

A uniform reconstruction formula in integral geometry

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Abstract: A general method for analytic inversion in integral geometry is proposed. All classical and some new reconstruction formulas of Radon-John type are obtained by this method. No harmonic analysis and no PDE is used.

Key words: Hypersurface family, Minkowski-Funk transform, Principal value integral, Reconstruction, Hyperbolic domain

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1 Introduction

We present a uniform reconstruction method for a class of Funk-Radon type integral transforms extending results of [19] for arbitrary dimensions. The reconstruction does not include summation of an infinite series and looks as the standard (John type) inversion for the Radon transform. We specify this method for classical and new regular acquisition geometries. The condition of regularity (see the next Section) is necessary for an inversion operator to be bounded in a Sobolev space scale, but it is not sufficient. We shall see that the existence of a elementary exact reconstruction formula depends on vanishing of some integrals of rational forms on an algebraic manifold. In the last Section, we discuss reconstruction in Euclidean space from data of integrals over a family of spheres. This subject is in focus of recent research, see papers [11],[12],[15],[10],[16] and references therein.

2 Geometry and integrals

Let X and Σ be smooth n dimensional manifolds ($n > 1$), let Z be a smooth closed hypersurface in $X \times \Sigma$ and $p : Z \rightarrow X$, $\pi : Z \rightarrow \Sigma$ be natural projections. We suppose that there exists a real smooth function Φ in $X \times \Sigma$ (called generating function) such that $Z = \{(x, \sigma) ; \Phi(x, \sigma) = 0\}$ and $d_x \Phi \neq 0$ on Z . Suppose that

(i) The map π has rank n and the mapping $P : N^*(Z) \rightarrow T^*(X)$ is a local diffeomorphism. Here, $N^*(Z)$ denotes the conormal bundle of Z and $P(x, \sigma; \nu_x, \nu_\sigma) = (x, \nu_x) \in T^*(X)$. It follows that a set $Z(\sigma) = \pi^{-1}(\sigma) = \{x; \Phi(x, \sigma) = 0\}$ is a smooth hypersurface in X , and for any point $x \in X$ and for any tangent hyperplane $h \subset T_x(X)$ there is a locally unique hypersurface $Z(\lambda, \omega)$ through x tangent to h .

Proposition 2.1 For an arbitrary generating function Φ property (i) is equivalent to the condition: $\det(d_{x,t}d_{\sigma,\tau}\Psi) \neq 0$ where $\Psi(x, t; \sigma, \tau) = t\tau\Phi(x, \sigma)$, $t, \tau \in \mathbb{R}$, $t\tau > 0$ for any local coordinate system x_1, \dots, x_n in X and any local coordinate system $\sigma_1, \dots, \sigma_n$ in Σ .

For a proof see [18], Proposition 1.1.

Definition. We call a generating function Φ *regular* if it satisfies conditions (i) and (ii) there are no conjugated points, that is the equations $\Phi(x, \sigma) = \Phi(y, \sigma)$ and $d_\sigma\Phi(x, \sigma) = d_\sigma\Phi(y, \sigma)$ are fulfilled for no $x \neq y \in X$, $\sigma \in \Sigma$.

Suppose that X is oriented and g is a Riemannian metric in X ; let dV be the oriented Riemannian volume form. Consider the integral

$$M_\Phi f(\sigma) = \int \delta(\Phi(x, \sigma)) f dS = \int_{Z(\sigma)} \frac{f dV}{d\Phi(x, \sigma)} = \int_{Z(\sigma)} f q$$

for an arbitrary continuous function f compactly supported in X . The quotient $f dV/d\Phi$ denotes an arbitrary $n-1$ form q such that $d\Phi \wedge q = f dV$. It is defined up to a term $h d\theta$ where h is a continuous function. An orientation of a hypersurface $Z(\sigma)$ is defined by means of the form $d\theta$ and the integral of q over $Z(\sigma)$ is uniquely defined. We call the operator M_Φ *Minkowski-Funk* transform generated by Φ . This transform can be written in terms of Riemannian integral as follows:

$$M_\Phi f(\sigma) = \int_{Z(\sigma)} \frac{f d_g S}{|\nabla_g \Phi(x, \sigma)|} \quad (1)$$

Here, $d_g S$ is the Riemannian $n-1$ surface element and $|\nabla_g a| \doteq \sqrt{g(da)}$ is the Riemannian gradient of a function a . The function $|\nabla_g \Phi|$ vanishes nowhere since of (i). Suppose that the gradient factorizes through X and Σ that is $|\nabla_g \Phi(x, \sigma)| = m(x)\mu(\sigma)$ for some positive continuous functions m in X and μ in Σ . Then data of the Minkowski-Funk transform is equivalent to data of Riemann hypersurface integrals

$$R_g f(\sigma) = \int_{Z(\sigma)} f d_g S, \sigma \in \Sigma$$

since $R_g f(\sigma) = \mu(\sigma) M_\Phi(mf)(\sigma)$. The reconstruction problem of a function f from Riemann integrals $R_g f$ is then reduced to inversion of the operator M_Φ .

We say that a generating function Φ is *resolved* if $\Sigma = \mathbb{R} \times S^{n-1}$, and Φ has the form $\Phi(x; \lambda, \omega) = \theta(x, \omega) - \lambda$, $\lambda \in \mathbb{R}$, $\omega \in S^{n-1}$ for a smooth function θ on $X \times S^{n-1}$ where S^{n-1} denotes the unit sphere in an n dimensional space. Note that the map $p : Z \rightarrow X$

is always proper for a resolved generating function. This property guarantees that the functions $M_\Phi f$ and $R_g f$ have compact support in Σ . The operator M_Φ fulfils the range conditions similar to that of the Radon transform.

Proposition 2.2 *Let $\Phi = \theta - \lambda$ be a resolved regular generating function and $\theta(x, \omega)$ be a polynomial function of ω of order m . Then for an arbitrary integrable function f in X with compact support and for an arbitrary polynomial $p(\lambda)$ of order k the integral*

$$\int p(\lambda) M_\Phi f(\lambda, \omega) d\lambda$$

is a polynomial of ω of order $\leq mk$.

Proof. We have

$$\int p(\lambda) M_\Phi f(\lambda, \omega) d\lambda = \int p(\lambda) \int_{\theta=\lambda} \frac{f dV}{d\theta} d\lambda = \int_X p(\theta(x, \omega)) f(x) dV$$

where $p(\theta(x, \omega))$ is a polynomial of ω of order $\leq mk$. ►

3 Main theorem

For a real smooth function f in a manifold X and a natural n we consider singular integrals

$$I_{n\pm}(\rho) = \int_X \frac{\rho}{(f \pm i0)^n} = \lim_{\varepsilon \searrow 0} \int_X \frac{\rho}{(f \pm i\varepsilon)^n},$$

for a smooth density ρ with compact support. If $df \neq 0$ on the zero set of f , then these limits exist and the functionals $I_{n\pm}$ are generalized functions in X . The functional

$$(P) \int_X \frac{\rho}{f^n} \doteq \operatorname{Re} I_{n+}(\rho) = \operatorname{Re} I_{n-}(\bar{\rho})$$

is called a *principal value* integral. For a resolved regular generating function $\Phi = \theta - \lambda$ we define the function on $X \times X \setminus \{\text{diag}\}$

$$\Theta_n(x, y) = \int_{S^{n-1}} \frac{d\omega}{(\theta(x, \omega) - \theta(y, \omega) - i0)^n}$$

where $d\omega$ is the Euclidean volume form on S^{n-1} . The singular integral converges since by (ii) the $d_\omega(\theta(x, \omega) - \theta(y, \omega)) \neq 0$ as $\theta(x, \omega) - \theta(y, \omega) = 0$.

