Time reversal in photoacoustic tomography and levitation in a cavity

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Time reversal in photoacoustic tomography and levitation in a cavity

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Abstract
A class of photoacoustic acquisition geometries in $\mathbb{R}^n$ is considered such that the spherical mean transform admits an exact filtered back projection reconstruction formula. The reconstruction is interpreted as a time reversion mirror that reproduces exactly an arbitrary source distribution in the cavity. A series of examples of non-uniqueness of the inverse potential problem is constructed based on the same geometrical technique.

Keywords: photoacoustic tomography, time reversal, inverse potential problem

1. Introduction

Reconstruction of a function from its spherical means is a mathematical tool for photo- and thermo-acoustic tomography. This method is also applied in geophysics (travel time inversion). Typically, the integral data is available for all spheres centered at a set $Z \subset \mathbb{R}^n$ (e.g. an array of transducers) and an unknown function is supported by an open set $H \subset \mathbb{R}^n \backslash Z$. Closed reconstruction formulae of filtered back projection (FBP) types are known for a few types of central sets Z. We state here that such a reconstruction is possible for a class of algebraic central sets Z called oscillatory § 2.

For general information on the photoacoustic and other hybrid tomographic methods, we refer to surveys [12] and [11].

Next, we show that the reconstructions can be interpreted as application of a universal time reversal method for the wave equation in the sense of Fink [8]. A signal is recorded by an array of transducers (mirror), time-reversed and retransmitted into the medium. The retransmitted signal propagates back through the same medium and refocuses on the source. The time reversal method is an effective tool in acoustical imaging even if the array of transducers is aside of a radiation source and no data from the region of interest is available, see [8].

We show that for a free space and any oscillatory array set Z, the time reversal is a perfect mirror providing the exact reconstruction of any source distribution supported by the cavity H § 5. The cavity can be open; the method works for a paraboloid, a half-space and a two sheet hyperboloid giving an approximate reconstruction with a finite array of transducers. In spite of the time reversal method appearing as quite natural in our problem, it only works accurately for a very special class of center sets Z.

The method of oscillatory geometry can be applied also for a generalization of Newton's famous attraction theorem (Principia, 1687). This theorem states that a mass uniformly distributed over a thin sphere exerts zero gravitation field inside (levitation). This is an example of the non-uniqueness of the interior inverse potential problem. We show that there are many sets Z in $\mathbb{R}^3$ supporting a mass distribution which generate levitation in an open set § § 7–9. See [22] and [10] for surveys of inverse problems of the potential theory.

2. Oscillatory sets

Definition. Let p be a real polynomial in $\mathbb{R}^n$ of degree m with the zero set Z. We call p and Z *oscillatory* with respect to a point $a \in \mathbb{R}^n \backslash \mathrm{Z}$, if p has m simple zeros in L for almost any line $L \subset \mathbb{R}^n$ through a. The point a will be called *hyperbolic* point of p and Z. A point $x \in \mathrm{Z}$ is called *regular* if $\mathrm{d}p(x) \neq 0$.

Theorem 1. *Let p be a polynomial in $\mathbb{R}^n$, $n > 1$ with a compact oscillatory zero set Z with respect to a point a. We have $\mathrm{Z} = \mathrm{Z}_1 \cup \ldots \cup \mathrm{Z}_\mu$, where $\mathrm{Z}_1, \ldots, \mathrm{Z}_\mu$, $\mu = m/2$ are ovals (homeomorphic images of a sphere). They are nested in the sense that a set of regular points of Z_i is contained in the interior of Z_{i+1} for $i = 1, \ldots, \mu$. Moreover, $\mathrm{Z}_1 = \partial \mathrm{H}$ where H is the set of all hyperbolic points. It is a convex component of $\mathbb{R}^n \backslash \mathrm{Z}$.*

Proof. Let a be a hyperbolic point of p and S be a sphere in $\mathbb{R}^n$ with the center at the origin. For any $\omega \in \mathrm{S}$ we numerate zero points of p by $x_k(\omega) = a + t_k \omega$, $k = 1, \ldots, \mu(\omega)$ counting with multiplicity in such a way that:

$$t_{-\sigma(\omega)} \leqslant \ldots \leqslant t_{-2} \leqslant t_{-1} < 0 < t_1 \leqslant t_2 \leqslant \ldots \leqslant t_{\tau(\omega)}. \quad (1)$$

By the classical Rouche theorem for an arbitrary disc $D \subset \mathbb{C}$ and a holomorphic function f in D that is continuous in the closed disc, the number of zeros of f in D is equal to the number of zeros of $f + h$ provided h is holomorphic in D, continuous up to the boundary and satisfies:

$$\max_{\partial D} |h| < \min_{\partial D} |f|.$$

Fix an arbitrary $\omega_0 \in \mathrm{S}$ and arbitrary real $u < v$ such that $p(a + t\omega_0) \neq 0$ for $t = u, v$. The holomorphic function $f(t) = p(a + t\omega_0)$ does not vanish in the circle C such that segment $[u, v]$ is a diameter since p is oscillatory and $f(t)$ has only real zeros. By the Rouche theorem the number of zeros $t = t_k(\omega)$ of $p(a + t\omega)$ such that $u \leqslant t_k(\omega) \leqslant v$ is constant for all ω in a neighborhood U of ω_0 since the $p(a + t\omega)$ has no complex zeros. It follows that the number of zeros of $p(a + t\omega)$ is equal to m for all $\omega \in \mathrm{S}$ since Z is compact. This implies $\sigma(\omega) + \tau(\omega) = m$ for any ω. Therefore we have $\sigma = \tau = \mu \doteqdot m/2$ and $t_{-k}(\omega) = t_k(-\omega)$, $k = 1, \ldots, \mu$ for $\omega \in \mathrm{S}$ since the sphere S is connected. For any $k = 1, \ldots, \mu$, the function $x_k(\omega) = a + t_k(\omega)\omega$ is defined and continuous for $\omega \in \mathrm{S}$, hence $\mathrm{Z}_k = \left\{x = x_k(\omega),\ \omega \in \mathrm{S}\right\}$ is an oval. We have $\mathrm{Z} = \bigcup_1^\mu \mathrm{Z}_k$ and the variety $\mathrm{Z}_k \cap \mathrm{Z}_j$ has dimension $< n - 1$ for any $j \neq k$. Therefore the sets Z_k are nested and a belongs to the interior H of the oval Z_1.

