the continuous map

$$H_{\text{comp}}^\alpha (X, \Omega^r) \times H^{-\alpha} \left(X, \Omega^{1-r}\right) \to \mathbb{C}. \tag{2}$$

It is a topological duality.

We simplify the notation for the two cases: $H^\alpha (X) \doteq H^\alpha (X, \Omega^0)$, $H^\alpha (X, \Omega) \doteq H^\alpha (X, \Omega^1)$.

Proposition 3.1 *For any hypersurface $F \subset X \times \Sigma$ that satisfies (*), an arbitrary compact set $K \subset X$, any real α, any $\psi \in D(\Sigma)$ and $\mathrm{f} \in H_K^\alpha (X, \Omega)$ the inequality*

$$\|\psi M \mathrm{f}\|^{\alpha + (n-1)/2} \leq C_{K,\psi} \|\mathrm{f}\|^\alpha \tag{3}$$

holds, where $C_{K,\psi}$ *does not depend on* f. *It follows that the Funk transform generates a bounded operator*

$$M : H_K^{\alpha}(X, \Omega) \to H^{\alpha+(n-1)/2}(\Sigma).$$

Proof. The Funk transform can be expressed as the oscillatory integral

$$M\mathrm{f}(\sigma) = \int_K \int_{\mathbb{R}} \exp(2\pi\imath\tau\Phi(x,\sigma))\, \mathrm{f}(x)\, \mathrm{d}\tau.$$

The critical set of the phase function $\tau\Phi(x,\sigma)$ is the hypersurface $F(\sigma)$ and the condition $\mathrm{d}_x\Phi \neq 0$ implies that the phase function is non-degenerate. The corresponding conic Lagrange manifold is

$$\mathrm{L} = \{(x,\sigma,\xi,\rho) \in T^*(X \times \Sigma), \Phi(x,\sigma) = 0, \rho = \lambda \mathrm{d}_\sigma \Phi, \ \xi = \lambda \mathrm{d}_x \Phi, \ \lambda \neq 0\}.$$

Lemma 3.2 *Rank of the matrix* $\frac{\partial(x,\xi)}{\partial(\sigma,\rho)}$ *is equal to* $2n$ *in any point of* Λ.

Proof. Suppose that the rank is less $2n$. Then there exists a vector $t = (t_x, t_\sigma, t_\xi, t_\rho)$ in $T^*(X \times \Sigma)$ tangent to L such that $t_\sigma = 0, t_\rho = 0$. This yields

$$t_x(\Phi) = 0, \ t_\rho = t_\lambda \mathrm{d}_\sigma \Phi + \lambda t_x(\mathrm{d}_\sigma \Phi), \ t_\xi = t_\lambda \mathrm{d}_x \Phi + \lambda t_x(\mathrm{d}_x \Phi)$$

for a tangent vector t_λ to $\mathbb{R}$. The first equation implies that the vector $(\lambda t_x, t_\lambda)$ fulfils $(\lambda t_x, t_\lambda)\Phi = 0$. By Proposition 1.1 this vector vanishes, that is $t_x = 0, \ t_\lambda = 0$. The second equation gives $t_\xi = 0$. ▶

By this Lemma the projections of L to $T^*(X)$ and to $T^*(\Sigma)$ are submersions, that is L is locally the graph of a canonical transformation. The symbol $a(x,\sigma,\xi,\rho) = 1$ is a homogeneous function of ξ, ρ of order 0. The order m of the Fourier integral operator M satisfies the equation $m + \dim X \times \Sigma/4 - N/2 = 0$, where $\dim X \times \Sigma = 2n$ and $N = 1$ is the number of variables τ. This yields $m = (1-n)/2$, which means that the functional

$$\psi \mapsto \int_\Sigma \int_X \int_{\mathbb{R}} \exp(2\pi\imath\tau\Phi(x,\sigma))\, \psi(x,\sigma)\, \mathrm{d}\tau$$

defined for a smooth density $\psi \in D(X \times \Sigma)$, is a distribution of the class $I^{(1-n)/2}(X \times \Sigma, \mathrm{L})$ in the sense of Hörmander [6]. By Corollary 25.3.2 loc. cit. the operator ψM defines a continuous map $H_K^\alpha(X, \Omega^{1/2}) \to H_\Lambda^{\alpha+(n-1)/2}(\Sigma, \Omega^{1/2})$ for any real α, where $\Lambda = \operatorname{supp}\psi$. ▶