Theorem 3.1 *Let $\Phi = \theta - \lambda$ be a regular resolved generating function in $X \times \Sigma$ and $f \in L_2(X)$ be an arbitrary function with compact support. If n is even and $\text{Re } \Theta_n(x, y) = 0$ for any $x \neq y \in X$, a reconstruction from data of $M_\Phi f$ is given by the formula:*

$$\begin{aligned} f(x) &= -\frac{(n-1)!}{(2\pi i)^n} \frac{1}{D_n(x)} (P) \int_\Sigma \frac{M_\Phi f(\lambda, \omega) d\lambda d\omega}{(\theta(x, \omega) - \lambda)^n} \\ &= -\frac{1}{(2\pi i)^n D_n(x)} \int_{S^{n-1}} (P) \int_{\mathbb{R}} \frac{\partial^{n-1}}{\partial \lambda^{n-1}} M_\Phi f(\lambda, \omega) \frac{d\lambda d\omega}{\theta(x, \omega) - \lambda} \end{aligned} \quad (2)$$

If n is odd and $\text{Im } \Theta_n(x, y) = 0$ for $x \neq y$, the function can be reconstructed by

$$\begin{aligned} f(x) &= \frac{1}{2(2\pi i)^{n-1} D_n(x)} \int_\Sigma \delta^{(n-1)}(\theta(x, \omega) - \lambda) M_\Phi f(\lambda, \omega) d\lambda d\omega \\ &= \frac{1}{2(2\pi i)^{n-1} D_n(x)} \int_{S^{n-1}} \left. \frac{\partial^{n-1}}{\partial \lambda^{n-1}} M_\Phi f(\lambda, \omega) \right|_{\lambda=\theta(x, \omega)} d\omega \end{aligned} \quad (3)$$

where

$$D_n(x) = \frac{1}{|S^{n-1}|} \int_{S^{n-1}} \frac{d\omega}{|\nabla_g \theta(x, \omega)|^n}$$

The integrals (2) and (3) converge in mean on any compact set in X .

Remark 1. A more invariant form of (2) or (3) is the reconstruction of the form $f dV$:

$$f dV = -\frac{1}{(2\pi i)^n} \frac{dV}{D_n} \int \dots d\omega$$

The quotient dV/D_n depends only on the conformal class of the Riemannian metric g .

Remark 2. The condition (ii) is not required when $n = 2$.

Remark 3. G. Beylkin studied "the generalized Radon transform" [6], which coincides with the operator M_Φ in the Euclidean space. He constructed a Fourier integral operator parametrix for this operator and reduced inversion of this operator to solution of a Fredholm equation. Two dimensional case of Theorem 3.1 was stated in [19].

Remark 4. When the condition on Θ_n fails, the right-hand side R_n of (2), respectively (3), gives anyway a reconstruction up to a compact operator. More precisely, let K be an arbitrary compact set in X and I_K be the indicator function of K . Then the equation holds $I_K R f = f + C_K f$ for any function f with support in K where C_K is a compact operator in $L_2(K)$.

Proof of Theorem.

Lemma 3.2 *The integral transform*

$$\begin{aligned} I_n f(x) &= (P) \int_{S^{n-1}} \int_{\mathbb{R}} \frac{M_\Phi f(\lambda, \varphi)}{\Phi^2(x; \lambda, \varphi)} d\lambda d\varphi, \text{ for even } n \\ I_n f(x) &= \int_{S^{n-1}} \left. \frac{\partial^{n-1}}{\partial \lambda^{n-1}} M f(\lambda, \omega) \right|_{\lambda=\theta(x, \omega)} d\omega, \text{ for odd } n \end{aligned}$$

is a continuous operator $L_2(X)_{\text{comp}} \rightarrow L_2(X)_{\text{loc}}$.

A proof of the Lemma is given in [18]. For even n and an arbitrary $x \in X$ and a function f that vanishes in a neighborhood of x we calculate

$$\begin{aligned} I_n f(x) &\doteq (P) \int_{\Sigma} \frac{M_{\Phi} f(\lambda, \omega) d\lambda d\omega}{\Phi^n(x; \lambda, \omega)} = \int_{S^{n-1}} d\omega (P) \int_{\mathbb{R}} \int_{\theta(y, \omega) = \lambda} f(y) q \frac{d\lambda}{(\theta(x, \omega) - \lambda)^n} \\ &= \int_X \left((P) \int_{S^{n-1}} \frac{d\omega}{(\theta(x, \omega) - \theta(y, \omega))^n} \right) f d\theta \wedge q = \int_X \operatorname{Re} \Theta_n(x, y) f(y) dV(y) \end{aligned}$$

Here, the relation $d\lambda = d\theta$ holds in Z and the equation $d\theta \wedge q = dV$ is fulfilled in X by definition. Thus the function Θ_n is the off-diagonal kernel of the operator I_n . It vanishes since of the assumption. Therefore $\Theta_n(x, y)$ is supported in the diagonal and according to Lemma 3.2 we have $\Theta_n(x, y) = a_n(x) \delta_x(y)$ for a locally bounded function a_n in X .

If n is odd we have

$$\begin{aligned} I_n f(x) &\doteq \int_{S^{n-1}} \int_{\mathbb{R}} \delta^{(n-1)}(\theta(x, \omega) - \lambda) M_{\Phi} f(\lambda, \omega) d\lambda d\omega \\ &= \int_S d\omega \frac{\partial^{n-1}}{\partial \lambda^{n-1}} \left(\int_{Z(\lambda, \omega)} f q \right) \Big|_{\lambda=\theta(x, \omega)} = \int \Theta_n(x, y) f(y) dV(y) \\ \int_{Z(\lambda, \omega)} f(y) q &= \frac{1}{\pi} \operatorname{Im} \int \frac{f(y) dV}{\lambda - \theta(y, \omega) - i0} \end{aligned}$$

$$\begin{aligned} \int \frac{\partial^{n-1}}{\partial \lambda^{n-1}} \left(\int_{Z(\lambda, \omega)} f q \right) \Big|_{\lambda=\theta(x, \omega)} dV &= \frac{(n-1)!}{\pi} \operatorname{Im} \int \left[\frac{1}{(\theta(x, \omega) - \theta(y, \omega) - i0)^n} \right] d\omega f dV \\ &= \frac{(n-1)!}{\pi} \int \operatorname{Im} \Theta_n(x, y) f dV \end{aligned}$$

that is $\pi^{-1} (n-1)! \operatorname{Im} \Theta_n$ is the kernel of I_n which vanishes off the diagonal by the assumption. By Lemma 3.2 we again conclude that $\pi^{-1} (n-1)! \operatorname{Im} \Theta_n = a_n(x) \delta_x(y)$ for a locally bounded function a_n .

