On reconstruction of strain fields from tomographic data

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Abstract. A method of reconstruction of a strain tensor ε in a solid body was described that uses non redundant data of the axial (logitudinal) and of the traceless normal ray integrals.

1 Introduction

Information about residual elastic strain is of primary importance in understanding deformation and stress within structural material and components. The evaluation of this strain requires imaging a six-component tensor quantity in three dimensions. An application of tomographic ideas to reconstruction of small residual strain fields in a body from data of diffraction pattern under penetrated X-ray or neutron radiation was proposed by A. Korsunsky *et al* [3],[4]. The mathematical model is the longitudinal (axial) line transform of a strain tensor ε [4]. Note that all the integrals vanish if ε is a potential tensor that is $\varepsilon = \mathrm{D}u$ for a small deformation u, where D is the differential for symmetric tensors. Adhemar de Saint-Venant (1860) introduced the differential consistency equation $\mathrm{V}\varepsilon = 0$ for a small tensor field called later by his name. Boussinesq (1871), Beltrami (1889), Cesaro (1906) proved its sufficiency for a 2-tensor to be potential in a simply connected domain. The general case was considered by Volterra [14]. Sharafutdinov [11] studied a more general situation. He has proved that a tensor field g in $\mathbb{R}^{n:}$ of rank m with compact support is potential if (and only if) the axial line transform $\mathrm{L}g$ of g vanishes for all lines. Recently, Paternain, Salo and Uhlmann proved that this property holds for the geodesic transform in simple surfaces with boundary (for any pair of points on the boundary, there is only one connecting geodesic). See [6] for further results in this direction.

A reconstruction of a 2-tensor field from only knowledge of $\mathrm{V}g$ is impossible, since the Saint-Venant tensor vanishes for any potential field. The solenoidal part of a tensor field is the unique solution of the system $\mathrm{D}^* ({}^s g) = 0,\ \mathrm{V} ({}^s g) = \mathrm{V}g$ in $\mathbb{R}^n$ that tends to zero at infinity. Sharafutdinov [11] and Denisjuk [2] gave integral formulas expressing the solenoidal part ${}^s g$ of the tensor g in terms of $\mathrm{V}g$. Denisjuk [2] has developed an algorithm for reconstruction of the field $\mathrm{V}g$ from data of axial line integrals over rays in $\mathbb{R}^n$ with sources on a curve. Other methods of reconstruction of the solenoidal part were given in [12]. Note that the support of the solenoidal gauge ${}^s g$ is not compact unless g fulfils an infinite number of orthogonality conditions. Thereby the information contained in the axial line integrals of g and inherited in ${}^s g$ is spread over the whole space.

This paper is focused on the case $n = 3$. In Sec. 3 we give a new method of recovering of $\mathrm{V}\varepsilon$ from data of axial line integrals for a 2-tensor field ε. The integral data is collected from rays emanated from a curve Γ fulfilling Tuy's completeness condition [13]. In particular, any curve Γ in the improper sphere can be taken if the end points of Γ are antipodal. In this case the integrals are taken over lines in E parallel to one of vectors $\gamma \in \Gamma$. For determination of a 2-tensor field,

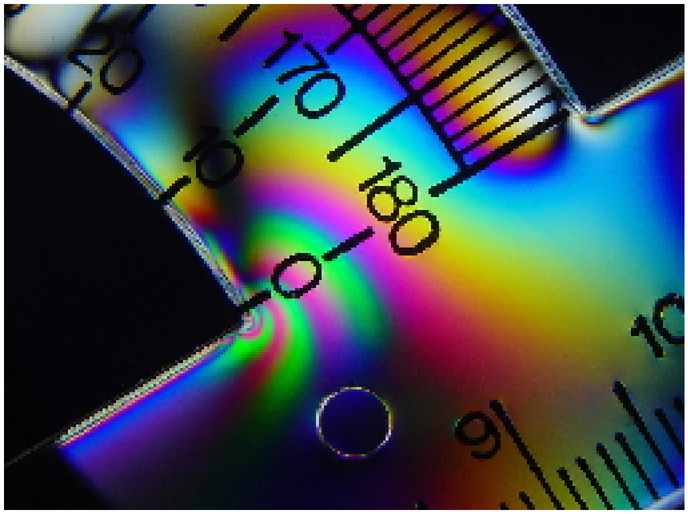

Figure 1: Strain in plastic protractor seen under polarized light

we need measurements of second order derivatives of the ray integrals with respect to the source point.

The polarization tomography is another method of reconstruction of a strain field in a transparent solid see Fig.1. It is based on measurements of transformation of the polarization ellipse of the penetrating light. The mathematical model is the line integral $\mathrm{T}\varepsilon$ of the traceless normal (truncated transverse) part of the stress field ε. Paper [1] is focused on a practical implementation, see also the list of references therein. For the mathematical background, see [11]. Puro [10] developed a method of reconstruction of the stress tensor by means of photoelasticity method using of magnetic fields for obtaining additional information. In [5] a method of complete reconstruction of a traceless 2-tensor ε is proposed form knowledge of the line integral $\mathrm{T}\varepsilon$ over all lines orthogonal to three or six vectors in general position. We give a simple method of reconstruction of the displacement form φ from data of $\mathrm{T}\varepsilon$ for tensor any tensor ε whose axial line integrals vanish (§5). The acquisition geometry is the same as above. The methods of Sec. 3-5 are combined in Sec. 6 for an algorithm of complete reconstruction of an arbitrary strain tensor ε from non redundant data of ray integrals $\mathrm{X}\varepsilon$ and $\mathrm{T}\varepsilon$.