We say that F is *proper over* X, if the natural map $p : F \to X$ is proper. If a density f is supported in a compact set $K \Subset X$, then Mf is supported in the compact set $\Lambda \doteq \pi(p^{-1}(K)) \Subset \Sigma$. Proposition 3.1 now yields

Corollary 3.3 *If* F *fulfils (*) and is proper over* X, *the Funk transform can be extended to a bounded operator* $M : H_K^\alpha(X, \Omega) \to H_\Lambda^{\alpha+(n-1)/2}(\Sigma)$ *for any* $\alpha \in \mathbb{R}$, *an arbitrary compact set* $K \subset X$ *and* $\Lambda = \pi(p^{-1}(K))$.

4 Reciprocal Funk transform and backprojection

Define the reciprocal Funk transform for a density φ in Σ with compact support as follows

$$M^{\circ}\varphi(x) = \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{|\Phi(x,\cdot)|\le\varepsilon}\varphi = \int_{F(x)}\frac{\varphi}{\mathrm{d}_{\sigma}\Phi}. \tag{4}$$

By Proposition 3.1 it can be extended to a continuous operator $M^{\circ}: H_{\Lambda}^{\alpha}(\Sigma,\Omega) \to H^{\alpha+(n-1)/2}(X)$ for any α.

Corollary 4.1 *If F fulfils (*) and is proper over X, the reciprocal transform defines for any α a continuous operator $M^{\circ}: H^{\alpha}(\Sigma,\Omega) \to H^{\alpha+(n-1)/2}(X)$.*

Example 1 (revisited). The reciprocal operator M° for the Minkowski-Funk transform M coincides with M through the natural isomorphism $X \cong \Sigma$. The reciprocal operator R° for the Radon transform is

$$R^{\circ}\phi(x) = \int_{p=\langle x,\omega\rangle}\frac{\phi}{\mathrm{d}_{\sigma}(p-\langle x,\omega\rangle)} = \int_{S}\phi(\langle x,\omega\rangle,\omega)\,\mathrm{d}\omega,$$

where $\phi = \phi\mathrm{d}p\mathrm{d}\omega$ and $S = \{\omega : |\omega| = 1\}$.

Proposition 4.2 *The operator $-M^{\circ}$ is dual to $M: H_{\mathrm{comp}}^{-\alpha-(n-1)/2}(X,\Omega) \to H^{-\alpha}(\Sigma)$.*

Proof. We have

$$\langle M\mathrm{f},\varphi\rangle = \int_{\Sigma} M(\mathrm{f})\,\bar{\varphi} = \int_{\Sigma}\bar{\varphi}\int_{F(\sigma)}\frac{\mathrm{f}}{\mathrm{d}_x\Phi} = \int_{F}\frac{\mathrm{f}\wedge\bar{\varphi}}{\mathrm{d}_x\Phi} = -\int_{F}\mathrm{f}\wedge\frac{\bar{\varphi}}{\mathrm{d}_{\sigma}\Phi},$$

since $\mathrm{d}\Phi = \mathrm{d}_x\Phi + \mathrm{d}_{\sigma}\Phi = 0$ on F. The right-hand side equals

$$-\int_{X}\mathrm{f}\int_{F(x)}\frac{\bar{\varphi}}{\mathrm{d}_{\sigma}\Phi} = -\int_{X}\mathrm{f}\,M^{\circ}(\bar{\varphi}) = -\langle\mathrm{f},M^{\circ}\varphi\rangle. \blacktriangleright$$

Definition. Fix some volume forms $\mathrm{d}X$ in X and $\mathrm{d}\Sigma$ in Σ. The *back projection* operator

$$M^{*}: g \mapsto M^{\circ}(g\mathrm{d}\Sigma)\,\mathrm{d}X$$

transforms a function g defined in Σ to a density $M^{*}g$ in X.