Next we calculate the function a_n . Choose a smooth function e_0 of one variable with support in $[-1, 1]$ such that $e_0(0) = 1$ and set $e_{\varepsilon}(x) = e_0(|x|^2/\varepsilon^2)$ for $x \in \mathbb{R}^n$ and any $\varepsilon > 0$. Take a point $x_0 \in X$ and show that

$$\operatorname{Re} \int_X dV \int_{S^{n-1}} \frac{e_{\varepsilon}(x - x_0) d\omega}{(\theta(x, \omega) - \theta(x_0, \omega) - i0)^n} \rightarrow a_n(x_0)$$

for n even and

$$\frac{(n-1)!}{\pi} \operatorname{Im} \int_X dV \int_{S^{n-1}} \frac{e_{\varepsilon}(x - x_0) d\omega}{(\theta(x, \omega) - \theta(x_0, \omega) - i0)^n} \rightarrow a_n(x_0)$$

for n odd as $\varepsilon \rightarrow 0$. We can change order of integrals and integrate first over X .

Lemma 3.3 *If n is even, we have for any $x_0 \in X$, arbitrary $\omega \in \mathbb{S}^{n-1}$ and small ε*

$$a_n(x_0, \omega) \doteq \operatorname{Re} \int_X \frac{e_\varepsilon(x - x_0) dV}{(\theta(x, \omega) - \theta(x_0, \omega) - i0)^n} = \frac{(-1)^{n/2-1} \pi^{(n+1)/2}}{\Gamma((n+1)/2)} \frac{1}{|\nabla\theta(x_0, \omega)|^n} + o(1) \quad (4)$$

where $o(1) \leq C\varepsilon^{1/2} \log 1/\varepsilon$ where C does not depend on ω . For odd n we have

$$a_n(x_0, \omega) \doteq \frac{(n-1)!}{\pi} \operatorname{Im} \int_X \frac{e_\varepsilon(x - x_0) dV}{(\theta(x, \omega) - \theta(x_0, \omega) - i0)^n} = \frac{(-1)^m (2\pi)^n}{(n-1)!} \frac{1}{|\nabla\theta(x_0, \omega)|^n} + O(\varepsilon)$$

Taking the limit and integrating (4) over \mathbb{S}^{n-1} yields for even n the equation

$$\begin{aligned} a_n(x_0) &= \lim_{\varepsilon \rightarrow 0} \int a_n(x_0, \omega) d\omega = (-1)^{n/2-1} \frac{\pi^{(n+1)/2}}{\Gamma((n+1)/2)} \int \frac{d\omega}{|\nabla\theta(x_0, \omega)|^n} \\ &= -\frac{(2\pi i)^n}{(n-1)! |\mathbb{S}^{n-1}|} \int \frac{d\omega}{|\nabla\theta(x_0, \omega)|^n} = -\frac{(2\pi i)^n}{(n-1)!} D_n(x_0) \end{aligned}$$

which implies (2). For odd n we obtain

$$a_n(x_0) = \lim_{\varepsilon \rightarrow 0} \int a_n(x_0, \omega) d\omega = 2(2\pi i)^{n-1} \frac{1}{|\mathbb{S}^{n-1}|} \int \frac{d\omega}{|\nabla_{\mathbf{g}}\theta(x_0, \omega)|^n} = 2(2\pi i)^{n-1} D_n(x_0)$$

This yields (3). This completes the proof of Theorem 3.1. ►

Proof of Lemma. We show first that the θ can be replaced by a linear function. Assume for simplicity that $x_0 = 0$ and $\partial\theta(x_0, \omega)/\partial x_1 = |\nabla_{\mathbf{g}}\theta|$, $\partial\theta(x_0, \omega)/\partial x_2 = \dots = \partial\theta(x_0, \omega)/\partial x_n = 0$ for a coordinate system x_1, \dots, x_n in a neighborhood of x_0 such that $g(x_0; \partial/\partial x_i, \partial/\partial x_j) = \delta_{ij}$. We have then $dV = v dx$, $dx = dx_1 \dots dx_n$ where v is a smooth function such that $v(x_0) = 1$.

If n is even, we integrate by parts n times with respect to x_1 :

$$\begin{aligned} \operatorname{Re} \int \frac{e_\varepsilon(x - x_0) dV}{(\theta(x, \omega) - \theta(x_0, \omega) - i0)^n} &= \frac{1}{(n-1)} \operatorname{Re} \int \frac{d_1(x) dx}{(\theta(x, \omega) - \theta(x_0, \omega) - i0)^{n-1}} \quad (5) \\ &\dots = \frac{1}{(n-1)!} \int \log |\theta(x, \omega) - \theta(x_0, \omega)| d_n(x) dx \end{aligned}$$

$$d_1 = \frac{\partial}{\partial x_1} \frac{e_\varepsilon(x - x_0) v(x)}{|\nabla_{\mathbf{g}}\theta(x)|} = \frac{\partial e_\varepsilon}{\partial x_1} \frac{v}{|\nabla_{\mathbf{g}}\theta(x, \omega)|} - \frac{e_\varepsilon (\partial_1 |\nabla_{\mathbf{g}}\theta(x, \omega)| - v \partial_1 |\nabla_{\mathbf{g}}\theta(x, \omega)|)}{|\nabla_{\mathbf{g}}\theta(x, \omega)|^2}$$

...

$$d_n = \frac{\partial}{\partial x_1} \frac{d_{n-1}(x)}{|\nabla_{\mathbf{g}}\theta(x)|} = \frac{\partial^n e_\varepsilon}{\partial x_1^n} \frac{v}{|\nabla_{\mathbf{g}}\theta(x, \omega)|^n} + \dots$$

where omitted terms only include derivatives of e_ε of order $< n$. Changing the variables $x = \varepsilon y$ we get

$$d_n(y) = \varepsilon^{-n} \frac{\partial^n e}{\partial y_1^n} \frac{v(\varepsilon y)}{|\nabla_{\mathbf{g}}\theta(\varepsilon y)|^n} + O(\varepsilon^{1-n}) \quad (6)$$

By Lagrange's theorem we can write

$$\theta(\varepsilon y, \omega) - \theta(0, \omega) = \varepsilon \rho_\varepsilon(y), \rho_\varepsilon(y) = \int_0^1 \langle y, \nabla \theta(\varepsilon t y) \rangle dt \quad (7)$$

This yields

$$\int \log |\theta(x, \omega) - \theta(x_0, \omega)| d_n(x) dx = \int (\log \varepsilon) d_n(x) dx + \int \log |\rho_\varepsilon(y)| d_n(x) dx$$

The first integral vanishes since d_n is equal to x_1 -derivatives of a function with compact support. It follows that the left hand side of (5) equals

$$\frac{1}{(n-1)!} \int \log |\rho_\varepsilon(y)| d_n(x) dx = \int \log |\rho_\varepsilon(y)| \frac{1}{|\nabla_{\mathbf{g}} \theta(x_0, \omega)|^n} \frac{\partial^n e}{\partial y_1^n} dy + O(\varepsilon)$$

since the logarithmic factor is absolutely integrable. By (7) we have C^1 -convergence $\rho_\varepsilon \rightarrow \langle y, \nabla \theta(0) \rangle$ as $\varepsilon \rightarrow 0$ in a neighborhood of the origin. This implies the inequality

$$- \int_{|y| \leq 1, |y_1| < \delta} \log |y_1| dy - \int_{|y| \leq 1, \rho_\varepsilon(y) < \delta} \log |\rho_\varepsilon(y)| dy \leq C \delta \log |\delta|$$

where δ , $0 < \delta \leq 1$ is arbitrary and C does not depend on ε and δ . On the other hand, $\log |\rho_\varepsilon(y)| \rightarrow \log |y_1|$ everywhere as $\varepsilon \rightarrow 0$. Therefore

$$\int \log |\rho_\varepsilon(y)| \frac{\partial^n e}{\partial y_1^n} dy \rightarrow \int \log |y_1| \frac{\partial^n e}{\partial y_1^n} dy$$

and

$$\operatorname{Re} \int \frac{e_\varepsilon(x - x_0) dV}{(\theta(x, \omega) - \theta(x_0, \omega) - i0)^n} \rightarrow \frac{1}{(n-1)! |\nabla_{\mathbf{g}} \theta(x_0, \omega)|^n} \int \log |y_1| \frac{\partial^n e}{\partial y_1^n} dy$$