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2 Symmetric tensors and differentials

Let E be an Euclidean space of dimension 3. Choose an euclidean coordinate system $\left(x^1, x^2, x^3\right)$. We will denote by $y^i, z^i, u^i, v^i, \ldots\ i = 1, 2, 3$, components of points or vectors $y, z, u, v, \ldots \in E$, respectively. Tensor fields of rank 1 and 2 are expressions of the form

$$f = \sum f_i \mathrm{d}x^i,\ \ g = \sum g_{ij} \mathrm{d}x^i \cdot \mathrm{d}x^j,$$

respectively, where the symbol $\cdot$ means a symmetric product that is $\mathrm{d}x^j \cdot \mathrm{d}x^i = \mathrm{d}x^i \cdot \mathrm{d}x^j$. The components f_i and $g_{ij} = g_{ji}$ are functions in E which transform as vectors and bivectors, respectively, under coordinate changes. We can write any 2-tensor field g defined in $U \subset E$ as a function in $U \times E \times E$ bilinear and symmetric in vector variables

$$g\left(x;u,v\right)=\sum g_{ij}\left(x\right)u^{i}v^{j},\ u=\left(u^{1},u^{2},u^{3}\right),\ v=\left(v^{1},v^{2},v^{3}\right).$$

We will also use the abbreviation $g\left(y;v\right) \doteqdot g\left(y;v,v\right)$. The spaces of smooth or singular (generalized) tensor fields in E of ramk 1 and 2 we denote by Σ^1 and Σ^2 respectively. The symmetric differential $\mathrm{D}:\Sigma^1 \to \Sigma^2$ reads

$$\mathrm{D}f=g,\ g_{ii}=\partial_i f_i,\ g_{ij}=\frac{1}{2}\left(\partial_i f_j+\partial_j f_i\right),\ \partial_i=\partial/\partial x^i, \tag{1}$$

where no summation on repeating indices is assumed. Let Λ^2 be the space (bundle) of skew symmetric differential forms of degree 2 and $B^4 \doteqdot \Lambda^2 \otimes_S \Lambda^2$ be the symmetric square of this bundle. An element $b \in B^4$ is a tensor field whose components $b_{ij,kl}$ are functions in E that are skew symmetric in (i,j) and in (k,l) and symmetric with respect to permutation $(i,j) \leftrightarrow (k,l)$. The *Saint-Venant* operator $\mathrm{V}:\Sigma^2 \to B^4$ is defined for a tensor field $g \in \Sigma^2$ by

$$\left(\mathrm{V}g\right)_{ij,kl} \doteqdot \partial_i\partial_k g_{jl}-\partial_i\partial_l g_{jk}-\partial_j\partial_k g_{il}+\partial_j\partial_l g_{ik}.$$

The fields $\partial_i,\partial_j,\partial_k,\partial_l$ can be replaced here by arbitrary tangent vectors $\alpha,\beta,\gamma,\delta$ in E:

$$\left(\mathrm{V}g\right)_{\alpha\beta,\gamma\delta}=\partial_\alpha\partial_\gamma g\left(\beta,\delta\right)-\partial_\alpha\partial_\delta g\left(\beta,\gamma\right)-\partial_\beta\partial_\gamma g\left(\alpha,\delta\right)+\partial_\beta\partial_\delta g\left(\alpha,\gamma\right). \tag{2}$$

Tensor $\mathrm{V}g$ vanishes for any potential field $g=\mathrm{D}f$, since $\mathrm{VD}=0$.

3 Reconstruction from ray integrals

For a 2-tensor field g in E with compact support and bounded components, ay and line integrals

$$\mathrm{X}g\left(y;v\right)=\int_0^\infty g\left(y+tv;v\right)\mathrm{d}t,\ \mathrm{L}g\left(y;v\right)=\int_{-\infty}^\infty g\left(y+tv;v\right)\mathrm{d}t$$

are well defined for any point $y \in E$ and any unit vector v; y is called source point for the ray integral. For an arbitrary vector $\alpha \in E$, we denote

$$\partial_\alpha \mathrm{X}g\left(y;v\right)=\left\langle \alpha,\nabla_y\right\rangle \mathrm{X}g\left(y;v\right),\ \partial_{;\alpha}\mathrm{X}g\left(y;v\right)=\frac{\partial}{\partial t}\,\mathrm{X}g\left(y;v+t\alpha\right)\Big|_{t=0}.$$

Let S^2 denote the unit sphere in E and Ω be area form in S^2. For $p \in \mathbb{R},\ \omega \in \mathrm{S}^2$, $H\left(p,\omega\right)$ will mean the plane $\left\{\left\langle \omega,x\right\rangle=p\right\}$ in E;

Theorem 1 *Let $\Gamma \subset E$ be a piecewise smooth curve and g be a smooth 2-tensor g with compact support in $E\backslash\Gamma$. For an arbitrary point $x \in E\backslash\Gamma$ such that any plane H through x meets Γ transversely at a point y, tensor $\mathrm{V}g\left(x\right)$ can be reconstructed from data of ray integrals with source points in Γ by*

$$\left(\mathrm{V}g\right)_{\alpha\beta;\gamma\delta}\left(x\right)=-\frac{1}{8\pi^2}\int_{\omega\in\mathrm{S}^2}\partial_p \mathrm{R}_{H(p,\omega)}\mid_{p=\langle x,\omega\rangle}\left(\alpha,\beta;\gamma,\delta\right)\Omega, \tag{3}$$

where $\alpha,\beta,\gamma,\delta \in E$ are arbitrary, and for any plane H through the point $y \in \Gamma$