Definition. We say that points $x,y\in X$ are *conjugate in* F, if $x\neq y$ and the form $\mathrm{d}_{\sigma}\Phi(x,\sigma)\wedge\mathrm{d}_{\sigma}\Phi(y,\sigma)$ defined in Σ vanishes. This condition is equivalent to the "Bolker condition" for a general double fibration in the sense of V. Guillemin [5].

Theorem 4.3 *Suppose that F fulfils (*), is proper over X and has no conjugate points. Then the composition*

$$M^{*}M: H_{\mathrm{comp}}^{\alpha}(X,\Omega) \to H^{a+n-1}(\Sigma)$$

is an elliptic PDO of order $1-n$.

A similar statement is due to Guillemin [5].

Proof. Write $\mathrm{f}(y) = f_0(y)\,\mathrm{d}Y$ and calculate

$$\frac{M^* M \mathrm{f}}{\mathrm{d}X} = \int_{F(x)} \frac{\mathrm{d}\Sigma}{\mathrm{d}_\sigma \Phi} \int_{F(\sigma)} \frac{\mathrm{f}(y)}{\mathrm{d}_y \Phi(y,\sigma)}$$
$$= \int_{\Phi(x,\sigma)=0} \frac{\mathrm{d}\Sigma}{\mathrm{d}_\sigma \Phi(x,\sigma)} \int_{\Phi(y,\sigma)=0} \frac{\mathrm{d}Y}{\mathrm{d}_y \Phi(y,\sigma)} f_0(y)$$
$$= -\int_{\Phi(x,\sigma)=0} \int_{\Phi(y,\sigma)=0} \frac{\mathrm{d}\Sigma}{\mathrm{d}_\sigma \Phi(x,\sigma) \wedge \mathrm{d}_\sigma \Phi(y,\sigma)} f_0(y)\,\mathrm{d}Y,$$

since $\mathrm{d}\Phi = \mathrm{d}_y \Phi + \mathrm{d}_\sigma \Phi = 0$ in Y. We can write the right-hand side as $\int A(x,y)\,\mathrm{f}(y)$, where

$$A(x,y) = -\int_{F(x)\cap F(y)} \frac{\mathrm{d}\Sigma}{\mathrm{d}_\sigma \Phi(x,\sigma) \wedge \mathrm{d}_\sigma \Phi(y,\sigma)}. \tag{5}$$

The dominator $\mathrm{d}_\sigma \Phi(x,\sigma) \wedge \mathrm{d}_\sigma \Phi(y,\sigma)$ does not vanishes for $y \neq x$, since conjugate points are absent. Therefore the quotient in the right-hand side of (5) is a well defined smooth function except for the diagonal. Near the diagonal we can write $\Phi(y,\sigma) = \Phi(x,\sigma) + \sum (y_i - x_i)\,\partial I(x,\sigma)/\partial x_i + O\left(|y-x|^2\right)$ and

$$\mathrm{d}_\sigma \Phi(x,\sigma) \wedge \mathrm{d}_\sigma \Phi(y,\sigma) = \sum_i (y_i - x_i)\,\mathrm{d}_\sigma \Phi(x,\sigma) \wedge \mathrm{d}_\sigma \frac{\partial \Phi(x,\sigma)}{\partial x_i} + O\left(|y-x|^2\right).$$

The forms $\mathrm{d}_\sigma \Phi(x,\sigma) \wedge \mathrm{d}_\sigma \partial \Phi(x,\sigma)/\partial x_i, i = 1,...,n$ are linearly independent, because of Proposition 1.1. Therefore the product $\mathrm{d}_\sigma \Phi(x,\sigma) \wedge \mathrm{d}_\sigma \Phi(y,\sigma)$ is bounded by $c\,|x-y|$ from below as $y \to x$. Therefore the integral (5) is well defined. By the conditions the projection $q : F \to G^{n-1}T(X)$ is a proper submersion. Therefore we have $A(x,y) = a(x)\,|x-y|^{-1} + O(1)$ near the diagonal, where a is a smooth non-vanishing function. This implies that $M^* M$ is a classical integral operator on K with weak singularity, moreover it is a pseudodifferential operator of order $1-n$. It is an elliptic operator, since the symbol $a(x)$ does not vanish. ▶