More detailed arguments show that the difference is equal to $O(\varepsilon^{1/2} \log \varepsilon)$. The same is true for the linear function $\theta(x, \omega) = x_1$ that is

$$\int_X \frac{e_\varepsilon(x - x_0) dV}{(\theta(x, \omega) - \theta(x_0, \omega) - i0)^n} \rightarrow \frac{1}{|\nabla_{\mathbf{g}} \theta(x_0, \omega)|^n} \operatorname{Re} \int_{\mathbb{R}^n} \frac{e(y) dy}{(y_1 - i0)^n}, \quad \varepsilon \rightarrow 0$$

Calculate the integral in the right hand side by partial integration:

$$\begin{aligned} \operatorname{Re} \int_{\mathbb{R}^n} \frac{e dy}{(y_1 - i0)^n} &= \frac{1}{n-1} \operatorname{Re} \int_{\mathbb{R}^n} \frac{dy}{(y_1 - i0)^{n-1}} \frac{\partial e}{\partial y_1} \\ &= \frac{1}{n-1} \operatorname{Re} \int_{S^{n-1}} (\cos \omega_1 - i0)^{2-n} d\omega \int_0^\infty \frac{\partial e_0}{\partial r^2} dr^2 \\ &= -\frac{1}{n-1} \operatorname{Re} \int_{S^{n-1}} (\cos \omega_1 - i0)^{2-n} d\omega \\ &= \frac{|S^{n-2}|}{n-1} \operatorname{Re} \int_0^\pi \sin^{n-2} \omega_1 (\cos \omega_1 - i0)^{2-n} d\omega_1 \end{aligned} \quad (8)$$

where $y = r \cos \omega_1$ since

$$\frac{\partial e}{\partial y_1} = 2y_1 \frac{\partial e}{\partial r^2}, \quad y_1 - i0 = (\cos \omega_1 - i0) r, \quad \int_0^{\infty} \partial e_0 / \partial r^2 dr^2 = -e(0) = -1$$

By substituting $s = \cos^2 \omega_1$ we get

$$\operatorname{Re} \int_0^\pi \left(\frac{\sin \omega_1}{\cos \omega_1 + i0} \right)^{n-2} d\omega_1 = \frac{1}{2} B \left(\frac{n-1}{2}, \frac{3-n}{2} \right) = (-1)^{n/2-1} \frac{\pi}{2}$$

We use the formula for Beta-function extended for all complex (non negative integers) values of arguments. The exponent $\lambda = 1/2 - n/2$ is a regular point and we can use a classical formula. The right hand side of (8) equals

$$(-1)^{n/2-1} \frac{\pi |S^{n-2}|}{2(n-1)} = (-1)^{n/2-1} \frac{\pi^{(n+1)/2}}{\Gamma((n+1)/2)}.$$

which yields (4).

In the case of odd n we integrate by parts as in the previous case and get

$$\operatorname{Im} \int \frac{e_\varepsilon(x-x_0) dV}{(\theta(x, \omega) - \theta(x_0, \omega) - i0)^n} = \frac{\pi}{(n-1)!} \int_{\theta(x, \omega) > \theta(x_0, \omega)} d_n(x) dx$$

Taking in account (6) and convergence $\rho_\varepsilon \rightarrow y_1$ the limit of the right hand side is

$$\frac{\pi}{(n-1)! |\nabla_g \theta(x_0, \omega)|^n} \int_{y_1 > 0} \frac{\partial^n e}{\partial y_1^n} dy \quad (9)$$

Integrating by parts backward gives the equation

$$\int_{y_1 > 0} \frac{\partial^n e}{\partial y_1^n} dy = - \int_{y_1=0} \frac{\partial^{n-1} e}{\partial y_1^{n-1}} dy' = -2^{m-1} (n-2)!! |S^{n-2}| \int_0^\infty \frac{\partial^m e_0(s)}{\partial s^m} s^{m-1} ds$$

where $y' = (y_2, \dots, y_n)$, $m = (n-1)/2$, $s = r^2$. Here, we applied the formula

$$\left. \frac{\partial^{n-1} e(y)}{\partial y_1^{n-1}} \right|_{y_1=0} = 2^m (n-2)!! \frac{\partial^m e_0(s)}{\partial s^m}$$

Integrating by parts $m-1$ times in the interior integral we get the quantity

$$\int_0^1 \frac{\partial^m e_0(s)}{\partial s^m} s^{m-1} ds = (-1)^{m-1} (m-1)! \int_0^1 \partial e_0 / \partial s ds = (-1)^m (m-1)!$$

This implies that (9) equals

$$\frac{(-1)^m 2^{m-1} \pi (m-1)! (n-2)!! |S^{n-2}|}{(n-1)!} \frac{1}{|\nabla_g \theta(x_0, \omega)|^n} = \frac{2(2\pi i)^{n-1}}{(n-1)!} \frac{1}{|S^{n-1}| |\nabla_g \theta(x_0, \omega)|^n}$$

For odd n we have

$$a_n(x_0, \omega) \doteq \frac{(n-1)!}{\pi} \lim_{\varepsilon \rightarrow 0} \operatorname{Im} \int_X \frac{e_\varepsilon(x-x_0) dV}{(\theta(x, \omega) - \theta(x_0, \omega) - i0)^n} = \frac{2(2\pi i)^{n-1}}{|S^{n-1}| |\nabla_{\mathbf{g}} \theta(x_0, \omega)|^n}$$

Integrating over S^{n-1} we get

$$a_n(x_0) = \int a_n(x_0, \omega) d\omega = \frac{2(2\pi i)^{n-1}}{|S^{n-1}|} \int \frac{d\omega}{|\nabla_{\mathbf{g}} \theta(x_0, \omega)|^n}$$

and (3) follows. \blacktriangleright

4 Integrals of rational trigonometric functions

We focus now on the condition $\Theta(x, y) = 0$. A function of the form

$$t(\varphi) = \sum_{j=0}^k a_j \cos j\varphi + b_j \sin j\varphi$$

is called trigonometric polynomial of degree k if $a_k \neq 0$ or $b_k \neq 0$. Any trigonometric polynomial is 2π -periodic and is well-defined and holomorphic on the cylinder $\mathbb{C}/2\pi\mathbb{Z}$. It always has $2k$ zeros in the cylinder. If the polynomial is real, number of real zeros is even.

