

Reconstruction of a differential form from Doppler transform

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Abstract: A vector field in n -space is reconstructed from a n -dimensional family of ray or line integrals. A similar problem is considered for differential forms of higher order.

Key words: Differential form, ray integral, k -plane integral, reconstruction, support condition

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Abbreviated title: Doppler transform

1 Introduction

Integrals of a vector field model measurements in tomographic imaging of moving mediums, in particular, imaging of liquid or gas flows, tumor detection, optics and plasma physics etc.; see Braun and Hauck [1], Sparr *et al* [13], Osman and Prince [9], Wells *et al* [15], Sielschott [11] and references therein. In this context, the data of longitudinal line integrals of a vector field form is called the Doppler transform according to the Doppler shift law. See some basic mathematical properties in Natterer and Wübbeling [8]. The longitudinal measurement of a field can be interpreted as a line integral of the corresponding 1-differential form f . The Doppler transform is invariant with respect to the gauge transformation $f + da$, since all line integrals vanish for the exact form da (irrotational part of the field). The differential df of the form f is gauge invariant. By means the Helmholtz theorem reconstruction of the form df this is equivalent to reconstruction of the solenoidal part of the corresponding vector field. In 2D case the reconstruction problem is immediately reduced to inversion of the Radon transform; see Norton [7], Howard [6].

In the 3D case the complete 4D-data of line integrals are redundant; see Gelfand *et al* [3] for consistency conditions. The problem is to find a 3D subvariety in the data space that is sufficient for a reconstruction of df . For example, the variety of lines that are parallel to either of three (or of two) given planes has dimension 3; a reconstruction can be done by reduction to 2D case; see Schuster [10]. Vertgeim [14] has shown that df is uniquely determined from ray integrals $g(L)$ of f for the family of rays L with vertices on a curve Γ of arbitrary shape that touches any plane in, at least, three points. Two intersection points are sufficient for reconstruction according to Denisjuk [2] (p. 405), see more details in Sharafutdinov [12]. We show here that even one intersection point is sufficient, if the first derivatives of g are known. In particular, the reconstruction is possible on any chord of a curve Γ provided the derivatives of g are known for all lines that meet Γ . The main step in our method is reconstruction of hyperplane integrals of components of the form df , a similar method was used by Grangeat [4] in the scalar case. We extend this result for differential forms of degree 2.

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2 Line integrals of differential forms

Fix a real vector space $\mathbf{V} \cong \mathbb{R}^n$. Let $f = \sum f_{i_1 \dots i_k} dx_{i_1} \wedge \dots \wedge dx_{i_k}$ be a k -differential form in \mathbf{V} . We write this form as the function $f(x; \theta_1, \dots, \theta_k) = \sum f_{i_1 \dots i_k}(x) \theta_1^{i_1} \dots \theta_k^{i_k}$ defined for $x \in \mathbf{V}$ and $\theta_i = (\theta_i^1, \dots, \theta_i^n) \in \mathbf{V}$, $i = 1, \dots, k$. It is a polylinear and skew-symmetric function of $\theta_1, \dots, \theta_k$.

We say a function a defined on \mathbf{V} is fast decreasing, if $a(x) = O(|x|^{-q})$ for $q = 0, 1, 2, \dots$ and a norm $|\cdot|$ on \mathbf{V} . We say that a form f belongs to the class \mathbf{S}_m for some natural m , if the function $f(x; \theta_1, \dots, \theta_k)$ has continuous fast decreasing derivatives up to the order m for any fixed $\theta_1, \dots, \theta_k$.

The integral transform of a 1-form $f \in \mathbf{S}_0$ is the function

$$g(L) = \int_L f$$

defined on the set of oriented lines $L \subset \mathbf{V}$, we write also $I(f) \doteq g$. We have $I(da) = 0$ for any fast decreasing function a . Let $L = L(x, \theta)$ be the line given by the equation $y = x + t\theta$, $t \in \mathbb{R}$ oriented by the vector $\theta \in \mathbf{V}$; we shall write $g(x; \theta) \doteq g(L(x, \theta))$. The inversion problem for the operator I can be formulated as follows:

Problem: to recover the form df from knowledge of values of $I(f)$ on a n -dimensional manifold Σ of straight lines in \mathbf{V} .

The case $n = 2$.