Remark. The operator $-M^* M$ is positive with respect to the scalar product

$$\langle \mathrm{f}, \mathrm{g} \rangle = \int \frac{\mathrm{f}\bar{\mathrm{g}}}{\mathrm{d}X}$$

defined on compactly supported densities in X. Indeed by Proposition 4.2, we have

$$-\langle M^* M \mathrm{f}, \mathrm{f} \rangle = -\int M^* M \mathrm{f} \frac{\bar{\mathrm{f}}}{\mathrm{d}X} = -\int M^\circ (\mathrm{d}\Sigma M \mathrm{f})\,\bar{\mathrm{f}} = \int_\Sigma |M\mathrm{f}|^2\,\mathrm{d}\Sigma \geq 0.$$

5 Estimates from below

Theorem 5.1 *Suppose that a hypersurface F satisfies (*), is proper over X and has no conjugate points. Then for an arbitrary compact set $K \subset X$ and arbitrary $\alpha > \beta$ the* *estimate*

$$\|\mathrm{f}\|^{\alpha} \leq C_{\alpha} \|M\mathrm{f}\|^{\alpha+(n-1)/2} + C_{\beta} \|\mathrm{f}\|^{\beta} \tag{6}$$

holds for densities f *supported in* K *for some constants* C *and* C'.

Proof. Take a function $\phi \in D(X)$ such that $\phi = 1$ on K. By Proposition 3.1 ϕM^{*} is $(n-1)/2$-smoothing operator, which yields

$$\|\phi M^{*} M\mathrm{f}\|^{\alpha+n-1} \leq C \|M\mathrm{f}\|^{\alpha+(n-1)/2}. \tag{7}$$

By Theorem 4.3 the operator $\phi M^{*} M$ is elliptic in K, therefore the standard inequality holds

$$\|\mathrm{f}\|^{\alpha} \leq C_{\alpha,\beta} \|\phi M^{*} M\mathrm{f}\|^{\alpha+n-1} + C'_{\alpha,\beta} \|\mathrm{f}\|^{\beta}$$

for an arbitrary β and some constants $C_{\alpha,\beta}, C'_{\alpha,\beta}$. Taking in account (7) yields (6). ▶

Remark. Suppose that for some $\beta < \alpha$ the equation $M\mathrm{f} = 0, \mathrm{f} \in H_{K}^{\beta}(X, \Omega)$ implies $\mathrm{f} = 0$. Then the term $\|\mathrm{f}\|^{\beta}$ in (6) can be removed and the two-side estimate holds:

$$c_{\alpha} \|\mathrm{f}\|^{\alpha} \leq \|M\mathrm{f}\|^{\alpha+(n-1)/2} \leq C_{\alpha} \|\mathrm{f}\|^{\alpha}. \tag{8}$$

Mukhometov's result [8] contains the estimate

$$\|\mathrm{f}\|^{0} \leq C \|M\mathrm{f}\|^{1},$$

which implies injectivity of M for any family F that satisfies the following condition: any curve $F(\sigma) \cap X_0$ reaches ∂X_0 in two points in non tangent directions. However the order of smoothness is not sharp.

Estimates for the Sobolev norms are well known for the Radon transform. Inequalities for lateral derivatives of order $\alpha + 1/2$ were obtained by several authors for the plane case. Natterer [9] has shown that (3) holds also for angular derivatives. For the attenuated Radon transform see Rullgard [15].

Our approach is similar to that of Lavrent'ev and Bukhgeim [7], where the composition $M^{*}M$ was described as an integral operator in the local case. Guillemin [5] has defined a generalized Radon transform R for an arbitrary double fibration. This transform is treated as an elliptic Fourier integral operator and $R^{*}R$ is shown to be an elliptic PDO under the "Bolker condition". More details are given in the paper of Quinto [14]. D. Popov [12] studied the problem of inversion of a generalized Radon transform for a family of curves close to the family of plane affine lines.