$$\begin{aligned}\mathrm{R}_H\left(\alpha,\beta;\gamma,\delta\right)=&\ \mathrm{R}_H\left(\left\langle\beta,\omega\right\rangle\alpha,\omega;\left\langle\delta,\omega\right\rangle\gamma,\omega\right)-\mathrm{R}_H\left(\left\langle\alpha,\omega\right\rangle\beta,\omega;\left\langle\delta,\omega\right\rangle\gamma,\omega\right)\\&-\mathrm{R}_H\left(\left\langle\beta,\omega\right\rangle\alpha,\omega;\left\langle\gamma,\omega\right\rangle\delta,\omega\right)+\mathrm{R}_H\left(\left\langle\alpha,\omega\right\rangle\beta,\omega;\left\langle\gamma,\omega\right\rangle\delta,\omega\right),\end{aligned} \tag{4}$$

where
$$\mathrm{R}_H(\alpha,\omega;\beta,\omega)=\frac{1}{2}\int_0^{2\pi}\partial_\alpha\partial_\beta\partial^3_{;\omega}\mathrm{X}g(y;v)\,\mathrm{d}\theta. \tag{5}$$

Remark. This acquisition geometry is essentially the same as in paper [8], where a differential 1-form is reconstructed from first derivatives of its ray integrals. For reconstruction of a 2-tensor, we need the second derivatives $\partial^2\mathrm{X}g(y;v)/\partial y^i\partial y^j$ of the ray integrals. This is a non redundant volume of the integral data. In medical tomography the similar acquisition geometry is implemented by rotation of the pair consisting of a X-ray source and a plate of detectors on opposite side of the patient. This complicated machinery is not necessary for strain tomography of solid samples. Here the X-ray or the neutron source and the detectors may be fixed, whereas a specimen is mechanically manipulated following the given *relative* acquisition geometry.

Lemma 2 *For an arbitrary 2-tensor g, any plane $H=H(p,\omega)$, any point $y\in H$ and arbitrary vectors α,β parallel to H, we have*
$$\partial_p\int_{H(p,\omega)}(\mathrm{V}g)_{\omega\alpha,\omega\beta}\ \mathrm{d}H=\frac{1}{2}\int_{v\in\mathrm{S}(\omega)}\partial_\alpha\partial_\beta\partial^3_{;\omega}\mathrm{X}g(y;v)\,\mathrm{d}\theta, \tag{6}$$
where $\mathrm{S}(\omega)$ is the unit circle in $H(0,\omega)$, $\mathrm{d}\theta$ is the angular measure in $\mathrm{S}(\omega)$ and $\mathrm{d}H$ is the area density in the plane $H(p,\omega)$.

Proof of Lemma. By (2)
$$\int_{H(p,\omega)}(\mathrm{V}g)_{\omega\alpha,\omega\beta}\,\mathrm{d}H=\partial_p^2\int_{H(p,\omega)}g(x;\alpha,\beta)\,\mathrm{d}H, \tag{7}$$
Here and below g is written as a bilinear function of vector variables. For arbitrary $y\in H$ and any unit vector v parallel to H,
$$\partial_{;\omega}\mathrm{X}g(y;v)=\int_0^\infty\partial_\omega g(y+tv;v,v)\,t\mathrm{d}t+2\int_0^\infty g(y+tv;\omega,v)\,\mathrm{d}t,$$
$$\ldots$$
$$\begin{aligned}\partial^3_{;\omega}\mathrm{X}g(y;v)&=\int_0^\infty\partial^3_\omega g(y+tv;v,v)\,t^3\mathrm{d}t\\&+6\int_0^\infty\partial^2_\omega g(y+tv;\omega,v)\,t^2\mathrm{d}t+6\int_0^\infty\partial_\omega g(y+tv;\omega,\omega)\,t\mathrm{d}t.\end{aligned} \tag{8}$$
Take an arbitrary point $y\in\Gamma$ and a plane $H(p,\omega)$ that meets Γ transversely in y. Choose a basis e_2,e_3 in $H(0,\omega)$ and have $v=\cos\theta e_2+\sin\theta e_3$ for some $\theta\in\mathrm{S}^1$. We have
$$g(y+tv;\omega,v)\,t=g(x;\omega,e_2)\left(x^2-y^2\right)+g(x;\omega,e_3)\left(x^3-y^3\right)$$
where $x\doteqdot y+tv\in H(p,\omega)$, $x^2-y^2=t\cos\theta,\ x^3-y^3=t\sin\theta$. Substitute this equation in (8), integrate against the form $\mathrm{d}\theta$ and take into account that $t\mathrm{d}t\mathrm{d}\theta=\mathrm{d}H$ and $\partial_\alpha=\langle\alpha,\nabla_x\rangle$:
$$\begin{aligned}\int_0^{2\pi}\partial_\alpha\partial^3_{;\omega}\mathrm{X}g(y;v)\,\mathrm{d}\theta&=\sum_{i,j=2}^3\int_H\partial_\alpha\partial^3_\omega g(x;e_i,e_j)(x-y)^i(x-y)^j\,\mathrm{d}H\\&+6\sum_{j=2}^3\int_H\partial_\alpha\partial^2_\omega g(x;\omega,e_j)(x-y)^j\,\mathrm{d}H\\&+6\int_H\partial_\alpha\partial_\omega g(x;\omega,\omega)\,\mathrm{d}H.\end{aligned}$$