Proposition 1 Let f be a 1-form of the class \mathbf{S}_1 on an oriented Euclidean plane \mathbf{V} . We have for any unit vector $\theta \in \mathbf{V}$ and $p \in \mathbb{R}$

$$\int_{L(x,\theta)} df/dp = \partial_p g(x, \theta), \quad (1)$$

where $p = x \times \theta$.

◀ Apply Cauchy-Green formula for the halfplane H bounded by the line $L(x, \theta) : \int_H df = g(x, \theta)$. Taking the derivative with respect to p yields (1). ▶

Write $df = FdS$ for some Euclidean structure in \mathbf{V} . The left-hand side of (1) can be written as $\int_L Fds$, where ds is the Euclidean line element. The function F can be reconstructed from (1) by inversion of the plane Radon transform.

The case $n > 2$. For a 1-form $f \in \mathbf{S}_0$ in \mathbf{V} we consider the ray integral

$$h(x; \theta) = \int_0^\infty f(x + r\theta; \theta) dr$$

defined for $x, \theta \in \mathbf{V}$. We have $h(x; \theta) - h(x; -\theta) = g(x; \theta)$. For any vectors $\xi, \omega \in \mathbf{V}$ we denote

$$f_\xi = \langle \xi, \nabla_x \rangle f, \quad h_\xi(x; \theta) = \langle \xi, \nabla_x \rangle h(x; \theta), \quad \partial_{;\omega} h(x; \theta) = \langle \omega, \nabla_\theta \rangle \partial_\tau h(x; \theta).$$

The operators ∇_x and $\partial_{;\omega}$ commute. We use the similar notations for derivatives of the line integral function g .

Introduce an Euclidean structure in \mathbf{V} and denote by $H_{p,\omega}$ the hyperplane $\{y; \langle \omega, y \rangle = p\} \subset \mathbf{V}$, where $\omega \in \mathbf{V}$ is a unit vector and $p \in \mathbb{R}$.

Proposition 2 Let f be a 1-form of the class \mathbf{S}_n in a space \mathbf{V} of dimension $n \geq 2$. For an arbitrary hyperplane $H = H_{p,\omega}$, an arbitrary point $x \in H$ and any vector $\xi \in \{\omega\}^\perp$ we have

$$\partial_p^{n-2} \int_{H_{p,\omega}} df(y; \xi, \omega) dH = \int_S \partial_{;\omega}^{n-1} h_\xi(x; \theta) dS(\theta), \quad (2)$$

where the integral is applied to the form df componentwise, S is the unit sphere in $\{\omega\}^\perp$ and dH, dS are the Euclidean volume forms in H and S respectively.

If n is even this formula can be written in terms of the line integrals

$$\partial_p^{n-2} \int_H df(y; \xi, \omega) dH = \int_{S_+} \partial_{;\omega}^{n-1} g_\xi(x; \theta) dS, \quad (3)$$

where S_+ is an arbitrary hemisphere in S .

◀ The differential df is the 2-form given by the formula $df(x; \theta, \eta) = f_\theta(x, \eta) - f_\eta(x, \theta)$ for $x, \theta, \eta \in \mathbf{V}$. Integrating by part we get the equation

$$\int_H df(y; \theta, \omega) dH = - \int f_\omega(y; \theta) dH. \quad (4)$$

Lemma 3 *We have for any $x, \xi, \omega \in \mathbf{V}$*

$$\begin{aligned} \partial_{;\omega}^{n-1} h(x; \theta) &= \int f_\omega^{(n-1)}(x + r\theta; \theta) r^{n-1} dr \\ &+ (n-1) \int f_\omega^{(n-2)}(x + r\theta; \omega) r^{n-2} dr. \end{aligned}$$

◀ A proof is by induction. ▶

Applying the derivative ∂_ξ to both parts we get

$$\begin{aligned} \partial_{;\omega}^{n-1} h_\xi(x; \theta) &= \int (f_\omega^{(n-1)})_\xi(x + r\theta; \theta) r^{n-1} dr \\ &+ (n-1) \int (f_\omega^{(n-2)})_\xi(x + r\theta; \omega) r^{n-2} dr. \end{aligned}$$