Check that any $b \in \mathrm{H}$ is a hyperbolic point. Let reg Z be the set of all regular points of Z. The set $\mathrm{sing}\, \mathrm{Z} = \mathrm{Z} \backslash \mathrm{reg}\, \mathrm{Z}$ is a closed algebraic subset of Z of dimension $< n - 1$.

Consider the central projection π_b: $Z \to S$, where $\pi_b(x) = (x - b)/|x - b|$ and S is the unit $n - 1$ sphere. It is proper, since Z is compact and $b \notin Z$. The image $\pi_b(\text{sing } Z)$ is a closed algebraic set of dimension $<n - 1$. If $\omega \in S\backslash\pi_b(\text{sing } Z)$, the line $L(b, \omega) = \{x = b + t\omega,\ t \in \mathbb{R}\}$ does not meet sing Z and intercepts each closed hypersurface $Z_1, ..., Z_\mu$ at least two times. The total number of intersections is $\geqslant 2\mu = m$. It is equal to m, since p is a polynomial of degree m. This shows that b is a hyperbolic point. It follows that H is the set of all hyperbolic points.

Check that H is convex. If it is not the case, then a sequence of lines exists $L_j \to L$ such that L contains a concave point $b \in \partial H$ but L_j does not meet the set $U\backslash Z$ for a neighborhood U of b. By the Rouche theorem the number of zeros of p in $L\backslash U$ is the same as in $L_j\backslash U$, which is equal to m. The point $b \in Z$ makes the number of zeros of p in L larger m, which is impossible. □

Definition. We call a *hyperbolic cavity* of an oscillatory set Z any maximal connected set H of points a such that Z is oscillatory with respect to a. By theorem 1 there is only one hyperbolic cavity, if the zero set is compact. The zero set of an oscillatory polynomial p is compact if and only if p is elliptic, that is the principal part p_m does not vanish in $\mathbb{R}^n\backslash 0$.

3. Photoacoustic inversion for oscillatory acquisition geometry

Consider the spherical mean transform in an Euclidean space E^n with a central set $Z \subset E^n$.

$$\mathrm{R}_Z f(r, \xi) = \int_{|x-\xi|=r} f(x)\mathrm{d}S, \quad \xi \in Z, \quad r > 0$$

Theorem 2. *Let p be a polynomial in E^n with a compact regular oscillatory zero set* Z *and* H *be the hyperbolic cavity. An arbitrary function f in E^n with support in* H *can be reconstructed from its spherical means by*

$$f(x) = -\frac{p(x)}{\mathrm{j}^{n-1}} \int_Z \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{n-1} \frac{\mathrm{R}f(r, \xi)}{r}\bigg|_{r=|x-\xi|} \frac{\mathrm{d}\xi}{\mathrm{d}p} \tag{2}$$

for odd n, and

$$f(x) = -\frac{2p(x)}{\mathrm{j}^n} \int_Z \left(\int_0^\infty \frac{\mathrm{d}r^2}{|x-\xi|^2 - r^2}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{n-1} \frac{\mathrm{R}f(r, \xi)}{r}\right)\frac{\mathrm{d}\xi}{\mathrm{d}p} \tag{3}$$

for even n, where Z *is oriented by the outward conormal and $\omega \doteqdot \mathrm{d}\xi/\mathrm{d}p$ is a differential form in E^n such that $\mathrm{d}p \wedge \omega = \mathrm{d}\xi = \mathrm{d}\xi_1 \wedge ... \wedge \mathrm{d}\xi_n$.*

A proof is given in § 6.

4. Examples

Agranovski and Quinto [1] studied the structure of injectivity sets for the circle mean transform in a plane. They showed that a plane set is injectivity set except for special unions of lines and points. On the other hand, a closed form reconstruction is only possible for very special injectivity sets.

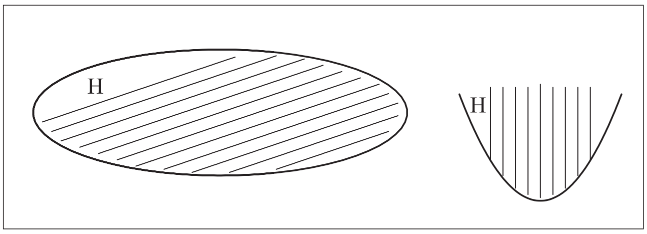

Figure 1. Hyperbolic cavity of ellipsoids and elliptic paraboloids

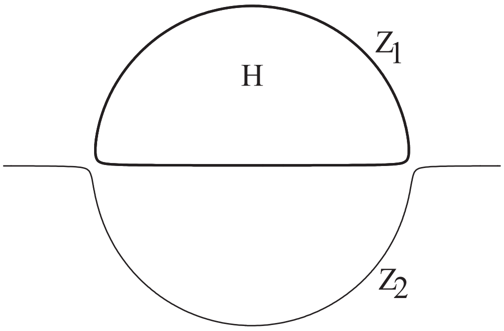

Figure 2. Hyperbolic cavity of a curve of degree three

Example 1. Half-space. A hyperplane Z is oscillatory with two hyperbolic cavities · Such acquisition geometry appears in geophysics. Reconstructions (2) and (3) for this case look different from that of [5] and [2].

2. Ellipsoids and paraboloids. Any ellipsoid, elliptic paraboloid, elliptic and parabolic cylinder is oscillatory with only one hyperbolic cavity: see figure 1 A two sheet hyperboloid is oscillatory with two hyperbolic cavities. A slab has three hyperbolic cavities.

One sheet hyperboloid and parabolic hyperboloid Z are not oscillatory since the curvature form of Z is not positively or negatively determined.