Integrating by parts yields

$$\int_0^{2\pi} \partial_\alpha \partial^3_{;\omega} \mathrm{X} g\left(y; v\right) \mathrm{d}\theta = -2 \int_H \partial^3_\omega g\left(x; e_i, \alpha\right) \left(x-y\right)^i \mathrm{d}H - 6 \int_H \partial^2_\omega g\left(x; \omega, \alpha\right) \mathrm{d}H,$$

since the integral of $\partial_\alpha \partial_\omega g$ over H vanish. Integrating by parts once again we get

$$\begin{aligned}\int_0^{2\pi} \partial_\beta \partial_\alpha \partial^3_{;\omega} \mathrm{X} g\left(y; v\right) \mathrm{d}\theta &= -2 \int_H \partial_\beta \partial^3_\omega g\left(x; e_i, \alpha\right) y^i \mathrm{d}H - 6 \int_H \partial_\beta \partial^2_\omega g\left(x; \beta, \alpha\right) \mathrm{d}H \\ &= 2\partial^3_p \int_H g\left(x; \beta, \alpha\right) \mathrm{d}H\end{aligned}$$

where again $x = y + tv$. Finally

$$\partial_p \int_{H(p,\omega)} \left(\mathrm{V}g\right)_{\omega\alpha,\omega\beta} \mathrm{d}H = \frac{1}{2} \int_0^{2\pi} \partial_\alpha \partial_\beta \partial^3_{;\omega} \mathrm{X} g\left(y; v\right) \mathrm{d}\theta.$$

This together with (7) implies (6). $\square$

Proof of Theorem. For arbitrary $\alpha, \beta, \gamma, \delta \in E$ and plane $H \subset E$, we denote

$$\mathrm{R}_H\left(\alpha, \beta; \gamma, \delta\right) = \partial_p \int_H \left(\mathrm{V}g\right)_{\alpha\beta,\gamma\delta} \mathrm{d}H, \tag{9}$$

where the right hand side is known from Lemma 2 and equation (5) follows from (6). Check equation

$$\begin{aligned}\mathrm{R}_H\left(\alpha, \beta; \gamma, \delta\right) &= \mathrm{R}_H\left(\langle\beta, \omega\rangle \alpha, \omega; \langle\delta, \omega\rangle \gamma, \omega\right) - \mathrm{R}_H\left(\langle\alpha, \omega\rangle \beta, \omega; \langle\delta, \omega\rangle \gamma, \omega\right) \\ &\quad - \mathrm{R}_H\left(\langle\beta, \omega\rangle \alpha, \omega; \langle\gamma, \omega\rangle \delta, \omega\right) + \mathrm{R}_H\left(\langle\alpha, \omega\rangle \beta, \omega; \langle\gamma, \omega\rangle \delta, \omega\right),\end{aligned} \tag{10}$$

where ω is a unit normal vector to H. The left and the right hand sides are skew symmetric in (α, β) and symmetric in pairs (α, β) and (γ, δ). If α and β are parallel to H, then the right hand side of (10) vanish. The same is true for the left hand side since of (2). The same is true if γ and δ are parallel to H. In the general case we write $\alpha = \alpha' + a\omega$, $\beta = \beta' + b\omega$ where α', β' are parallel to H and $a, b \in R$. Then we have

$$\mathrm{R}_H\left(\alpha, \beta; \gamma, \delta\right) = \mathrm{R}_H\left(\alpha', \beta'; \gamma, \delta\right) + a\mathrm{R}_H\left(\omega, \beta'; \gamma, \delta\right) + b\mathrm{R}_H\left(\alpha', \omega; \gamma, \delta\right),$$

where the first term vanishes. A similar equation holds for the right hand side of (10). Finally, it is sufficient to consider the case $\alpha = \gamma = \omega$ and β and δ are parallel to H. Then all terms of the right hand side of (10) vanish except the last one which coincides with the left hand side. Therefore (10) holds for all $\alpha, \beta, \gamma, \delta$. Finally, we recover the Saint-Venant tensor at $x \in E$ by H. Lorentz's formula

$$\left(\mathrm{V}g\right)_{\alpha\beta;\gamma\delta}(x) = -\frac{1}{8\pi^2} \int_{\omega \in \mathrm{S}^2} \Omega\, \partial^2_p \int_{H(p,\omega)} \left(\mathrm{V}g\right)_{\alpha\beta,\gamma\delta} \Big|_{p=\langle\omega,x\rangle} \mathrm{d}H$$

and substitute (9), where the left hand side is already determined by means of (6). $\square$

4 A gauge field

Let ε be an unknown 2-tensor field with compact support in E such that the Saint-Venant tensor $\mathrm{V}\varepsilon$ is known. We want to construct a field g with compact support such that $\mathrm{V}g = \mathrm{V}\varepsilon$. We call g a *gauge* of ε. The field ε need not to be smooth e.g. it can has discontinuity on the boundary of the specimen. In this case the components of $\mathrm{V}\varepsilon$ are singular (generalized) functions in E.

Theorem 3 *For any 2-tensor field ε supported in convex compact K in E and a neighborhood U of K, there exists a gauge field e supported in U that fulfils equation $\mathrm{V}g = \mathrm{V}\varepsilon$ and can be explicitly constructed from data of $\mathrm{V}\varepsilon$.*

Proof. Set $h = \{h_{ij}\} \doteqdot \mathrm{V}\varepsilon$ and write the Saint-Venant system (2) for a gauge field $g = \{g_{ij}\}$:

$$\begin{aligned} \partial_{22}g_{11} - 2\partial_{12}g_{12} + \partial_{11}g_{22} &= h_{33}, \\ \partial_{33}g_{11} - 2\partial_{13}g_{13} + \partial_{11}g_{33} &= h_{22}, \\ -\partial_{23}g_{11} + \partial_{12}g_{13} + \partial_{13}g_{12} - \partial_{11}g_{23} &= h_{23}, \\ \partial_{33}g_{22} - 2\partial_{23}g_{23} + \partial_{22}g_{33} &= h_{11}, \\ -\partial_{12}g_{33} + \partial_{13}g_{23} + \partial_{23}g_{13} - \partial_{33}g_{12} &= h_{12}, \\ -\partial_{13}g_{22} + \partial_{12}g_{23} + \partial_{23}g_{12} - \partial_{22}g_{13} &= h_{13}, \end{aligned} \tag{11}$$

where $\partial_{ij} = \partial_i\partial_j,\ i,j = 1,2,3$ and $g_{ij} = (\mathrm{V}\varepsilon)_{ki,kj}$.