Lemma 4.1 *Let $t(\varphi)$ and $s(\varphi)$ be real trigonometric polynomials such that $\deg s < \deg t$ and all the roots of t are real. Then for $r = s/t$ and arbitrary natural n*

$$(P) \int_0^{2\pi} r^n(\varphi) d\varphi \doteq \frac{1}{2} \int_0^{2\pi} (r(\varphi) + i0)^n d\varphi + \frac{1}{2} \int_0^{2\pi} (r(\varphi) - i0)^n d\varphi = 0. \quad (10)$$

Proof. Suppose first that all roots of t are simple. Let $\alpha_1 < \alpha_2 < \dots < \alpha_m$ be all roots of $\partial t / \partial \varphi$ on the circle $\mathbb{R}/2\pi\mathbb{Z}$. Let $\varepsilon_k = \operatorname{sgn} \partial t / \partial \varphi$ on the interval (α_k, α_{k+1}) for $k = 1, \dots, m$, where $\alpha_{m+1} = \alpha_1$. The function $r(\zeta)$ is meromorphic for $\zeta = \varphi + i\tau \in \mathbb{C}/2\pi\mathbb{Z}$ and has no poles in the half-cylinder $\{\tau > 0\}$ since of the assumption. Take a continuous function $\lambda = \tau(\varphi)$ defined on the circle that vanishes in all points α_k and is positive otherwise. We have for any k

$$\int_{\alpha_k}^{\alpha_{k+1}} (r(\varphi) + \varepsilon_k i0)^n d\varphi = \int_{\alpha_k}^{\alpha_{k+1}} r^n(\varphi + i0) d\varphi.$$

Take the sum

$$\sum_k \int_{\alpha_k}^{\alpha_{k+1}} (r(\varphi) + \varepsilon_k i0)^n d\varphi = \sum_k \int_{\alpha_k}^{\alpha_{k+1}} r^n(\varphi + i0) d\varphi = \int_0^{2\pi} r^n(\varphi + i0) d\varphi \quad (11)$$

Now we can replace the form $r^n(\varphi + i0) d\varphi$ by $r^n(\zeta) d\zeta$ for $\zeta = \varphi + i\eta$ for an arbitrary $\eta > 0$ without changing the integral in the right-hand side. We have $|r(\zeta)| \rightarrow 0$ as $\eta \rightarrow \infty$ hence the right hand side of (11) vanishes. The real part of the left hand side is equal to the left hand side of (10) which also vanishes.

In the general case, we can approximate the polynomial t with real roots by polynomials \tilde{t} with real simple roots. The equation (10) holds for $\tilde{r} = s/\tilde{t}$ hence it is true for $r = s/t$. ►

Proposition 4.2 *Let $v \in \mathbb{R}^n$ and $a \in \mathbb{R}$ be such that $|a| < |v|$. Then for arbitrary even $n \geq 2$*

$$\operatorname{Re} \int \frac{d\omega}{(\langle \omega, v \rangle - a - i0)^n} = 0$$

and for arbitrary odd $n \geq 3$

$$\operatorname{Im} \int \frac{d\omega}{(\langle \omega, v \rangle - a - i0)^n} = 0.$$

Proof. We may assume that $|v| = 1$. For even n we have

$$\operatorname{Re} \int \frac{d\omega}{(\langle \omega, v \rangle - a - i0)^n} = \operatorname{Re} \int \frac{d\omega}{(\cos \varphi - a - i0)^n}$$

where φ is the spherical distance between ω and v . We have $d\omega = \sin^{n-2} \varphi d\varphi d\omega'$ where $d\omega'$ is the area of a unit sphere $S^{n-2} = \{\omega \in S^{n-1}; \varphi = \pi/2\}$. Integrating over $n - 2$ -spheres $\varphi = \text{const}$ we get

$$\operatorname{Re} \int \frac{d\omega}{(\cos \varphi - a - i0)^n} = \frac{|S^{n-2}|}{2} (P) \int_0^{2\pi} \frac{\sin^{n-2} \varphi d\varphi}{(\cos \varphi - a)^n}$$

since the integrand is π -periodic. The right hand side vanishes by Lemma 4.1.

For odd n we have

$$\begin{aligned} \operatorname{Im} \int \frac{d\omega}{(\cos \varphi - a - i0)^n} &= \frac{|S^{n-2}|}{2i(n-1)!} \left[\int_0^\pi \frac{\sin^{n-2} \varphi d\varphi}{(\cos \varphi - a - i0)^n} - \frac{\sin^{n-2} \varphi d\varphi}{(\cos \varphi - a + i0)^n} \right] \\ &= \frac{|S^{n-2}|}{2i(n-1)!} \int_{|\varphi - \alpha| = \varepsilon} \frac{\sin^{n-2} \varphi d\varphi}{(\cos \varphi - a)^n} = \frac{\pi |S^{n-2}|}{(n-1)!} \operatorname{res}_\alpha \frac{\sin^{n-2} \varphi d\varphi}{(\cos \varphi - a)^n} \end{aligned}$$

where $\alpha = \arccos a \in [0, \pi]$. Changing variable $\zeta = \cos \varphi$ and omitting the constant coefficient we get the quantity

$$\operatorname{res}_a \frac{(1 - \zeta^2)^m d\zeta}{(\zeta - a)^n}$$

where $m = (n - 3)/2$. The residue is equal to zero since the numerator has order $2m = n - 3$. ►

Corollary 4.3 *For any regular resolved generating function $\Phi = \theta - \lambda$ such that for any pair of points $x \neq y$ in an open set $\Omega \subset X$ we have $\theta(x, \omega) - \theta(y, \omega) = \langle v, \omega \rangle + a$, $|a| < |v|$, formulas (2) and (3) hold for any function $f \in L_2(X)$ with support in Ω .*

5 Reconstruction in spaces of constant curvature

We apply the above results to recover few known and unknown inversion formulas for geodesic integral transforms in spaces of curvature $\kappa = 0, 1, -1$.

Euclidean space. Take the generating function $\Phi(x; \lambda, \omega) = \langle \omega, x \rangle - \lambda$ in $\mathbb{R}^n \times \Sigma$, $\Sigma = \mathbb{R} \times S^{n-1}$. We have $|\nabla\theta| = D_n(x) = 1$. Then (2),(3) coincide with the classical reconstruction in Euclidean space from data of hyperplane integrals.

Elliptic space. Funk [2] inspired by the seminal paper of Minkowski [1] found a reconstruction formula of an even function f on the unit sphere S^2 from integrals Gf for the family of big circles. His method was adapted by Radon for Euclidean plane. A generalization of Funk's formula for arbitrary dimension is due to Helgason [4].

A reconstruction formula of a different form can be obtained if we apply Theorem 3.1 to a generating function $\Phi(x; \lambda, \omega) = \langle \omega, \xi \rangle - \lambda$ defined in $X \times \Sigma$ where $X = \{(x_0, x) \in E^{n+1}, x_0^2 + |x|^2 = 1, x_0 \geq 0\}$, $\Sigma = \{\xi \in E^n : |\xi| < 1\}$ and g is the metric in X induced from the Euclidean space E^{n+1} . Omitting some simple calculations, we come to

Theorem 5.1 *If n is even then any function $f \in L_2(X)_{\text{comp}}$ can be reconstructed from its integrals $Gf(\sigma)$ over big spheres $S(\sigma) = \{x \in X, \langle \sigma, x \rangle = 0\}$, $\sigma \in S_+^n$ by*

$$f(x) = -\frac{(n-1)!}{(2\pi i)^n} \int_{S_+} \frac{Gf(\sigma) d\sigma}{\langle \sigma, x \rangle^n}$$

where $S_+^{n-1} = \{\sigma \in \mathbb{R}^{n+1}; |\sigma| = 1, \sigma_0 \geq 0\}$ is a hemisphere. If n is odd we have

$$f(x) = \frac{1}{2(2\pi i)^{n-1}} \int_{S_+^{n-1}} \delta^{(n-1)}(\langle \sigma, x \rangle) Gf(\sigma) d\sigma.$$

Hyperbolic space ($\kappa = -1$). Take the generating function $\Phi(x; \lambda, \omega) = \theta - \lambda$, $\theta = -2(|x|^2 + 1)^{-1} \langle \omega, x \rangle$, $-1 < \lambda < 1$ in the unit ball $X \subset \mathbb{R}^n$. The hypersurfaces $Z(\lambda, \omega)$ are fully geodesics for the hyperbolic metric $d_g s = 2(1 - |x|^2)^{-1} ds$. By a similar rearrangement we obtain

$$f(x) = -\frac{(n-1)!}{(2\pi i)^n} \int_{Q_+} \frac{Gf(\sigma) d\sigma}{\langle \sigma, x \rangle^n} \quad (12)$$

for even n and

$$f(x) = \frac{1}{2(2\pi i)^{n-1}} \int_{Q_+} \delta^{(n-1)}(\langle \sigma, x \rangle) Gf(\sigma) d\sigma \quad (13)$$

for odd n where $Q_+ = \{\sigma \in \mathbb{R}^{n+1}; \sigma_0^2 - \sigma_1^2 - \dots - \sigma_n^2 = -1, \sigma_0 \geq 0\}$ is the dual one-fold hyperboloid.