Reconstructions of FBP types for case Z as a sphere were conducted by Finch *et al* in [6, 7]. The authors' method is based on the wave operator. Xu and Wang [21] obtained explicit reconstructions when Z was a sphere, a circular cylinder and a slab. Kunyansky [13] applied an approach based on the Helmholtz operator. Recently, a reconstruction of an FBP type was done in [16] for ellipsoids Z in 3D, and in [18] in the form (2–3) for arbitrary dimension, see also [9]. Note that the reconstruction formulas obtained in the above papers are different from [18]. Kunyansky [14] also found reconstructions for some polygonal curves and polyhedral surfaces as center sets. The case of variable velocity is addressed in [19].

3. Half-ellipsoid. Let p_2 be an elliptic second order polynomial. The set $p_3 = 0$ where $p_3 = x_1 p_2$ is the union of an ellipsoid and a hyperplane. If the set $\mathrm{H} = \{p_2(x) < 0, x_1 > 0\}$ is not empty it is a hyperbolic cavity for p_3. By proposition 9 for any $\varepsilon > 0$, there exists a polynomial $\tilde{p}_3$ with a regular oscillatory zero set $\tilde{\mathrm{Z}}$ such that $\left|p_3 - \tilde{p}_3\right| < \varepsilon$ in $\mathrm{Z} \cup \tilde{\mathrm{Z}}$. The set $\tilde{\mathrm{Z}}$ consists of an oval $\mathrm{Z}_1 = \partial \mathrm{H}$ and an unbounded component Z_2: see figure 2.

Note that contribution of points $\xi \in \mathrm{Z}_2$ to (3) decreases fast as $\xi \to \infty$.

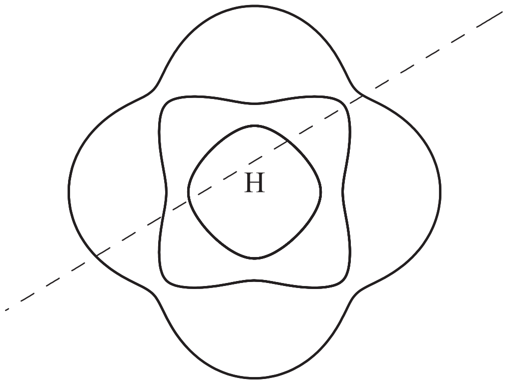

Figure 3. Oscillatory zero set of degree six

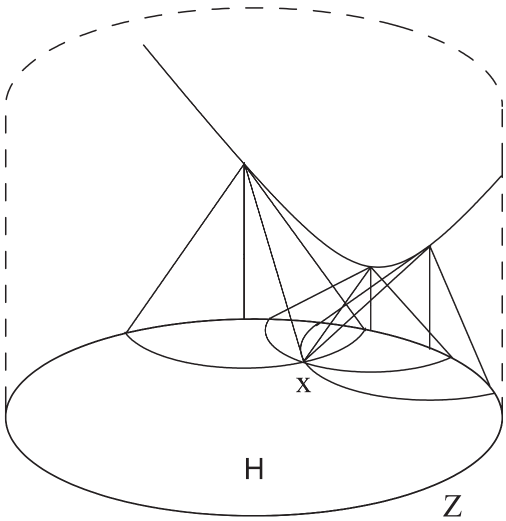

Figure 4. Geometry of the time reversal

4. The zero set Z of the polynomial $p = (x^2 + y^2)^3 - 12\,(x^2 + y^2)^2 + 7x^2y^2 + 30(x^2 + y^2) - 20$ is compact since the principal part of p is the elliptic polynomial $(x^2 + y^2)^3$. It is shown in figure 3 drawn by Maple. It is clear that Z is regular and oscillatory.

5. Time reversal structure

Let E^n be an Euclidean space; consider the Cauchy problem for the wave equation in the space-time $\mathbb{R} \times E^n$

$$\left(\frac{\partial^2}{\partial t^2} - \Delta\right) u = 0$$
$$u(0, x) = 0, \quad u'_t(0, x) = f(x). \tag{4}$$

for a function f in E^n. Let $E_{n+1}(t, x)$ be the forward propagator for (4).

Theorem 3. *Formulae (2) and (3) are equivalent to the time reversal method for (4) acting in the following steps:*

(i) transmission (forward propagation) of a function f supported in H *to the mirror manifold* $\mathbb{R} \times \mathrm{Z}$*:*

$$f \mapsto u(t, \xi) = \int_{\mathrm{H}} E_{n+1}(t, x - \xi) f(x) \mathrm{d}x, \quad \xi \in \mathrm{Z},$$

(ii) filtration:

$$v = \mathrm{F}u \doteqdot - \frac{\partial}{\partial t} \frac{4}{t} \frac{\partial}{\partial t} u,$$

(iii) time reversion and retransmission:

$$g(x) = \int_{\mathrm{Z}} \int_0^{\infty} E_{n+1}(t, x - \xi) v(-t, \xi) \mathrm{d}t \frac{\mathrm{d}\xi}{\mathrm{d}p},$$

(iv) reconstruction:

$$f(x) = -p(x) g(x).$$

See figure 4.

Remark 1. In the general setting steps (i–iv) correspond to the conception of time reversal due to Fink [8]. The system of transducers is assumed transparent for the wave field and no data in $\mathbb{R} \times \mathrm{H}$ is used for the reconstruction.