Choose an orthonormal frame $\varepsilon_1, \dots, \varepsilon_{n-1}$ in the space $\{\omega\}^\perp$ and set $\theta = \sum \sigma_i \varepsilon_i$. We have $f(y; \theta) = \sum \sigma_i f(y; \varepsilon_i)$, since f is linear in the second argument. Integrating this equation against the form $d\sigma$ over the unit sphere S in H and replacing $r^{n-2} dr d\sigma = dH$ gives

$$\begin{aligned} \int_S \partial_{;\omega}^{n-1} h_\xi(x; \theta) d\sigma &= \sum \int_H (f_\omega^{(n-1)})_\xi(x + r\theta; \varepsilon_i) y_i dH \\ &+ (n-1) \int_H (f_\omega^{(n-2)})_\xi(x + r\theta; \omega) dH, \end{aligned}$$

where $y_i = \sigma_i r$. Integrating by part in the right-hand side yields

$$\int_S \partial_{;\omega}^{n-1} h_\xi(x; \theta) d\sigma = - \int_H f_\omega^{(n-1)}(y; \xi) dH,$$

since $\sum \partial_\xi(y_i) \theta_i = \xi$, $\partial_\xi(1) = 0$. This implies (2), if we take in account (4).

For any odd k we have

$$\partial_{;\omega}^k h(x; \theta) + \partial_{;\omega}^k h(x; -\theta) = \partial_{;\omega}^k g(x; \theta),$$

which implies that for even n

$$\int_{S_+} \partial_{;\omega}^{n-1} g_\xi(x; \theta) d\sigma = - \int_H f_\omega^{(n-1)}(y; \xi) dH,$$

where S_+ is an arbitrary hemisphere in S . This yields (3). ▶

Theorem 4 *Let f be a 1-form of the class S_n in a vector space \mathbf{V} of dimension n , Γ be a set in a space \mathbf{V} such that any hyperplane H that meets the support of f has, at least, one common point with Γ . The form df can be reconstructed from data of first derivatives of the function $h(L)$ for rays $L = L(x, \theta)$, $x \in \Gamma$. If n is even, the form df can be reconstructed from the data of line integrals $g(L)$.*

◀ Denote for arbitrary vectors $\eta, \xi \in \mathbf{V}$ and a hyperplane H

$$I_H(\eta, \xi) = \partial_p^{n-2} \int_H df(y; \eta, \xi) dH \quad (5)$$

and show that this quantity can be determined from the given integral data. If both vectors η, ξ are parallel to H , the equation $I_H(\eta, \xi) = 0$ follows from partial integration. If η parallel to H and $\xi = \omega$ it is known by the formula (2) applied to a point $x \in H \cap \Gamma$. Take an arbitrary pair of vectors (η, ξ) one of which, say ξ , is not parallel to H . We can write $\xi = a\omega + \xi'$, and $\eta = b\xi + \eta'$, for some numbers $a \neq 0$ and b , where ξ', η' are parallel to H . We have the equation

$$I_H(\eta, \xi) = I_H(\eta', \xi) = I_H(\eta', a\omega) = aI_H(\eta', \omega),$$

where the right-hand side is already known.

The form df can be reconstructed from data of integrals (5) by means of any inversion formula for the Radon transform. In particular in the case $n = 3$ one can apply the classical formula of K. Lorentz

$$df(x) = -\frac{1}{8\pi^2} \int_{|\omega|=1} \partial_p^2 \int_{H_{\omega,p}} df \Big|_{p=\langle \omega, x \rangle} d\omega.$$

We only need to know these integrals for hyperplanes H that meet the support of df . Otherwise the integral vanishes. ▶

3 Reconstruction of a 2-form

Let f be a k -form in \mathbf{V} of the class S_0 , where $1 \leq k \leq n$. The integral transform $g = I(f)$ of f is the function

$$A \mapsto g(A) = \int_A f$$

defined on the manifold of oriented affine k -subspaces A of \mathbf{V} . We extend now the results of the previous Section differential forms f of degree 2. Consider the halfplane integral

$$h(x; \eta, \theta) = \int_0^\infty \int_{-\infty}^\infty f(x + s\eta + r\theta; \eta, \theta) ds dr$$

as a function on $\mathbf{V} \times \Lambda^2 \mathbf{V}$. Take one more vector $\omega \in \mathbf{V}$ and denote $\partial_{;\omega} h(x; \eta, \theta) = \langle \omega, \nabla_\theta \rangle h(x; \eta, \theta)$.