Step 1. Find functions $g_{ij},\ i,j = 1,2$ such that

$$h_{33} = \partial_{22}g_{11} - 2\partial_{12}g_{12} + \partial_{11}g_{22}. \tag{12}$$

Equations (2) mean that (11) hold with g_{ij} replaced by ε_{ij}. In particular the first line of (11) with g replaced by ε implies equations

$$\int_0^{a_2}\int_0^{a_1} h_{33}(x)\,\mathrm{d}x^1\mathrm{d}x^2 = \int\int x^1 h_{33}(x)\,\mathrm{d}x^1\mathrm{d}x^2 = \int\int x^2 h_{33}(x)\,\mathrm{d}x^1\mathrm{d}x^2 = 0 \tag{13}$$

which hold for any $x^3 \in \mathbb{R}$, since all ε_{ij} have compact support. Suppose for simplicity that $K = \left\{0 \le x^i \le a_i, i = 1,2,3\right\}$ and take a smooth function e_0 with compact support in $[0, a_1]$ such that $\int e_0 \mathrm{d}t = 1$. The function

$$f_0\left(x^1,x^2,x^3\right) = h_{33}\left(x^1,x^2,x^3\right) - e_0\left(x^1\right)\int_0^{a_1} h_{33}\left(t,x^2,x^3\right)\mathrm{d}t$$

fulfils

$$\int_0^{a_1} f_0\left(t,x^2,x^3\right)\mathrm{d}t = 0 \tag{14}$$

for any x^2, x^3. Functions

$$f_1(x) = \int_0^{x^1} f_0\left(t,x^2,x^3\right)\mathrm{d}t,\ \ f_2(x) = e_0\left(x^1\right)\int_0^{x^2}\int_0^{a_1} h_{33}\left(t,s,x^3\right)\mathrm{d}t\mathrm{d}s$$

satisfy $h_{33} = \partial_1 f_1 + \partial_2 f_2$, and are supported in K since of (14) and of the first equation (13), respectively. By (13)

$$\int_{\mathbb{R}^2} f_i \mathrm{d}x^1\mathrm{d}x^2 = \int_{\mathbb{R}^2} x^i h_{33}\mathrm{d}x^1\mathrm{d}x^2 = 0,\ \ i = 1,2.$$

By repeating the previous arguments we can write $f_i = \partial_1 f_{1i} + \partial_2 f_{2i}, i = 1, 2$ with some functions f_{ji} supported in K. Equation (12) holds with $g_{11} = f_{11},\ g_{22} = f_{22}$ and $g_{12} = -1/2\left(f_{12} + f_{21}\right)$.

Step 2. By the same method and the second line of (11) we find a solution $(g'_{11}, g'_{13}, g'_{33})$ of the equation

$$h_{22} = \partial_3\partial_3 g'_{11} - 2\partial_1\partial_3 g'_{13} + \partial_1\partial_1 g'_{33} \tag{15}$$

supported in K. Note that g'_{11} need not to coincide with g_{11}.

Step 3. Check that

$$\int_0^{a_1} \left(g'_{11}\left(t, x^2, x^3\right) - g_{11}\left(t, x^2, x^3\right)\right) \mathrm{d}t = 0. \tag{16}$$

By (12) and (15) we have

$$\partial_{22}\left(g_{11} - \varepsilon_{11}\right) - 2\partial_{12}\left(g_{12} - \varepsilon_{12}\right) + \partial_{11}\left(g_{22} - \varepsilon_{22}\right) = 0, \tag{17}$$

$$\partial_{33}\left(g'_{11} - \varepsilon_{11}\right) - 2\partial_{13}\left(g'_{13} - \varepsilon_{13}\right) + \partial_{11}\left(g'_{33} - \varepsilon_{33}\right) = 0, \tag{18}$$

and (17) implies

$$\partial_{22}\int_0^{a_1} \left(g_{11}\left(t, x^2, x^3\right) - \varepsilon_{11}\left(t, x^2, x^3\right)\right) \mathrm{d}t = 0.$$

The equation holds with the derivative ∂_{22} omitted, since $g_{11} - \varepsilon_{11}$ has compact support. By (18) and similar arguments we obtain

$$\int_0^{a_1} \left(g'_{11}\left(t, x^2, x^3\right) - \varepsilon_{11}\left(t, x^2, x^3\right)\right) \mathrm{d}t = 0.$$

Subtracting the previous equation from the former one we get (16). Therefore, function

$$d\left(x\right) = \int_0^{x^1} \left(g'_{11}\left(t, x^2, x^3\right) - g_{11}\left(t, x^2, x^3\right)\right) \mathrm{d}t$$

is supported in K and satisfies $\partial_1 d = g'_{11} - g_{11}$. Set $g_{13} = g'_{13} - 1/2\partial_3 d,\ g_{33} = g'_{33}$ and have $h_{22} = \partial_{33} g_{11} - 2\partial_{13} g_{13} + \partial_{11} g_{33}$, since of (15).