Remark 2. The filtration operator F is a positive self-adjoint differential operator in the Hilbert space $L_2(\mathbb{R}_+)$. We can replace the volume form $\mathrm{d}\xi/\mathrm{d}p$ by $q\mathrm{d}\xi/\mathrm{d}p$ where q is an arbitrary polynomial of degree $\leqslant m - 2$ preserving equations (2) and (3). If q is a strict separator of Z (see §7), the form $q\mathrm{d}x/\mathrm{d}p$ is a volume form in Z. Then the retransmitting operator E^* is adjoint to the transmitting operator E: $L_2(\mathrm{H}) \to L_2(\mathbb{R} \times \mathrm{Z})$ where the space $\mathbb{R} \times \mathrm{Z}$ is endowed with the volume form $q\mathrm{d}x/\mathrm{d}p$. The time reversal operator T can be written in the self-adjoint form

$$\mathrm{T} = |p|^{1/2}\, \mathrm{E}^*\mathrm{F}\mathrm{E}\, |p|^{1/2}.$$

Proof. For odd $n = 2m + 1$, the forward propagator is

$$E_{n+1}(t, x) = \frac{1}{2\pi} \left(\frac{\partial}{\pi \partial t^2} \right)^{m-1} \theta(t) \delta\left(t^2 - |x|^2 \right).$$

The solution of (4) equals:

$$u(t, \xi) = \frac{1}{4\pi} \left(\frac{\partial}{\pi \partial t^2} \right)^{m-1} \frac{1}{t} \int_{|x-\xi|=t} f \mathrm{d}S = \frac{1}{4\pi} \left(\frac{\partial}{\pi \partial t^2} \right)^{m-1} \frac{\mathrm{R}f(t, \xi)}{t}$$

and by (2):

$$
\begin{aligned}
-\frac{f(x)}{p(x)} &= \frac{2^{n-1}}{\mathrm{j}^{n-1}} \int_{\mathrm{Z}} \frac{\mathrm{d}\xi}{\mathrm{d}p} \left(\frac{\partial}{\partial t^2}\right)^{n-1} \frac{\mathrm{R}f(t,\xi)}{t}\Big|_{t=|x-\xi|} \\
&= (-1)^{(n-1)/2} \frac{1}{2\pi} \int_{\mathrm{Z}} \frac{\mathrm{d}\xi}{\mathrm{d}p} \left(\frac{\partial}{\pi \partial t^2}\right)^{m-1} \frac{1}{t}\frac{\partial}{\partial t}\frac{1}{t}\frac{\partial}{\partial t} u(t,\xi)\Big|_{t=|x-\xi|} \\
&= -\frac{1}{2\pi} \int_{\mathrm{Z}} \frac{\mathrm{d}\xi}{\mathrm{d}p} \int_0^\infty \left(\frac{\partial}{\pi \partial t^2}\right)^{m-1} \delta\left(t^2 - |x-\xi|^2\right) \frac{\partial}{\partial t}\frac{4}{t}\frac{\partial}{\partial t} u(t,\xi)\mathrm{d}t \\
&= \int_{\mathrm{Z}} \frac{\mathrm{d}\xi}{\mathrm{d}p} \int_{\mathbb{R}} \left(E_{n+1}(t, x-\xi) * v(-t,\xi)\right)\mathrm{d}t \\
&= \int_{\mathrm{Z}} \frac{\mathrm{d}\xi}{\mathrm{d}p} \int_{\mathbb{R}} \left(E_{n+1}(-t, x-\xi) * v(t,\xi)\right)\mathrm{d}t
\end{aligned}
$$

is the retransmission of $v = \mathrm{F}u$.

For even $n = 2m$, solving the Cauchy problem by means of the propagator:

$$
E_{n+1}(t,x) = \frac{1}{2\pi}\left(\frac{\partial}{\pi \partial t^2}\right)^{m-1} \frac{\theta(t-|x|)}{\left(t^2-|x|^2\right)^{1/2}} \tag{5}
$$

we obtain:

$$
u(t,\xi) = E_{n+1}(t,\cdot) * f(\,\cdot\,) = \frac{1}{4\pi^m} \int_{-\infty}^{\infty} \frac{\mathrm{d}\rho}{(\tau-\rho)_+^{1/2}} \left(\frac{\partial}{\partial\rho}\right)^{m-1} \mathrm{S}f(\rho,\xi), \tag{6}
$$

where $\tau = t^2$, and $\mathrm{R}f(r,\xi) = 0$ for $r < 0$. Inverting this equation by Abel's method we find

$$
\left(\frac{\partial}{\pi\partial\rho}\right)^{m-1} \mathrm{S}f(\rho,\xi) = D_\rho \mathrm{S}f(\rho,\xi) = 4\frac{\mathrm{d}}{\mathrm{d}\rho} \int \frac{u(t,\xi)}{(\rho-\tau)_+^{1/2}} \mathrm{d}\tau
$$

We write (3) in the form

$$
\begin{aligned}
-\frac{f(x)}{p(x)} &= (-1)^m \frac{2}{(2\pi)^n} \int_{\mathrm{Z}} \frac{\mathrm{d}\xi}{\mathrm{d}p} \left(\int_0^\infty \frac{\mathrm{d}r^2}{|x-\xi|^2 - r^2} \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{n-1} \mathrm{S}f(\rho,\xi)\right) \\
&= (-1)^m \frac{1}{\pi^n} \int_{\mathrm{Z}} \frac{\mathrm{d}\xi}{\mathrm{d}p} \left(\int_0^\infty \frac{\mathrm{d}\rho}{\eta-\rho} \left(\frac{\partial}{\partial\rho}\right)^{n-1} \mathrm{S}f(\rho,\xi)\right)
\end{aligned} \tag{7}
$$

where $\mathrm{S}f(\rho,\xi) = \mathrm{R}f(r,\xi)/r$, $\rho = r^2$ and $\eta = |x-\xi|^2$. Integrate by parts in (6) and apply lemma 4 (below)

$$
\int_{-\infty}^{\infty} \frac{\mathrm{d}\rho}{|x-\xi|^2 - \rho} \left(\frac{\partial}{\partial\rho}\right)^{n-1} \mathrm{S}f(\rho,\xi)
$$