A reconstruction in a different form was done first by Radon [3] ($n = 2$) and Helgason [4] ($n > 2$); formulas (12) and (13) are due to Gelfand-Graev-Vilenkin [5].

6 Equidistant spheres and horospheres in hyperbolic space

Equidistant spheres. Let X be again a unit n dimensional ball, $n \geq 2$ and

$$\Phi(x; \lambda, \omega) = \theta - \lambda, \quad \theta(x, \omega) = \frac{p - \langle \omega, x \rangle}{1 - |x|^2}, \quad \omega \in S^{n-1}$$

be a generating function where $0 \leq p < 1$. For a fixed ω and an arbitrary $\lambda \neq 0$ the hypersurface $Z(\lambda, \omega) = \{x; \Phi(x; \lambda, \omega) = 0\}$ is the intersection of X and of an $n-1$ sphere $S(\lambda)$ whereas $S(0) = \{\langle \omega, x \rangle - p = 0\}$ is a hyperplane; all the spheres $S(\lambda)$ contain $n-2$ sphere $S(0) \cap \partial X$. For arbitrary real λ and μ the hypersurfaces $S(\lambda) \cap X$, $S(\mu) \cap X$ are equidistant with respect to the hyperbolic metric. Check that Φ fulfils the conditions of Corollary 4.3 for arbitrary p , $0 \leq p \leq 1$. A proof of regularity is a routine. Further we have

$$\theta(x, \omega) - \theta(y, \omega) = - \left\langle \omega, \frac{x}{1 - |x|^2} - \frac{y}{1 - |y|^2} \right\rangle + p \left(\frac{1}{1 - |x|^2} - \frac{1}{1 - |y|^2} \right).$$

and we need to prove that

$$\left| \frac{x}{1 - |x|^2} - \frac{y}{1 - |y|^2} \right| > \left| \frac{1}{1 - |x|^2} - \frac{1}{1 - |y|^2} \right| \quad (14)$$

for arbitrary $x \neq y \in X$. Squaring both sides we reduce (14) to the obvious inequality $2(1 - xy) > (2 - |x|^2 - |y|^2)$. The proof is complete and reconstructions (2) and (3) follow. Write these formulas in terms of the geodesic integral transform

$$Hf(\sigma) = \int_{Z(\sigma)} f d_g S, \quad \sigma = (\lambda, \omega).$$

where $d_g S$ is the hyperbolic hypersurface element. The Minkowski-Funk operator M_Φ can be written in terms of the Euclidean integral transform $Gf(\sigma) = \int_{Z(\sigma)} f d_e S$ since of the factorization $|\nabla \theta(x, \omega)| = (1 - |x|^2)^{-1} \sqrt{4\lambda^2 - 4p\lambda + 1}$ (see (1)). On the other hand,

$$d_g S = \left(\frac{2}{1 - |x|^2} \right)^{n-1} d_e S$$

which yields

$$Mf(\lambda, \omega) = \frac{Gf_1(\lambda, \omega)}{\sqrt{\lambda^2 - p\lambda + 1/4}} = \frac{Hf_2(\lambda, \omega)}{\sqrt{\lambda^2 - p\lambda + 1/4}}$$

where

$$f_1(x) = \frac{(1 - |x|^2)}{2} f(x), \quad f_2(x) = \left(\frac{1 - |x|^2}{2} \right)^n f(x).$$

Corollary 6.1 For any function f with compact support in the unit ball, a reconstruction is given for even n by

$$f(x) = -\frac{1}{(4\pi i)^n} \frac{(1 - |x|^2)^{2n}}{D_n(x)} \int_{S^{n-1}} \int_{\mathbb{R}} \frac{Hf(\lambda, \omega) d\lambda}{\sqrt{\lambda^2 - p\lambda + 1/4}} \frac{d\omega}{(\langle \omega, x \rangle - \lambda(1 - |x|^2))^n} \quad (15)$$

and for odd n by

$$f(x) = \frac{1}{2(4\pi i)^n} \frac{(1 - |x|^2)^{2n}}{D_n(x)} \int_{S^{n-1}} \frac{\partial^{n-1}}{\partial \lambda^{n-1}} \frac{Hf(\lambda, \omega)}{\sqrt{\lambda^2 - p\lambda + 1/4}} \Big|_{\lambda=\theta(x, \omega)} d\omega. \quad (16)$$

Horospheres. Taking $p = 1$ in the above formulas we obtain the function

$$\theta(x, \omega) = \frac{1 - \langle \omega, x \rangle}{1 - |x|^2}$$

which defines the family of horospheres $\theta(x, \omega) = \lambda, 1/2 < \lambda < \infty$. The formulas (15) and (16) holds for horospheres if we substitute $\sqrt{4\lambda^2 - 4\lambda + 1} = 2\lambda - 1$ and integrate in (15) over the ray $(1/2, \infty)$ in the interior integral.

For the cases $n = 2$ and $n = 3$ reconstruction formulas (of different form) are contained in the book of Gelfand-Gindikin-Graev [8].

7 Isofocal hyperboloids

Hyperboloids. The equation $\lambda = |x| + \varepsilon x_1, \varepsilon > 1$ defines a fold of a two-fold hyperboloid

$$\left(\alpha x_1 - \frac{\varepsilon \lambda}{\alpha}\right)^2 - x_2^2 - \dots - x_n^2 = \frac{\lambda^2}{\alpha^2}, \quad \alpha = \sqrt{\varepsilon^2 - 1}$$

with a focus at the origin. The function $\Phi(x; \lambda, \omega) = \theta(x, \omega) - \lambda, \theta(x, \varphi) = |x| + \varepsilon \langle x, \omega \rangle$ generates the family of all one-fold hyperboloids in $X = \mathbb{R}^n \setminus 0$ with a focus at the origin. The function

$$\theta(x, \omega) - \theta(y, \omega) = \varepsilon \langle \omega, x - y \rangle + |x| - |y|, \quad \omega \in S^{n-1}$$

satisfies the conditions of Proposition 4.2 since $||x| - |y|| < \varepsilon |x - y|$ for any $x, y \in \mathbb{R}^n, x \neq y$. Therefore Theorem 3.1 holds for this family. We have

$$|\nabla \theta(x, \omega)|^2 = 1 + \varepsilon^2 + 2\varepsilon |x|^{-1} \langle \omega, x \rangle$$

and

$$D_n = \int_{S^{n-1}} \frac{d\omega}{(1 + \varepsilon^2 + 2\varepsilon |x|^{-1} \langle \omega, x \rangle)^{n/2}} = |S^{n-2}| \int_0^\pi \frac{\sin^{n-2} \varphi d\varphi}{(1 + \varepsilon^2 - 2\varepsilon \cos \varphi)^{n/2}}$$

where φ is the angle between ω and $|x|^{-1}x$. This integral does not depend on x .