Step 4. Now we only need to find a component g_{23} that fulfils the third line of (11). Consider the 2-field $\tilde{g} = \left\{g_{ij}, (i, j) \neq (2, 3)\right\}$, where $g_{23} = 0$ and other components g_{ij} are found in Steps 1-3. The differential operator

$$\left(\mathrm{Z}r\right)_i = \sum_{j=1}^{3} \partial_j r_{ji}$$

satisfies $\mathrm{ZV} = 0$ which implies $\mathrm{Z}r = 0$ for $r = \mathrm{V}\tilde{g} - \mathrm{V}\varepsilon$. This system is reduced to

$$\begin{aligned} \partial_1 r_{11} + \partial_2 r_{12} + \partial_3 r_{13} &= 0, \\ \partial_1 r_{21} + \partial_3 r_{23} &= 0, \\ \partial_1 r_{31} + \partial_2 r_{32} &= 0 \end{aligned} \tag{19}$$

because of $r_{33} = r_{22} = 0$. It follows that $2\partial_{23} r_{23} = -\partial_{11} r_{11}$, hence

$$\partial_{23}\int_0^{a_1} r_{23}\left(x\right) \mathrm{d}x^1 = \partial_{23}\int_0^{a_1} r_{23}\left(x\right) x^1 \mathrm{d}x^1 = 0$$

for any x^2 and x^3. The same is true with the derivative ∂_{23} omitted. Hence we can find a function g_{23} supported in K that satisfies $\partial_{11} g_{23} = -r_{23}$ by double integration in x^1. Set $g = \tilde{g} + g^*$, where

g^* is the tensor with only one non zero component equal to g_{23}. Tensor $r = \mathrm{V}g - \mathrm{V}\varepsilon$ fulfils (19) where $r_{33} = r_{22} = r_{23} = 0$. This implies $\partial_1 r_{21} = \partial_1 r_{31} = \partial_{11} r_{11} = 0$ and $r_{11} = r_{21} = r_{31} = 0$, since r has compact support. This means that the last three equations (11) are fulfilled and $\mathrm{V}g = \mathrm{V}\varepsilon$. For an arbitrary convex compact K, a proof follows from [7] §7.8. □

Proposition 4 *For any 2-tensor field e with compact support in E such that $\mathrm{V}e = 0$, there exists a unique displacement field φ that fulfils equation $\mathrm{D}\varphi = e$ such that $\varphi = 0$ in the unbounded connected component of $E \backslash \operatorname{supp} \varepsilon$.*

Proof. We are going to solve system (11) with $r_{ij} = 0$. By the first equation, we have

$$\partial_{22} \int e_{11}\left(t, x^2, x^3\right) \mathrm{d}t = -\int \partial_{12} e_{12}\left(t, x^2, x^3\right) \mathrm{d}t + \int \partial_{11} e_{22}\left(t, x^2, x^3\right) \mathrm{d}t = 0$$

which implies $\int e_{11}\left(t, x^2, x^3\right) \mathrm{d}t = 0$ for any x^2 and x^3, since this function has compact support. It follows that the function

$$\varphi_1\left(x^1, x^2, x^3\right) = \int_{-\infty}^{x^1} e_{11}\left(t, x^2, x^3\right) \mathrm{d}t$$

has compact support and fulfils $\partial_1 \varphi_1 = e_{11}$. In the similar way, we find solutions φ_i of $\partial_i \varphi_i = e_{ii}$, $i = 2, 3$ with compact support. Set $\varphi = \sum \varphi_i \mathrm{d}x^i$ and $e' = e - \mathrm{D}\varphi$. This tensor field satisfies (11) with $h = 0, g = e'$ and $e'_{ii} = 0$ for $i = 1, 2, 3$. Equations (11) yield $e' = 0$, since e' has compact support. Thus $\mathrm{D}\varphi = e$. Operator D is elliptic and has constant coefficients. Therefore the field φ is analytic in $E \backslash \operatorname{supp} \varepsilon$. The field φ vanishes in the unbounded connected component of this set, since φ has compact support. □

5 Recovering of a displacement field

The method of integrated polarization tomography is based on measurement of the motion of the polarization ellipse of propagating light. For a small stress tensor σ, this transformation is related to the line integral of the traceless normal part of σ [1]. Given a 2-tensor field ε and a vector v, the traceless normal part of ε is the 2-tensor $Q_v \varepsilon$ defined in any plane P orthogonal to v :

$$Q_v \varepsilon \mid_P = \varepsilon \mid_P - \frac{1}{2} \operatorname{tr}\left(\varepsilon \mid_P\right) \mathrm{i} \mid_P,$$

where $\mathrm{i} = \left(\mathrm{d}u^1\right)^2 + \left(\mathrm{d}u^2\right)^2 + \left(\mathrm{d}u^3\right)^2$. We have

$$Q_v \varepsilon = \frac{1}{2}\left(\varepsilon_{11} - \varepsilon_{22}\right)\left(\mathrm{d}x^1\right)^2 + \varepsilon_{12} \mathrm{d}x^1 \cdot \mathrm{d}x^2 + \frac{1}{2}\left(\varepsilon_{22} - \varepsilon_{11}\right)\left(\mathrm{d}x^2\right)^2$$

for any euclidean coordinate system $\left(x^1, x^2, x^3\right)$ such that $v = (0, 0, 1)$. The traceless normal ray integral of ε is 2-tensor

$$\mathrm{T}_v \varepsilon\,(y; u, w) = \int_0^\infty Q_v \varepsilon\,(y + tv; u, w)\, \mathrm{d}t, \;\; u, w \in P$$

defined for any $v \in \mathrm{S}^2$ in a plane P orthogonal v.