$$
\begin{aligned}
&= \int_{\mathbb{R}} \frac{\mathrm{d}\rho}{\eta - \rho}\left(\frac{\partial}{\partial\rho}\right)^{m}\left(\frac{\partial}{\partial\rho}\right)^{m-1} \mathrm{S}f(\rho, \xi)\\
&= 4\pi^{m-1}\int_{\mathbb{R}} \frac{\mathrm{d}\rho}{\eta - \rho}\left(\frac{\partial}{\partial\rho}\right)^{m+1} \int \frac{u(t, \xi)}{(\rho - \tau)_+^{1/2}}\mathrm{d}\tau\\
&= 4\pi^{m-1}\left(\frac{\partial}{\partial\eta}\right)^{m+1}\int_{\mathbb{R}} u(t, \xi)\mathrm{d}\tau \int \frac{\mathrm{d}\rho}{\eta - \rho}\frac{1}{(\rho - \tau)_+^{1/2}}\\
&= 4\pi^{m}\left(\frac{\partial}{\partial\eta}\right)^{m+1}\int_{\mathbb{R}} \frac{u(t, \xi)\mathrm{d}\tau}{(\tau - \eta)_+^{1/2}}\\
&= (-1)^{m-1}4\pi^{m}\int_{\mathbb{R}}\left(\frac{\partial}{\partial\tau}\right)^{2} u(t, \xi)\left(\frac{\partial}{\partial\tau}\right)^{m-1}\frac{\mathrm{d}\tau}{(\tau - \eta)_+^{1/2}}\\
&= (-1)^{m-1}\pi^{m}\int_{\mathbb{R}}\frac{\partial}{\partial t}\frac{2}{t}\frac{\partial}{\partial t}u(t, \xi)\left(\frac{\partial}{\partial\tau}\right)^{m-1}\frac{\mathrm{d}t}{(\tau - \sigma)_+^{1/2}}\\
&= (-1)^{m-1}\pi^{m}2^{-1}\int_{\mathbb{R}} v(t, \xi)\left(\frac{\partial}{\partial\tau}\right)^{m-1}\frac{\mathrm{d}t}{\left(t^2 - |x - \xi|^2\right)^{1/2}}\\
&= (-1)^{m-1}\pi^{n}\int v(t, \xi)E_{n+1}(-t, x - \xi)\mathrm{d}t.
\end{aligned}
$$

This together with (7) gives

$$
-\frac{f(x)}{p(x)} = -\int v(t, \xi)E_{n+1}(-t, x - \xi)\mathrm{d}t,
$$

where again

$$
v(t, \xi) = -\frac{\partial}{\partial t}\frac{4}{t}\frac{\partial}{\partial t}u(t, \xi).
$$

This completes the proof. □

Lemma 4. *We have*

$$
\int_{-\infty}^{\infty}\frac{\mathrm{d}\rho}{(\sigma - \rho)(\rho - \tau)_+^{1/2}} = \frac{\pi}{(\tau - \sigma)_+^{1/2}} \tag{8}
$$

where $t_\pm \doteq \max\{\pm t, 0\}$.

Proof. The integral is the convolution

$$
\int_{-\infty}^{\infty}\frac{\mathrm{d}\rho}{(\sigma - \rho)(\rho - \tau)_+^{1/2}} = \left(\rho^{-1} * \rho_+^{-1/2}\right)(\sigma - \tau)
$$

of distributions $\rho^{-1}\mathrm{d}\rho$ and $\rho_+^{-1/2}\mathrm{d}\rho$. The Fourier transform of the convolution is equal to the product $F(\rho^{-1} * \rho_+^{-1/2}) = F(\rho^{-1})F(\rho_+^{-1/2})$ where

$$F\left(\rho_+^{-1/2}\right) = (2\pi)^{-1/2}\left(i^{1/2}r_+^{-1/2} + i^{-1/2}r_-^{-1/2}\right),\ F\left(\rho^{-1}\right) = \pi i\,\mathrm{sgn}\,r$$
$$F\left(\rho^{-1}\right)F\left(\rho_+^{-1/2}\right) = \pi(2\pi)^{-1/2}\left(i^{-1/2}r_+^{-1/2} + i^{1/2}r_-^{-1/2}\right) = \pi F\left(\rho_-^{-1/2}\right)$$

This equation agrees with (8). □

6. Proof of the reconstruction

Check that the generating function $\Phi(x; \lambda, \xi) = \theta(x, \xi) - \lambda$, $\theta = |x - \xi|^2$ defined in $\mathrm{H} \times \Sigma$, $\Sigma = \mathbb{R} \times \mathrm{Z}$ satisfies conditions (**i,ii,iii**) of theorem 3.1 of [18] . Condition (**i**) is easy to verify. To prove (**ii**) we suppose that $y \neq x$ are conjugate points in H, which means

$$\theta(x, \xi) = \theta(y, \xi),\ \ \mathrm{d}_\xi\theta(x, \xi) = \mathrm{d}_\xi\theta(y, \xi) \tag{9}$$

for some $\xi \in \mathrm{Z}$. The first equation (9) implies that $|x - \xi| = |y - \xi|$. It follows that the line L through $s = 1/2(x + y)$ and ξ is orthogonal to $x - y$. By the second condition (9) we have $\langle x - y, \mathrm{d}\xi\rangle = 0$, hence vector $x - y$ is orthogonal to Z at ξ. Therefore L is tangent to Z at ξ which is impossible, since H is convex and no line through $s \in \mathrm{H}$ is tangent to Z. To check (**iii**) we consider an integral

$$\Theta_n(x, y) = \int_\mathrm{Z} \frac{\omega_\mathrm{Z}}{(\varphi(x, y; \xi) - i0)^n},$$

where

$$\varphi(x, y; \xi) = \theta(y, \xi) - \theta(x, \xi) = 2\langle x - y, \xi\rangle + |y|^2 - |x|^2 = \langle z, \xi - s\rangle,\ \ z = 2(x - y).$$

Lemma 5. *We have* $\mathrm{Re}\, i^n\Theta_n(x, y) = 0$ *for arbitrary* $y \neq x$.