Proposition 5 *Let \mathbf{V} be an Euclidean space of dimension $n \geq 3$. Let $H = H_{p,\omega}$ be a hyperplane in \mathbf{V} , $x \in H$ and $L = L(x; \lambda) = \{y = x + t\lambda, t \in \mathbb{R}\}$ be a line in H . Then for an arbitrary 2-form $f \in \mathcal{S}_{n-1}$ in \mathbf{V} and arbitrary vector $\xi \in \{\lambda, \omega\}^\perp$ the equation holds*

$$\partial_p^{n-3} \int_{H_{p,\omega}} df(y; \lambda, \xi, \omega) dH = - \int_S \partial_{;\omega}^{n-2} h_\xi(x; \lambda, \theta) dS(\theta), \quad (6)$$

where S is the unit sphere in $\{\lambda, \omega\}^\perp$.

If n is odd, one can reconstruct df by means of plane integrals as follows

$$\partial_p^{n-3} \int_{H_{p,\omega}} df(y; \lambda, \xi, \omega) dH = - \int_{S_+} \partial_{;\omega}^{n-2} g_\xi(x; \lambda, \theta) dS(\theta), \quad (7)$$

where S_+ is an arbitrary hemisphere in S .

◀ We have $df(x; \theta, \eta, \xi) = f_\theta(x; \eta, \xi) - f_\eta(x; \theta, \xi) + f_\xi(x; \theta, \eta)$ and

$$\begin{aligned} \int df(x; \theta, \eta, \omega) dH &= \int_H f_\omega(x; \theta, \eta) dH, \\ \int df(x; \theta, \eta, \xi) dH &= 0, \end{aligned} \quad (8)$$

if $\theta, \eta, \xi \in \{\omega\}^\perp$. As in Lemma 3 we have

$$\partial_{;\omega}^{n-2} h(x; \lambda, \theta) = \int f_\omega^{(n-2)}(x + s\lambda + r\theta; \theta) r^{n-2} dr + (n-1) \int f_\omega^{(n-3)}(x + s\lambda + r\theta; \omega) r^{n-3} dr.$$

Take the derivative in a direction ξ orthogonal to ω :

$$\begin{aligned} \partial_{;\omega}^{n-2} h_\xi(x; \lambda, \theta) &= \int (f_\omega^{(n-2)})_\xi(x + s\lambda + r\theta; \lambda, \theta) r^{n-2} dr \\ &\quad + (n-1) \int (f_\omega^{(n-3)})_\xi(x + s\lambda + r\theta; \lambda, \omega) r^{n-3} dr \\ &= \sum_i \int (f_\omega^{(n-2)})_\xi(x + s\lambda + r\theta; \lambda, \varepsilon_i) \sigma_i r^{n-2} dr ds \\ &\quad + (n-1) \int (f_\omega^{(n-3)})_\xi(x + s\lambda + r\theta; \lambda, \omega) r^{n-3} dr ds. \end{aligned}$$

Integrating this equation against the form dS over the unit $n - 2$ -sphere S and replacing $r^{n-3}drd\sigma = dH$ yields

$$\begin{aligned} \int_S \partial_{;\omega}^{n-2} h_\xi(x; \lambda, \theta) dS &= \sum \int_H \partial_\xi f_\omega^{(n-2)}(y; \lambda, \theta_i) y_i dH \\ &+ (n-1) \int_H \partial_\xi f_\omega^{(n-3)}(y; \lambda, \omega) dH, \end{aligned}$$

where $y_i = \sigma_i r$. Integrate by parts in the right-hand side:

$$\begin{aligned} \sum \int_H (f_\omega^{(n-2)})_\xi(y; \lambda, \theta_i) y_i dH &= - \int_H \sum f_\omega^{(n-2)}(y; \lambda, \theta_i) \xi(y_i) dH \\ &= - \int_H f_\omega^{(n-2)}(y; \lambda, \xi) dH, \\ \int_H (f_\omega^{(n-3)})_\xi(y; \lambda, \omega) dH &= 0, \end{aligned}$$

since $\sum \xi(\theta_i) y_i = \xi$, $\partial_\xi(1) = 0$. It follows that

$$- \int_S \partial_{;\omega}^{n-2} h_\xi(x; \lambda, \theta) d\sigma = \int_H f_\omega^{(n-2)}(y; \lambda, \xi) dH.$$