Corollary 7.1 For any smooth function f with compact support in \mathbb{R}^n and even n , the equation holds

$$f(x) = -\frac{1}{(2\pi i)^n D_n} \int_{S^{n-1}} \int_{\mathbb{R}} \frac{Mf(\lambda, \omega) d\lambda d\omega}{(|x| + \varepsilon \langle \omega, x \rangle - \lambda)^n}, \quad n \text{ even}$$

$$f(x) = \frac{1}{2(2\pi i)^{n-1} D_n} \int_{S^{n-1}} \frac{\partial^{n-1}}{\partial \lambda^{n-1}} Mf(\lambda, \omega)|_{\lambda=|x|+\varepsilon\langle x, \omega \rangle} d\omega, \quad n \text{ odd.}$$

8 Photoacoustic geometries

Consider a resolved generating function

$$\Phi(x; \lambda, \omega) = |x - \xi(\omega)|^2 - \lambda, \quad \omega \in S^{n-1}$$

where $\xi : S^{n-1} \rightarrow \mathbb{R}^n$ is a smooth map. We call the image \mathbf{C} of ξ central set. Any surface $Z(\lambda, \omega)$ is a sphere of radius $\sqrt{\lambda}$ with the center $\xi(\omega) \in \mathbf{C}$ and

$$M_{\Phi} f(\lambda, \omega) = \frac{Gf(\lambda, \omega)}{2\sqrt{\lambda}}$$

where $Gf(\lambda, \omega)$ is the Euclidean integral over this sphere. Inversion of the operator M_{Φ} implies inversion of the spherical integral transform G for the given central surface \mathbf{C} (and vice versa). This subject is of special interest in view of application to the photoacoustic (thermoacoustic) tomography, see surveys [13],[14]. Inversion formulas for a function supported in a half space with the hyperplane central set is found by Fawcett [7]. In [17] a reconstruction was done by reduction to the Radon transform. For a spherical central surface Finch with coauthors [9],[11] found a reconstruction formula of type (17)-(18) in the physical domain for arbitrary dimension. Another reconstruction formula was proposed by Kunyanski [12]; it is similar to (17)-(18) after a simplification. An inversion of the spherical mean operator in three dimensional space was constructed by Xu and Wang [10] in terms of a Dirichlet-Green function. This approach gives an explicit reconstruction in the frequency domain when \mathbf{C} is a sphere or a circular cylinder. Kunyanski [15] construct inversion for polyhedral center sets with special symmetries. Natterer [16] has found an explicit inversion in the physical domain when the central surface \mathbf{C} is an arbitrary ellipsoid. Surveys of related results are done in [11],[10],[15]. We show here that for an arbitrary ellipsoid (elliptical cylinder) in R^n a simple reconstruction is given by Theorem 3.1.

Ellipsoids. Set $\xi(\omega) = (a_1\omega_1, \dots, a_n\omega_n)$ where a_1, \dots, a_n are positive constants. The central hypersurface $\{x = \xi(\omega), \omega \in S^{n-1}\}$ is the boundary of an ellipsoid E_a with half axes a_1, \dots, a_n . Then

$$\theta(x, \omega) - \theta(y, \omega) = 2 \langle \xi(\omega), y - x \rangle + |x|^2 - |y|^2 = 2 \langle \omega, z \rangle + |x|^2 - |y|^2$$

where $z = (a_1(y_1 - x_1), \dots, a_n(y_n - x_n))$. The inequality holds

$$\left| |x|^2 - |y|^2 \right| = \left| \sum (y_i - x_i)(y_i + x_i) \right| \leq \sum |a_i(y_i - x_i)| \sum |a_i^{-1}(y_i + x_i)| \leq \|z\| \|w\|$$

where $w = (a_1^{-1}(x_1 + y_1), \dots, a_n^{-1}(x_n + y_n))$. Suppose that $x, y \in E_a$ and $x \neq y$ then the point $(x + y)/2$ belongs to the interior of E_a which implies $\|w\| < 2$. It follows that the right hand side of is strictly bounded by $2\|z\|$. By Proposition 4.2 Theorem 3.1 holds for any $n \geq 2$. Note that $|\nabla\theta| = 2|x - \xi(\omega)| = 2\sqrt{\lambda}$. It follows that any function f supported in the closed ellipsoid E_a can be reconstructed by the formula

$$f(x) = \frac{1}{(2\pi i)^n D_n(x)} \int_{S^{n-1}} (P) \int_{\mathbb{R}} \frac{Gf(\rho^2, \omega) d\rho d\omega}{(|x - \xi(\omega)|^2 - \rho^2)^n} \quad (17)$$

for even n where we did the substitution $\lambda = \rho^2$, and by

$$f(x) = \frac{1}{4(2\pi i)^{n-1} D_n(x)} \int_{S^{n-1}} \left(\frac{\partial}{\partial \rho^2} \right)^{n-1} \frac{Gf(\rho^2, \omega)}{\rho} \Big|_{\rho=|x-\xi(\omega)|} d\omega \quad (18)$$

for odd n where

$$D_n(x) = \frac{1}{2^n |S^{n-1}|} \int \frac{d\omega}{|x - \xi(\omega)|^n} \quad (19)$$

Cylinders. If a central set \mathbf{C} is unbounded can not apply the same method since it can not be regularly parametrized. However one can write a reconstruction formula for any closed cylinder \mathbf{E} with an elliptic base. Indeed, \mathbf{E} is a union of the family of ellipsoids E_a as one or several half axes, say a_1, \dots, a_p , tend to infinity, a_{p+1}, \dots, a_n being fixed. One can come to limits in (17) and in (18). We omit details.

Algebraic plane curves. In the case $n = 2$, there are more geometries which admit exact reconstruction formulas. We call a curve $\mathbf{C} \subset \mathbb{R}^2$ *trigonometric* of degree k if it is given by a parametric equation

$$x_1 = \xi_1(\varphi), x_2 = \xi_2(\varphi), \varphi \in S^1$$

where ξ_1, ξ_2 are real trigonometric polynomials of degree k . A trigonometric curve is always a component of a real algebraic curve. A point $x \in \mathbb{R}^2$ is called *hyperbolic* with respect to a trigonometric curve \mathbf{C} if any straight line L through x meets the curve at $2k$ points (counting with multiplicities). It is easy to see that the set H (called hyperbolic set) of all hyperbolic points is always closed and convex. Introduce an Euclidean structure in \mathbb{R}^2 and consider a function

$$\theta(x, \varphi) = |x - \xi(\varphi)|^2, \xi(\varphi) = (\xi_1(\varphi), \xi_2(\varphi)), \varphi \in S^1, x \in \mathbb{R}^2.$$

Proposition 8.1 *Let H be the set of hyperbolic points with respect to a trigonometric curve of degree k . For arbitrary points $x, y \in H$, $x \neq y$, all roots of the polynomial $\theta(x, \varphi) - \theta(y, \varphi)$ (of order k) are real.*