Theorem 5 *Let K be a compact convex set E and $\Gamma \subset E \backslash K$ be a piecewise smooth curve such that any plane H that has a common point with K meets Γ transversely at a point y. For any 2-tensor ε with support in K satisfying $\mathrm{V}\varepsilon = 0$, the corresponding displacement field φ can be reconstructed from data of the second derivatives of the ray integrals $\mathrm{T}_v \varepsilon(y)$ for $y \in \Gamma$, $v \in \mathrm{S}^2$.*

Proof. Let $H = H(p, \omega)$ and $y \in \Gamma \cap H$ be as in the assumption. Choose a coordinate system $\left(x^1, x^2, x^3\right)$ in E such that $\omega = (1, 0, 0)$. For a ray $R \subset H$ with a direction vector $v = v(\theta) = (0, \cos\theta, \sin\theta)$, $0 \le \theta < 2\pi$, the vectors ω and $u = (0, -\sin\theta, \cos\theta)$ form an orthogonal basis in any plane P orthogonal to v. Tensor $\mathrm{T}_v \varepsilon(y)$ is known for $y \in \Gamma$ and $v \in H(0, \omega)$. It has two independent components in this basis:

$$\mathrm{T}_v \varepsilon(y; \omega, \omega) \doteqdot \frac{1}{2} \int_0^\infty \left(\varepsilon_{\omega\omega}(y + tv) - \varepsilon_{uu}(y + tv)\right) \mathrm{d}t,\ \mathrm{T}_v \varepsilon(y; \omega, u) \doteqdot \int_0^\infty \varepsilon_{\omega u}(y + tv)\, \mathrm{d}t.$$

By Proposition 4 we have $\varepsilon = \mathrm{D}\varphi$. Calculate the integrals

$$\mathrm{I}_1(y, p, \omega) \doteqdot \int_0^{2\pi} \partial^2_{;\omega} \mathrm{T}_{v(\theta)} \varepsilon(y, \omega, \omega)\, \mathrm{d}\theta,$$
$$\mathrm{I}_2(y, p, \omega) \doteqdot \int_0^{2\pi} \partial^2_{;\omega} \mathrm{T}_{v(\theta)} \varepsilon(y, \omega, u)\, \mathrm{d}\theta$$

in terms of components ε_{ij} of ε and components φ_i of φ. We have

$$\begin{aligned}
\varepsilon_{\omega u} &= -\sin\theta \varepsilon_{12} + \cos\theta \varepsilon_{13} \\
&= \partial_\omega\left(-\sin\theta \varphi_2 + \cos\theta \varphi_3\right) + \left(-\sin\theta \partial_2 + \cos\theta \partial_3\right)\varphi_1 \\
\int_0^\infty \varepsilon_{\omega u}(y + tv)\, \mathrm{d}t &= \partial_\omega \int_0^\infty \left(-\sin\theta \varphi_2 + \cos\theta \varphi_3\right) \mathrm{d}t + \int_0^\infty \left(-\sin\theta \partial_2 + \cos\theta \partial_3\right) \varphi_1 \mathrm{d}t, \\
\partial^2_{;\omega} \int_0^\infty \varepsilon_{\omega u}(y + tv)\, \mathrm{d}t &= \partial^3_\omega \int_0^\infty \left(-t \sin\theta \varphi_2 + t\cos\theta \varphi_3\right) t \mathrm{d}t + \partial^2_\omega \int_0^\infty \left(-t\sin\theta \partial_2 + t\cos\theta \partial_3\right) \varphi_1 t \mathrm{d}t \\
\mathrm{I}_2(y, p, \omega) &= \partial^3_\omega \int_0^{2\pi} \int_0^\infty \left(-t\sin\theta \varphi_2 + t\cos\theta \varphi_3\right) t \mathrm{d}t \mathrm{d}\theta \\
&+ \partial^2_\omega \int_0^{2\pi} \int_0^\infty \left(-t\sin\theta \partial_2 + t\cos\theta \partial_3\right) \varphi_1 t \mathrm{d}t \mathrm{d}\theta \\
&= \partial^3_p \int_H \left(-z^3 \varphi_2(x) + z^2 \varphi_3(x)\right) \mathrm{d}H + \partial^2_p \int_H \left(z^2 \partial_3 - z^3 \partial_2\right) \varphi_1(x)\, \mathrm{d}H,
\end{aligned}$$

where $H = H(p, \omega)$, $x = y + tv$ and $z^j = x^j - y^j$, $j = 2, 3$. The last integral vanishes after partial integration, hence

$$\mathrm{I}_2(y, p, \omega) = \partial^3_p \int_H \left(-z^3 \varphi_2 + z^2 \varphi_3\right) \mathrm{d}H$$

which implies

$$\frac{\partial}{\partial y^3} \mathrm{I}_2(y, p, \omega) = \partial^3_p \int_H \varphi_2 \mathrm{d}H,\ \frac{\partial}{\partial y^2} \mathrm{I}_2(y, p, \omega) = -\partial^3_p \int_{H(p,\omega)} \varphi_3 \mathrm{d}H.$$