6 Duality and range conditions

Theorem 6.1 *Suppose that* F *satisfies (*), is proper over* X *has no conjugate points. Then for an arbitrary compact set* K *in* X *and an arbitrary* α *the image of the Funk* *operator as above is closed and coincides with the subspace of functions* $\varphi \in H_{\Lambda}^{\alpha+(n-1)/2}(\Sigma)$ *such that* $\langle \mathrm{g}, \varphi \rangle = 0$ *for any solution* $\mathrm{g} \in H^{-\alpha-(n-1)/2}(\Sigma, \Omega)$ *of the equation*

$$M^{\circ}\mathrm{g}(x) = 0, \ x \in K. \tag{9}$$

Proof. By Theorem 5.1 the range of the operator M in Corollary 3.3 is closed, hence it coincides with the polar set of the kernel of the dual operator $M' : \left(H_{\Lambda}^{\alpha+n-1/2}(\Sigma)\right)' \to (H_K^{\alpha}(X, \Omega))'$. Any continuous functional on $H_{\Lambda}^{\alpha+(n-1)/2}(\Sigma)$ can be extended to a continuous functional G on $H_{\mathrm{comp}}^{\alpha+(n-1)/2}(\Sigma)$, which can be represented by a density $\mathrm{g} \in H^{-\alpha-(n-1)/2}(\Sigma, \Omega)$. By Proposition 4.2 the dual operator is equal to $-M^{\circ}$. The inclusion $\mathrm{g} \in \mathrm{Ker}\, M'$ means that $M^{\circ}\mathrm{g} \in H^{-\alpha}(X)$ vanishes on the subspace $H_K^{\alpha}(X, \Omega)$, which is equivalent to (9). ►

7 Thermoacoustic tomography

The mathematical problem of the thermoacoustic tomography is reconstruction of a function of two or three variables from data of spherical integrals, see the survey [11]. Consider first the case of complete acquisition geometry. Let X be the open ball of radius R in an Euclidean space $\mathbf{E}$ of dimension n, Y be the boundary of X, $\Sigma = Y \times \mathbb{R}_+$. Consider the family of spheres F given in $X \times \Sigma$ by the equation $\Phi(x; y, r) = 0$, where $\Phi(x; y, r) \doteq |x - y| - r$ as in Example 3. The hypersurface F is proper over X and there are no conjugate points.

For the Funk operator M defined in this geometry the range conditions of various form were given in the papers [10], [1], [3]. We write a series of necessary range conditions of explicit form that look different. Consider the reciprocal transform

$$M^{\circ}\phi(x) = \int \frac{\phi}{\mathrm{d}_{y,r}\Phi} = \int_{|x-y|=r} \varphi \mathrm{d}Y,$$

where $\phi = \varphi \mathrm{d}Y \mathrm{d}r$ and $\mathrm{d}Y$ denotes the Euclidean surface element of the sphere Y. Theorem 6.1 yields

Corollary 7.1 *For an arbitrary compact set* $K \subset X$ *and an arbitrary* $\alpha \in \mathbb{R}$ *the image of the Funk operator* $M : H_K^{\alpha}(X, \Omega) \to H_{\Lambda}^{\alpha+(n-1)/2}(\Sigma)$ *coincides with the set of functions* $g \in H_{\Lambda}^{\alpha+(n-1)/2}(\Sigma)$ *such that*

$$\int_{\Sigma} g\phi = 0 \tag{10}$$

for any density $\phi = \varphi \mathrm{d}Y \mathrm{d}r$*, where* $\varphi \in H^{-\alpha-(n-1)/2}(\Sigma)$ *and*

$$\int_{|x-y|=r} \varphi \mathrm{d}Y = 0, \ x \in \mathrm{int}K. \tag{11}$$

We take for K the closed ball of radius $\rho < R$, then the set $\Lambda = \pi\left(p^{-1}(K)\right) \subset \Sigma$ is given by $r \geq R - \rho$. The manifold $F(x)$ is the intersection of the cone surface $|y - x| = r$ with the cylinder Σ. This intersection is contained in the hyperplane

$$P(x) = \left\{y, s; 2\langle x, y\rangle + s = |x|^2 + R^2\right\},$$

where we use the variable $s = r^2$. Thus the condition (11) means vanishing of integrals of $\varphi \mathrm{d}Y$ over intersections of Σ with the hyperplanes $P(x)$, $|x| \leq R$ in the space $\mathbf{E} \times \mathbb{R}$. Note that the envelope of the family $\{P(x)\}$ is the paraboloid $\{y, s; s + |y|^2 = R^2\}$.