Taking in account (8), we see that the right-hand side is equal to the left-hand side of (6). If n is odd the contributions of the opposite points can be combined since of the equation $\partial_{;\omega}^{n-2} h(x; \lambda, \theta) + \partial_{;\omega}^{n-2} h(x; \lambda, -\theta) = \partial_{;\omega}^{n-2} g(x; \lambda, \theta)$. This yields 6). ►

Corollary 6 *Given a line $L = L(x, \lambda) \in H$, the integral*

$$I_H(\lambda, \eta, \xi) \doteq \partial_p^{n-3} \int_{H_{p,\omega}} df(y; \lambda, \eta, \xi) dH \quad (9)$$

can be reconstructed for arbitrary $\eta, \xi \in \mathbf{V}$ from knowledge of first derivatives of the integrals $h(P)$ of f over halfplanes P bounded by L .

◀ The formula (7) gives such reconstruction for arbitrary triple of vectors $(\lambda, \gamma, \omega)$, where γ is orthogonal to λ and ω . On the other hand, the integral of $df(y; \theta, \eta, \xi)$ vanishes if all three vectors are parallel to H . If one the vectors η, ξ is not parallel to H , say ξ , we can write $df(y; \lambda, \eta, \xi) = df(y; \lambda, \eta', \xi)$, where $\eta' = \eta - c\xi$ is parallel to H . Then we have

$$\int df(y; \lambda, \eta, \xi) dH = \int df(y; \lambda, \eta'', \xi') dH,$$

where ξ' is the orthogonal projection of ξ on ω and η'' is the projection of η' to $\{\lambda\}^\perp$. ►

Theorem 7 *Let Λ be a set of straight lines in a space \mathbf{V} such that any hyperplane $H \subset \mathbf{V}$ contains, at least, $n - 2$ lines $L \in \Lambda$ with linearly independent direction vectors $\lambda_1, \dots, \lambda_{n-2}$. Then for any 2-form $f \in \mathbf{S}_{n-1}$ in \mathbf{V} the form df can be recovered from knowledge of first derivatives of the function h evaluated on halfplanes P bounded by a line $L \in \Lambda$.*

◀ By Corollary 6 we know the quantity $I_H(\lambda, \eta, \xi)$ for an arbitrary hyperplane H , arbitrary vectors $\eta, \xi \in \mathbf{V}$ and some linearly independent vectors $\lambda = \lambda_1, \dots, \lambda_{n-2}$ parallel to H . It is enough to determine $I_H(\theta, \eta, \xi)$ for arbitrary vectors θ, η, ξ . If all the vectors are parallel to H , then $I_H(\theta, \eta, \xi) = 0$. Otherwise, one of these vectors, say ξ is not parallel to H . Then $I_H(\theta, \eta, \xi) = I_H(\theta', \eta', \xi)$, where $\theta' = \theta - a\xi$ and $\eta' = \eta - b\xi$ are parallel to H for some numbers a, b . If one of these vectors, say θ' belongs to the envelope of $\lambda_1, \dots, \lambda_{n-2}$, we can write $\theta' = \sum c_i \lambda_i$ and

$$I_H(\theta', \eta', \xi) = \sum c_i I_H(\lambda_i, \eta', \xi).$$

Otherwise we the vector $\eta'' = \eta' - e\theta'$ belongs to this envelope for some number e and we have $I_H(\theta', \eta', \xi) = -I_H(\eta'', \theta', \xi)$. The right-hand side is known from Corollary 6. ▶

Remark. The set Λ can be taken as the variety of lines that meet two given curves $\gamma_1, \gamma_2 \subset \mathbf{V}$. Note that the total dimension of the manifold of planes P such that $L \subset P$ for some $L \in \Lambda$ is equal to n .

4 A generalization

Let again \mathbf{V} be a real vector space of dimension n . Fix a number k , $1 \leq k \leq n$ and consider the following general

Problem: To reconstruct a k -form f in \mathbf{V} of the class \mathbf{S}_n from a data of its integrals $g(A) = \int_A f$ over a set Σ of affine subspaces $A \subset \mathbf{V}$ of dimension k . We mean, of course, a reconstruction *modulo* exact forms. The case when Σ is a manifold of dimension n is of special interest.