Proof. Let S be the unit sphere in E^n. We can write $\mathrm{d}p = p_t'\mathrm{d}t + \mathrm{d}_\omega p$ by means of spherical coordinates $\xi = a + t\omega,\ t \in \mathbb{R},\ \omega \in \mathrm{S}$. This yields

$$\mathrm{d}\xi = t^{n-1}\mathrm{d}t \wedge \Omega = t^{n-1}\frac{\mathrm{d}p}{p_t'} \wedge \Omega,\ \ \frac{\mathrm{d}\xi}{\mathrm{d}p} = \frac{t^{n-1}}{p_t'}\Omega, \tag{10}$$

where Ω is the volume form in the sphere S. We move the origin to the point $s \in \mathrm{H}$ and have $\varphi(\xi) = \langle z, \xi\rangle$. Let $t_{-\mu} < .. < t_{-1} < 0 < t_1 < ... < t_\mu$ be all zeros of $p(a + t\omega)$ as in theorem 1. For **even** n, we have

$$2\mathrm{Re}\Theta_n(x, y) = 2\int_\mathrm{Z} \frac{1}{\varphi^n}\frac{\mathrm{d}\xi}{\mathrm{d}p} = \lim_{\varepsilon\to 0+}\int_\mathrm{Z}\left[\frac{1}{\varphi(t\omega_+)^n} + \frac{1}{\varphi(t\omega_-)^n}\right]\frac{\mathrm{d}\xi}{\mathrm{d}p}, \tag{11}$$

where $\omega_\pm = \omega \pm i\varepsilon\,|z|^{-2} z$, $\varepsilon > 0$ is a small number. We have $\varphi\left(t\omega_\pm\right) = \left\langle tz, \omega_\pm\right\rangle = \langle tz, \omega\rangle \pm i\varepsilon\,\mathrm{sgn}\,t$ and

$$\frac{1}{\varphi(t\omega_+)^n} + \frac{1}{\varphi(t\omega_-)^n} = \frac{1}{t^n}\left(\frac{1}{\left\langle z, \omega_+\right\rangle^n} + \frac{1}{\left\langle z, \omega_-\right\rangle^n}\right)$$

for any $t \neq 0$. Taking into account (10), we integrate over Z and get:

$$2\mathrm{Re}\Theta_n(x,\, y) = \lim_{\varepsilon \to 0} \int_{\mathrm{S}_+} \sum_{k=-\mu}^{\mu} \frac{1}{t_k p_t'(a + t_k\omega)} \left[\frac{1}{\langle z,\, \omega_+ \rangle^n} + \frac{1}{\langle z,\, \omega_- \rangle^n} \right] \Omega, \quad (12)$$

where S_+ is an arbitrary hemisphere. The sum in (12) is equal to the sum of residues $\mathrm{res}_{t_k}\rho(t,\,\omega)$ of the form $\rho(t,\,\omega) = \mathrm{d}t/tp(a + t\omega)$. Integrate this form along a circle of radius $R > \max_{\mathrm{S}} \left| t_\mu(\omega) \right|$ and apply the Residue theorem:

$$\begin{aligned} \frac{1}{2\pi i} \int_{|t|=R} \rho(t,\,\omega) = & \ \mathrm{res}_0\rho(t,\,\omega) + \sum_{k=-\mu}^{\mu} \mathrm{res}_{t_k}\rho(t,\,\omega) = \frac{1}{p(a)} + \sum_{k=-\mu}^{\mu} \frac{1}{t_k p_t'(t_k\omega)} \\ = & \ -\mathrm{res}_\infty \rho(t,\,\omega) = 0. \end{aligned}$$

Here $t_k\omega \in \mathrm{Z}_{|k|},\ k = -\mu, \ldots, \mu$ and the residue at infinity vanishes since $\rho(t,\,\omega) = O(t^{-2})$. Therefore

$$\sum_{k=-\mu}^{\mu} \frac{1}{t_k p_t'(a + t_k\omega)} = -\frac{1}{p(a)} \quad (13)$$

and we come up with the equation:

$$\begin{aligned} 2\mathrm{Re}\Theta_n(x,\, y) = & \ -\frac{1}{p(a)} \lim_{\varepsilon \to 0} \int_{\mathrm{S}_+} \left(\frac{1}{\langle z,\, \omega_+ \rangle^n} + \frac{1}{\langle z,\, \omega_- \rangle^n} \right) \Omega \\ = & \ -\frac{2}{p(a)} \mathrm{Re} \int_{\mathrm{S}} \frac{\Omega}{(\langle z,\, \omega \rangle - i0)^n}. \end{aligned}$$

The right hand side vanishes by [18] proposition 4.3 hence $\mathrm{Re}\Theta_n(x,\, y) = 0$.

For **odd** n, we argue in a similar way:

$$\begin{aligned} 2i\mathrm{Im}\Theta_n(x,\, y) = & \ \lim_{\varepsilon \to 0} \sum_{k=-\mu}^{\mu} \int_{\mathrm{S}_+} \frac{1}{t_k p'(a + t_k\omega)} \left[\frac{1}{\langle z,\, \omega_- \rangle^n} - \frac{1}{\langle z,\, \omega_+ \rangle^n} \right] \Omega \\ = & \ -\frac{2i}{p(a)} \mathrm{Im} \int_{\mathrm{S}^{n-1}} \frac{\Omega}{(\langle z,\, \omega \rangle - i0)^n}. \end{aligned} \quad (14)$$

The right hand side vanishes according to [18] proposition 4.3. This together with (14) implies vanishing of $\mathrm{Im}\Theta_n(x,\, y)$ and completes the proof of lemma 5. □

Theorem 2 now follows from [18] theorem 3.1 applied for the generating function Φ as above and for the space $\Sigma = \mathbb{R} \times \mathrm{Z}$ endowed with the form $\mathrm{d}\xi/\mathrm{d}p$. We change the variable $\lambda = r^2$ and take into account that $|\nabla\theta| = 2\,|x - \xi| = 2r$ and $\mathrm{M}f(r^2,\, x) = (2r)^{-1}\mathrm{R}f(r,\, x)$ in loc. cit. To complete the proof we only need to calculate the dominator

$$D_n(x) = \frac{1}{\left| \mathrm{S}^{n-1} \right|} \int_{\mathrm{Z}} \frac{1}{|\xi - x|} \frac{\mathrm{d}\xi}{\mathrm{d}p(\xi)}$$

for an arbitrary $x \in \mathrm{H}$. For any $\xi \in \mathrm{Z}$, we can write $\xi = x + t_k(\omega)\omega$ for a unique $\omega \in \mathrm{S}$ and $t_k > 0$ and have $|\xi - x| = t_k(\omega)$. Replacing a by x in (10) yields

$$D_n(x) = \frac{1}{\left|S^{n-1}\right|}\int_S \sum_{k=1}^{\mu} \frac{\Omega}{t_k p_t'\left(x + t_k(\omega)\omega\right)}.$$