Suppose that φ is a polynomial in $s : \varphi(y, s) = \sum_k \varphi(y)\left(s - R^2\right)^k$. We have then

$$\int_{F(x)} \varphi \mathrm{d}Y = \sum_k \int_Y \varphi_k(y)\left(|x|^2 - 2\langle x, y\rangle\right)^k \mathrm{d}Y(y).$$

Set $x = tz$ for $|z| = 1$ and $0 \leq t < 1$ and develop the right-hand side in powers of $t :$

$$\begin{aligned}
\int_{F(x)} \varphi \mathrm{d}Y &= \sum \int_Y \varphi_k(y)\left(t^2 - 2t\langle z, y\rangle\right)^k \mathrm{d}Y \\
&= \int \varphi_0(y)\,\mathrm{d}Y - t \int 2\varphi_1(y)\langle z, y\rangle\,\mathrm{d}Y \\
&+ t^2 \int \left[\varphi_1(y) + 4\varphi_2(y)\langle z, y\rangle^2\right] \mathrm{d}Y \\
&- t^3 \int \left[4\varphi_2(y)\langle z, y\rangle + 8\varphi_3(y)\langle z, y\rangle^3\right] \mathrm{d}Y \\
&+ t^4 \int \left[\varphi_2(y) + 12\varphi_3(y)\langle z, y\rangle^2 + 16\varphi_4(y)\langle z, y\rangle^4\right] \mathrm{d}Y \\
+ \ldots &= 0.
\end{aligned}$$

The right-hand side must vanish for all t which implies a triangular system of equations. Setting $\varphi_i = 0$ for all $i > k$ for some k, we get a system of $k + 1$ equation with the unknowns $\varphi_0, \ldots, \varphi_k$, where only the moments

$$\int \varphi_k(y)\langle z, y\rangle^j \,\mathrm{d}Y, \;\; |z| = 1, \;\; j \leq k$$

are involved. There are only such $\binom{n+k-1}{n-1}$ linearly independent moments, hence one can find infinitely many independent solutions. Then the function

$$\varphi(y, s) = \sum_0^k \varphi_i(y)\left(s - R^2\right)^i$$

fulfils (11) and is orthogonal to the range of M. In particular, we can take for φ_0, φ_1 arbitrary functions on the sphere with zero average such that the function φ_1 has zero linear moments and set $\varphi_k = 0$ for $k > 1$ etc.

8 Partial scan

The case of non-complete acquisition geometry is more realistic in applications, but the analysis is more complicated. Let $X = \{x \in \mathbf{E},\ |x| < 1, x_1 \geq 0\}$ be the half-ball in an Euclidean space $\mathbf{E}$, $\Sigma \doteq S_\varepsilon \times \mathbb{R}_+$, where $S_\varepsilon = \{y, |y| = R, y_1 > -\varepsilon\}$ for some $R > 1$ and $\varepsilon > 0$. Let F be the family of spheres centered on S_ε This family satisfies (*), has no conjugate points but is not proper over X. Reconstruction algorithms were proposed by Popov and Sushko [13] and by Rullgård [15] for such acquisition geometry. Jan Boman [2] studied stability of reconstruction.

Take a smooth function $\psi \in D(\Sigma)$ such that $\psi(y,r) = 1$ for $-\varepsilon/2 \leq y_1 \leq 1$ and $\psi(y,r) = 0$ for $y_1 < -\varepsilon$. Arguing as in Theorems 4.3 and 5.1 we show the inequality

$$\|\mathrm{f}\|^{\alpha} \leq C \|\phi M^* \psi M \mathrm{f}\|^{\alpha+n-1} + C' \|\mathrm{f}\|^{\beta}$$

for an arbitrary $\mathrm{f} \in H^{\alpha}_{K}(X, \Omega)$ and any $\alpha > \beta$, where $\phi \in D(X)$, $\phi = 1$ in $\operatorname{supp} \mathrm{f}$. The Holmgren uniqueness theorem implies that $\operatorname{Ker} M = 0$, and the second term can be omitted. Then Proposition 3.1 gives the estimate

$$\|\mathrm{f}\|^{\alpha} \leq C \|\psi M \mathrm{f}\|^{\alpha+(n-1)/2},$$

which implies stability of reconstruction.