Summarize the answers we know:

1. The case $k = n$. The only data we have is the integral of f over \mathbf{V} . If the integral vanishes, then f is exact, that is $f = dh$, where h is a fast decreasing $n - 1$ -form in \mathbf{V} . This yields the positive answer.

2. The case $k = n - 1$. The answer is also positive, because of the following. Fix an orientation and Euclidean structure in \mathbf{V} and write $df = FdV$, where dV is the volume form. By the method of Proposition 1 one can reconstruct hyperplane

integrals of F :

$$\int_H F dS = \partial_p \int_H df \quad (10)$$

and recover the function F by inversion of the Radon transform, if we know all hyperplane integrals of f .

3. The cases $k = 1, 2$ were considered in previous Sections. The answers are again positive. Note that the formulae (2) for $n = 2$ and (6) for $n = 3$ are equivalent to (10). One can check it by partial integration.

4. These results can be generalized for differential forms $f \in \mathcal{S}_0$ in \mathbf{V} of degree $k < n$ as follows:

For arbitrary point $x \in \mathbf{V}$ and tangent vectors $\eta_1, \dots, \eta_k \in \mathbf{V}$ we consider the halfsubspace integral

$$h(x; \eta_1, \dots, \eta_k) = \int_0^\infty \int_{\mathbb{R}^k} f(x + s_1 \eta_1 + \dots + s_k \eta_k; \eta_1, \dots, \eta_k) ds_1 \dots ds_{k-1} ds_k$$

as a function on $\mathbf{V} \times \Lambda^k \mathbf{V}$. For an arbitrary $\omega \in \mathbf{V}$ we denote $\partial_{;\omega} h(x; \dots, \theta) = \langle \omega, \nabla_\theta \rangle h(x; \dots, \theta)$.

Proposition 8 *Let \mathbf{V} be an Euclidean space of dimension n and $0 < k < n$. Let $H = H_{p,\omega}$ be a hyperplane in \mathbf{V} , $x \in H$ and $\lambda_1, \dots, \lambda_{k-1}$ be some vectors in \mathbf{V} parallel to H . Then for an arbitrary k -form $f \in \mathcal{S}_{n-k}$ in \mathbf{V} and arbitrary vector $\xi \in G \doteq \{\lambda_1, \dots, \lambda_{k-1}, \omega\}^\perp$ the equation holds*

$$\partial_p^{n-k-1} \int_{H_{p,\omega}} df(y; \lambda_1, \dots, \lambda_{k-1}, \xi, \omega) dH = - \int_S \partial_{;\omega}^{n-k} h_\xi(x; \lambda_1, \dots, \lambda_{k-1}, \theta) dS(\theta),$$

where S is the unit sphere in G and dS is the Euclidean sphere density on it.

If $n-k$ is odd, we can replace the function h by k -plane integral $g = g(x; \lambda_1, \dots, \lambda_{k-1}, \theta)$ and the sphere S by an arbitrary hemisphere S_+ .

One can argue further as in Theorem 7. We leave details to a reader.

5 A support condition

Theorem 9 *Let \mathbf{V} be a vector space of dimension n , $0 < k < n$ and $K \subset \mathbf{V}$ be a compact set such that the intersection $A \cap K$ is convex for any affine supspace A of dimension $k+1$. The integral transform of a k -form f in \mathbf{V} of the class \mathcal{S}_1 vanishes on all k -spaces $L \subset \mathbf{V} \setminus K$ if and only if $\text{supp } df \subset K$.*

◀ Suppose that the support of df is contained in K and a line L is disjoint with K . Then $f|_L = dh$ for a function h on L of the class S_1 . Integrating yields $\int_L f = 0$.

Check the 'only if' part. Fix a point $x \in \mathbf{V} \setminus K$ and take an arbitrary $k + 1$ -space A through x . Fix an orientation in and an Euclidean structure. Write $df|_A = FdS$, where dS is the Euclidean area element in L . By the arguments of Proposition 1 we have for any hypersurface $H_{p,\omega}$ in A

$$\int_{H_{p,\omega}} F dS = \partial_p g(H_{p,\omega}).$$

The right-hand side vanishes for any hypersurface H disjoint with $A \cap K$. By Helgason's support theorem [5] the function F vanishes in $A \setminus K$. This implies that $df(x) = 0$, since A is arbitrary. ▶

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