Proof. We have

$$\begin{aligned}\theta(x, \varphi) - \theta(y, \varphi) &= |x - \xi(\varphi)|^2 - |y - \xi(\varphi)|^2 = |x|^2 - |y|^2 - 2\langle x - y, \xi(\varphi) \rangle \\ &= 2\langle x - y, \xi(\varphi) - s \rangle\end{aligned}\tag{20}$$

where $s = (x + y) / 2$. This point is contained in H since H is convex. Therefore, the line $L = \{z = s + rv, r \in \mathbb{R}\}$ has $2k$ common points $\xi(\varphi_1), \dots, \xi(\varphi_{2k})$ with \mathbf{C} for arbitrary vector $v \neq 0$. If v is orthogonal to $x - y$ the right hand side of (20) vanishes. The corresponding angles $\varphi_1, \dots, \varphi_{2k}$ are real roots of the polynomial $\theta(x, \varphi) - \theta(y, \omega)$. \blacktriangleright

The family of circles centered at the curve \mathbf{C} is generated by the function $\Phi(x; \lambda, \varphi) = \theta(x, \varphi) - \lambda$. Applying Lemma 4.1 we get

Corollary 8.2 *Let \mathbf{C} be a trigonometric curve with a hyperbolic set H . Theorem 3.1 holds for the family of circles centered at \mathbf{C} , arbitrary function f supported in H and for both integral transforms M_{Φ} and G .*

There is a large variety of trigonometric curves \mathbf{C} with non empty hyperbolic sets.

Examples. 1. Let $\xi_1(\varphi) = 2 \cos 2\varphi - \cos \varphi$, $\xi_2(\varphi) = 2 \sin 2\varphi + \sin \varphi$. The curve \mathbf{C} is shown in Fig.1

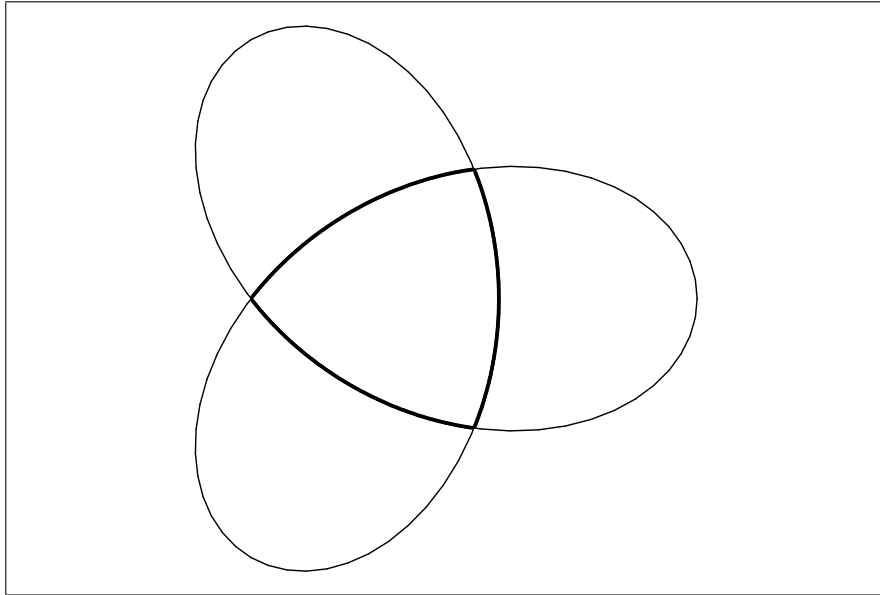


Fig.1

The hyperbolic set H is the triangle in the middle.

2. A hyperbolic "square" set is defined by the trigonometric curve $\xi_1(\varphi) = 2 \cos 3\varphi + \cos \varphi$, $\xi_2(\varphi) = 2 \sin 3\varphi - \sin \varphi$, see Fig.2

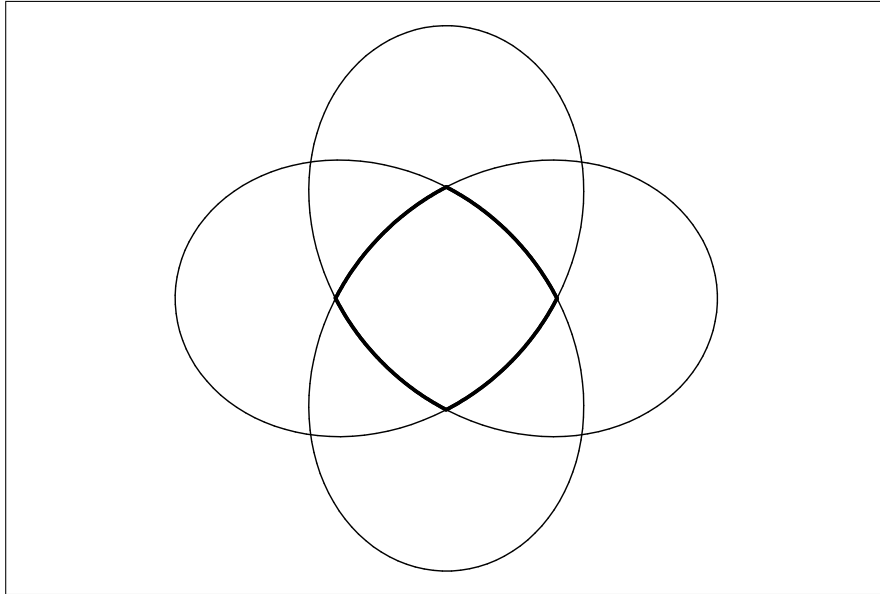


Fig.2

3. A "pentagon" is the hyperbolic set of the curve $\xi_1(\varphi) = 5 \cos 4\varphi + 4 \cos \varphi$, $\xi_2(\varphi) = 5 \sin 4\varphi - 4 \sin \varphi$, see Fig.3:

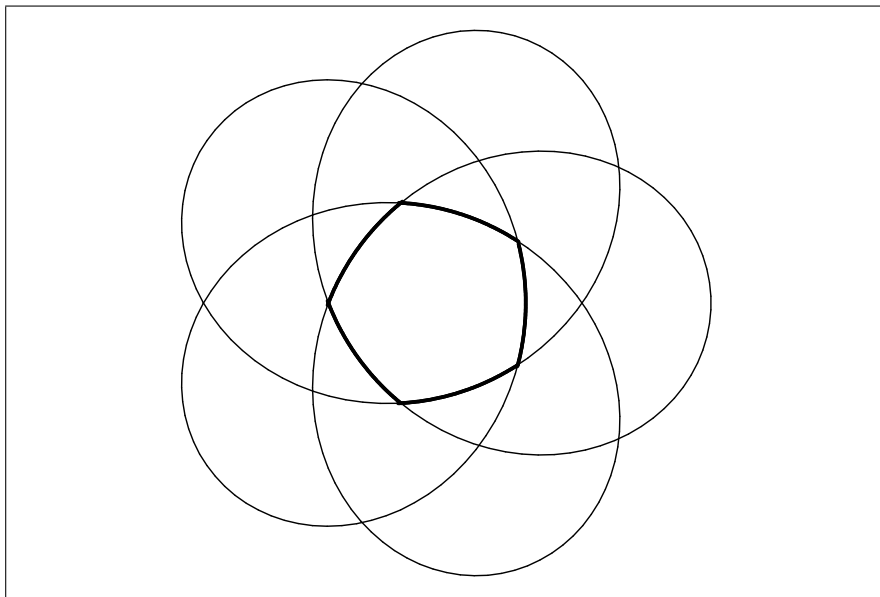


Fig.3

etc.

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