9 Kaczmarz method

The Kaczmarz method can be adopted for inversion of the general Funk transform. Let again $F \subset X \times \Sigma$ be a hypersurface that fulfils (*) and K be a compact set in X. We want to find a solution $\mathrm{f} \in L_2(K)$ of the equation

$$M\mathrm{f} = \varphi \tag{12}$$

for a function $\varphi \in L_2(\Lambda)$. Fix an area form $\mathrm{d}X$ in X and consider the operator $MM^* : L_2(\Lambda) \to L_2(\Lambda)$. It is non-positive; set $R = -MM^* + \varepsilon I$, where I is the identity operator. The operator R is positive and invertible. Following [9], we choose a real parameter ω and set $Q \doteq I - \omega M^* R^{-1} M$.

Lemma 9.1 *We have for* $0 < \omega < 2$ *and any* $\mathrm{g} \in L_2(K)$ *such that* $M\mathrm{g} \neq 0$

$$\|Q\mathrm{g}\| < \|\mathrm{g}\|$$

Proof. We have by Proposition 4.2

$$\begin{aligned}\|Q\mathrm{g}\|^2 &= \|\mathrm{g}\|^2 - 2\omega \left\langle \mathrm{g}, M^* R^{-1} M\mathrm{g} \right\rangle + \omega^2 \left\langle M^* R^{-1} M\mathrm{g}, M^* R^{-1} M\mathrm{g} \right\rangle \\ &= \|\mathrm{g}\|^2 - 2\omega \left\langle M\mathrm{g}, R^{-1} M\mathrm{g} \right\rangle - \omega^2 \left\langle R^{-1} M\mathrm{g}, M^* M R^{-1} M\mathrm{g} \right\rangle \\ &= \|\mathrm{g}\|^2 - 2\omega \left\langle M\mathrm{g}, R^{-1} M\mathrm{g} \right\rangle + \omega^2 \left\langle R^{-1} M\mathrm{g}, M\mathrm{g} \right\rangle - \omega^2 \varepsilon \left\langle R^{-1} M\mathrm{g}, R^{-1} M\mathrm{g} \right\rangle \\ &\leq \|\mathrm{g}\|^2 - \omega(2-\omega) \left\langle M\mathrm{g}, R^{-1} M\mathrm{g} \right\rangle.\end{aligned}$$

The last term does not vanish if $M\mathrm{g} \neq 0$. ▶

Take an arbitrary density f^0 and construct the sequence $\mathrm{f}^k, k = 1, 2, ...$ by means of the recurrent formula

$$\mathrm{f}^{k+1} = \mathrm{f}^k + \omega M^* R^{-1} \left(\varphi - M\mathrm{f}^k\right).$$

Theorem 9.2 *If M is injective and φ fulfils the range conditions, we have $\mathrm{f}^k \to \mathrm{f}$, where f is a solution of (12).*

Proof. We have

$$\begin{aligned} Q\left(\mathrm{f}^k - \mathrm{f}\right) &= \mathrm{f}^k - \omega M^* R^{-1} M\mathrm{f}^k - \mathrm{f} + \omega M^* R^{-1} M\mathrm{f} \\ &= \mathrm{f}^k + \omega M^* R^{-1} \left(\varphi - M\mathrm{f}^k\right) - \mathrm{f} = \mathrm{f}^{k+1} - \mathrm{f}. \end{aligned}$$

It follows that

$$\left\|\mathrm{f}^{k+1} - \mathrm{f}\right\| < \left\|\mathrm{f}^k - \mathrm{f}\right\| < ... < \left\|\mathrm{f}^0 - \mathrm{f}\right\|$$

and $\mathrm{f}^k \to \mathrm{g}$ strongly in $L_2(K)$. We have $\|Q(\mathrm{g} - \mathrm{f})\| = \|\mathrm{g} - \mathrm{f}\|$, which yields $\mathrm{g} = \mathrm{f}$ by Lemma. ▶

Remark. One can take for R any invertible operator that fulfils $R \geq -MM^*$.

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