The sum of contributions of opposite points $\omega \in S_+$ and $-\omega$ equals

$$\sum_{k=1}^{\mu} \frac{\Omega}{t_k(\omega) p_t'\left(x + t_k(\omega)\omega\right)} - \sum_{k=1}^{\mu} \frac{\Omega}{t_k(-\omega) p_s'\left(x - t_k(-\omega)\omega\right)}$$
$$= \sum_{k=-\mu}^{\mu} \frac{\Omega}{t_k(\omega) p_t'(x + t_k\omega)},$$

where $s = -t$ and $t_k(-\omega) = -t_{-k}(\omega)$, $k = 1, \ldots, \mu$. By (13) this sum is equal to $-1/p(x)$, hence

$$D_n(x) = -\frac{1}{\left|S^{n-1}\right| p(x)} \int_{S_+} \Omega = -\frac{1}{2p(x)},$$

which completes the proof of (2) and (3). □

Remark. Reconstructions (2)–(3) do not hold for an arbitrary convex set Z. Theorem 3 is equivalent to these reconstructions and consequently is not true for a general Z. However, it could work as a parametrics for arbitrary convex Z and may be for variable velocity with no traps, see [19].

7. Separators

Definition. Let p be an oscillatory polynomial of degree $m > 2$ with a hyperbolic point a. We say that a polynomial q *separates* p from a, if for almost any line L through a, each interval between consecutive zeros of p on L contains just one zero of q, except for the interval that contains a, where q does not vanish: see figure 5. It follows that q has, at least, $m - 2$ zero on L, hence $\deg q \geqslant m - 2$. We say that a separator q of a polynomial p is *strict* if $\deg q = m - 2$.

One can find a separator q by the following method.

Theorem 6. *For an arbitrary compact oscillatory set* Z *of degree* m*, there exists a polynomial* q *of degree* $<m$ *that separates* Z *from any hyperbolic point.*

Proof. The case $n = 1$ is trivial; we assume that $n > 1$. Take a hyperbolic point a and consider the Euler field $\mathbf{e}_a = \sum (x_i - a_i)\partial/\partial x_i$ centered at a. The polynomial $q_a \doteq \mathbf{e}_a(p) - mp$ has degree $\leqslant m - 1$ and for any unit ω,

$$q_a(a + t\omega) = t^{m+1}\frac{\mathrm{d}}{\mathrm{d}t}\left(t^{-m}p(a + t\omega)\right) = tp_t'(a + t\omega) - mp(a + t\omega). \quad (15)$$

We numerate roots of $p(a + t\omega)$ by $t_k = t_k(\omega)$, $k = \pm 1, \ldots, \pm \mu$ as in (1) so that t_k has the same sign as k. By (15) and Rolle's theorem $q_a(a + t\omega)$ has, at least, $m - 2$ roots $s_k = s_k(\omega)$, $k = \pm 1, \ldots, \pm (\mu - 1)$ such that

$$t_k \leqslant s_k \leqslant t_{k+1}, \quad t_{-k-1} \leqslant s_{-k} \leqslant t_{-k}, \quad k = 1, \ldots, \mu - 1. \quad (16)$$

By continuity it is true for all units ω, occasionally with non strict inequalities. Check that there are exactly $m - 2$ such zeros s_k counting with multiplicities. For any $\omega \in S$, except for a set of zero measures, all zeros t_k are simple and all inequalities (16) are strict.

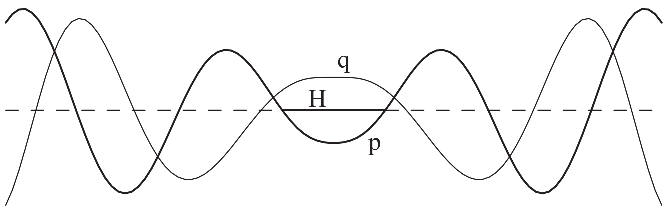

Figure 5. Polynomial p and a separator q

Suppose that one of the intervals (16), say $I_k \doteqdot (t_k, t_{k+1})$, contains more than one zero s_k. The total number r_k of zeros of q_a in I_k is odd, since the polynomial $t^{-m}p(a + t\omega)$ does not vanish in I_k. It follows that $r_k \geqslant 3$ and $r_i \geqslant 1$ for any i, which implies $\sum_i r_i \geqslant m$. This is not possible since $\deg q_a \leqslant m - 1$. By continuity (16) holds also for any unit ω in the sense that $q_a(a + s\omega)$ has a zero s of multiplicity $m - 1$, if this point is a root of $p(a + s\omega)$ of multiplicity m. It implies that all zeros $s = s_k(\omega)$, $k = 1,..., \mu - 1$ are unambiguously defined and by the Rouche theorem are continuous functions of ω. We have $s_k(-\omega) = -s_{-k}(\omega)$ and each variety

$$\mathrm{W}_k = \left\{x = a + s_k(\omega)\omega,\ \ \omega \in \mathrm{S}\right\},\ \ k = 1,..., \mu - 1$$

is closed and homeomorphic to a sphere. By (16) these varieties separate hypersurfaces Z_k, $k = 1,..., \mu$ constructed in theorem 1. Check that $q(a + t\omega)$ does not vanish for $t \in (t_{-1}, t_1)$. We can assume that $p < 0$ in H and have $q_a\left(a + t_{\pm 1}\omega\right) = t_{\pm}p_t'\left(a + t_{\pm 1}\omega\right) \geqslant 0$. If q vanishes at a point $s \in (t_{-1}, t_1)$, it must have $\geqslant 2$ zeros, which is impossible since $\deg q_a < m$. This shows that q_a separates p from a. It follows that q_a separates p from any other point $b \in \mathrm{H}$ since the hyperbolic cavity H is inside all ovals W_k. □