Integrating in p we get

$$\frac{\partial}{\partial y^3} \mathrm{J}_2(p, \omega) = \partial^2_p \int_{H(p,\omega)} \varphi_2 \mathrm{d}H,\ \frac{\partial}{\partial y^2} \mathrm{J}_2(p, \omega) = -\partial^2_p \int_{H(p,\omega)} \varphi_3 \mathrm{d}H,$$

where

$$\mathrm{J}_2(p,\omega) \doteqdot \int_{-\infty}^{p} \mathrm{I}_2(y,q,\omega)\,\mathrm{d}q, \tag{20}$$

since all the functions vanish for $p < p_0$ for some constant p_0 depending only on supp ε. Similarly

$$2\mathrm{T}_{v(\theta)}\varepsilon(y;\omega,\omega) = \int_0^\infty \left(\partial_1\varphi_1 - \left(\sin^2\theta\partial_2\varphi_2 - \sin\theta\cos\theta\left(\partial_2\varphi_3 + \partial_3\varphi_2\right) + \cos^2\theta\partial_3\varphi_3\right)\right)\mathrm{d}t,$$

$$\begin{aligned} 2\mathrm{I}_1(y,p,\omega) &= \partial_p^3 \int \left[\left(z^2\right)^2 + \left(z^3\right)^2\right]\varphi_1 \mathrm{d}H \\ &- \partial_p^2 \int \left[\left(z^3\right)^2 \partial_2\varphi_2 - z^2 z^3\left(\partial_2\varphi_3 + \partial_3\varphi_2\right) + \left(z^2\right)^2 \partial_3\varphi_3\right]\mathrm{d}H. \end{aligned}$$

The first integral is quadratic in y. By partial integration the second integral is reduced to the quantity

$$\partial_p^3 \int_{H(p,\omega)} \left(z^2\varphi_2 + z^3\varphi_3\right)\mathrm{d}H$$

which is linear in y. Therefore,

$$\frac{\partial^2}{\left(\partial y^2\right)^2}\mathrm{I}_1(y,p,\omega) = \frac{\partial^2}{\left(\partial y^3\right)^2}\mathrm{I}_1(y,p,\omega) = \partial_p^3 \int_{H(p,\omega)} \varphi_1 \mathrm{d}H$$

which implies

$$\mathrm{J}_1(p,\omega) \doteqdot \partial_p^2 \int_{H(p,\omega)} \varphi_1 \mathrm{d}H = \int_{-\infty}^{p} \frac{\partial^2}{\left(\partial y^2\right)^2}\mathrm{I}_1(y,q,\omega)\,\mathrm{d}q. \tag{21}$$

Now we write the result in the covariant form introducing the vector field $\mathrm{J}(p,\omega)$ defined in $\mathbb{R}\times\mathrm{S}^2$ such that for any vector $e \in E$

$$\langle \mathrm{J}(p,\omega), e\rangle = \langle e,\omega\rangle \mathrm{J}_1(p,\omega) + [\omega, e, \partial/\partial y]\,\mathrm{J}_2(p,\omega).$$

The functions $\mathrm{J}_1, \mathrm{J}_2$ are known from (21) and (20). Finally we apply Lorentz's formula and calculate the displacement term

$$\varphi(x) = -\frac{1}{8\pi^2}\int_{\mathrm{S}^2} \mathrm{J}(\langle x,\omega\rangle,\omega)\,\Omega$$

which completes the proof. □

Corollary 6 *If the specimen is homogeneous and coaxial, then the displacement φ can be reconstructed from traceless normal line integrals of the stress tensor σ.*

Proof. The strain and stress tensors are related by the constitutive equation $\sigma = \mathbf{c}\varepsilon$, where $\mathbf{c}$ is the tensor of elastic moduli that expresses Hooke's law. For a homogeneous coaxial solid, this equation is as follows

$$\sigma_{ij} = \lambda\delta_{ij}\mathrm{tr}\varepsilon + 2\mu\varepsilon_{ij},\ \mathrm{tr}\varepsilon = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}, \tag{22}$$

where λ and μ are positive Lamé's constants. We have

$$\mathrm{T}\varepsilon(\omega,\omega) = \frac{1}{\lambda + 2\mu}\mathrm{T}\sigma(\omega,\omega),\ \mathrm{T}\varepsilon(\omega,u) = \frac{1}{2\mu}\mathrm{T}\sigma(\omega,u),$$

hence the integrals $\mathrm{T}\varepsilon$ are known from data of $\mathrm{T}\sigma$. □

If Lamé's parameters are arbitrary C^1 functions, the method of Theorem 5 leads to the integral equation of Fredholm type

$$\varphi + G\varphi = L\left(\mathrm{J}\right),$$

where L is the operator in the right hand side of (9) and G is a bounded operator $L_2(K) \to H_1(K)$, hence a compact operator in $L_2(K)$.

6 Full reconstruction of a strain tensor

Theorem 7 *Let $\Gamma_i, i = 1, 2$ be piecewise smooth curves in E and K be a compact set in $E \backslash \Gamma_1 \cup \Gamma_2$ such that for any point $x \in K$ any plane H through x meets each Γ_i transversely at a point. Any 2-tensor field ε supported in the compact K can be reconstructed from*
(I) data of the axial ray integrals $\mathrm{X}\varepsilon$ with source points $y \in \Gamma_1$ and its second derivatives with respect to y, and
(II) data of ray integrals $\mathrm{T}\varepsilon$ of the traceless normal part of ε and its second derivatives for rays with sources points on a curve Γ_2.

Proof. The algorithm is as follows:
Step 1. Calculate $\mathrm{V}\varepsilon$ from data of the axial ray integrals $\mathrm{X}\varepsilon$ as in Sec. 3.
Step 2. Find a gauge field g with compact support in E as in Sec. 4.
Step 3. Set $e = \varepsilon - g$ and have $\mathrm{V}e = \mathrm{V}\varepsilon - \mathrm{V}g = 0$. By Proposition 4 we have $e = \mathrm{D}\varphi$ in E for a displacement field φ with compact support. Determine the field φ from knowledge of $\mathrm{T}e = \mathrm{T}\varepsilon - \mathrm{T}g$ by the algorithm of Sec. 5. The reconstruction reads $\varepsilon = g + \mathrm{D}\varphi$. $\square$

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