Corollary 7. *Any separator q of a compact oscillatory set Z of degree $m = 2\mu$ is an oscillatory polynomial with a hyperbolic cavity* $\mathrm{G} \supset \mathrm{H}$. *The zero set* W *of q is the union of continuous ovals* $\mathrm{W}_1, ..., \mathrm{W}_{\mu-1}$ *and of a closed unbounded component* W_μ *if q is not strict such that the sets* $\mathrm{H}, \mathrm{Z}_1, \mathrm{W}_1, \mathrm{Z}_2, ..., \mathrm{Z}_{\mu-1}, \mathrm{W}_{\mu-1}, \mathrm{Z}_\mu, \mathrm{W}_\mu$ *are nested.*

Proof. Any strict separator has no zeros $x = a + s\omega$ except for $s = s_k$ as in (16). If q is not strict it must have exactly one such real zero, say $s = s_\mu(\omega)$, which is defined and continuous for $\omega \in \mathrm{S}\backslash\mathrm{S}'$, where S' is a subset of dimension $<n - 1$. This function is odd, since any line $L_\omega = \{x = a + s\omega,\ \ s \in \mathbb{R}\},\ \ \omega \in \mathrm{S}\backslash\mathrm{S}'$ contains only one such zero. It follows that S' divides the unit sphere in two opposite parts $\mathrm{S}_\pm$ and $s_\mu(\omega) \to \infty$ as $\omega \to \mathrm{S}'$. The equation $x = a + s_\mu(\omega)\omega$ defines an unbounded component W_μ for $\omega \in \mathrm{S}_+$. □

Proposition 8. *Let p be an oscillatory polynomial of degree m such that the Taylor series of $p(a + y)$ does not contain terms of degree $m - 1$ for a hyperbolic point a. Then $q_a = \mathbf{e}_a(p) - mp$ is a strict separator of p. In particular, for any even polynomial p with a compact zero set the polynomial q_0 is a strict separator.*

Proof. The polynomial q_a is a strict separator since $\deg q_a \leqslant m - 2$. If p is even and Z is compact, the unique hyperbolic cavity H is symmetric with respect to the origin and contains the origin since it is convex. Therefore the second statement follows from the first one. □

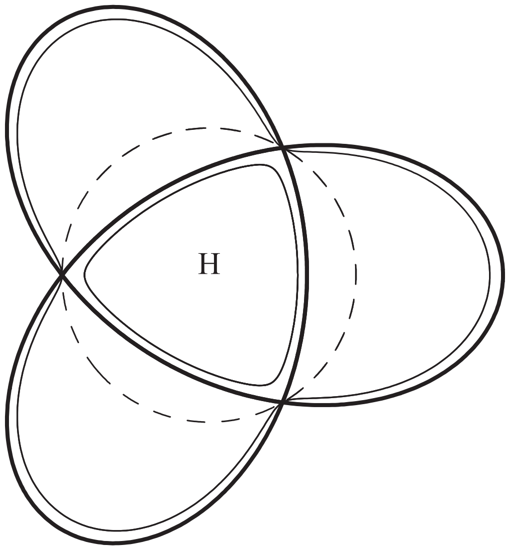

Figure 6. Hypotrochoid (thick), a regular approximation (thin)

Proposition 9. *For an arbitrary elliptic oscillatory polynomial p of degree m and arbitrary $\varepsilon > 0$, there exists an elliptic oscillatory polynomial p_ε of degree m such that the zero set Z_ε of p_ε is regular and $\left|p - p_\varepsilon\right| < \varepsilon$ in $\mathrm{Z} \cup \mathrm{Z}_\varepsilon$.*

Proof. Suppose that $p < 0$ in the hyperbolic camera H and $a \in \mathrm{H}$. The polynomial $q_a = \mathbf{e}_a(p) - mp$ separates p from a and $q_a > 0$ in H. Consider the polynomial $p_1 = p + \delta q_a$ for some small $\delta > 0$. It is easy to see that p_1 is an oscillatory polynomial with a hyperbolic camera H_1 that is close to H. For any unit ω, the multiplicity of any zero of $p_1(a + t\omega)$, $t \in \mathbb{R}$ is bounded by $\kappa - 1$, where $\kappa \leqslant \mu$ is the maximal multiplicity of zeros of $p(a + t\omega)$. We set $p_2 = p_1 + \delta_1 q_1$ where q_1 separates of p_1 from a and $\delta_1 > 0$. If δ_1 is sufficiently small, the maximal multiplicity of zeros of $p_2(a + t\omega)$ is bounded by $\kappa - 2$ and so on. A polynomial p_κ is again oscillatory and its zero set is regular. We have $\left|p - p_\kappa\right| < \varepsilon$, if the numbers $\delta, \delta_1, \ldots, \delta_{\kappa-1}$ chosen consecutively are sufficiently small. □

Example 5. A hypotrochoid given by the polynomial $p(x, y) = 4(x^2 + y^2)^2 - 4x^3 + 12xy^2 - 27(x^2 + y^2) + 27$ is oscillatory and $q = 4(x^2 + y^2) - 9$ is a strict separator: see figure 6.

Proposition 10. *Any compact convex domain Q in $\mathbb{R}^n$ can be approximated by regular hyperbolic cavities* H *of oscillatory sets admitting strict separators.*

8. A generalization of Newton's levitation theorem

I. Newton [17] proved that a mass uniformly distributed over a thin sphere $\mathrm{S} \subset E^3$ generates the zero gravitation field inside the sphere. The same is true for the cavity of a solid layer between two ellipsoids homothetic with respect to the center. Dive [4] called a monoid the layer between any two closed homothetic surfaces with respect to an interior point. He proved that any non-ellipsoidal monoid cannot create levitation in the cavity. Arnold [3] constructed a distribution of electric charge on a regular compact oscillatory set Z that generates the zero electrical field in the hyperbolic cavity H. This distribution, however, has a variable sign except if Z is not an ellipsoid. Similar problems for magnetic fields were studied in [20]. We show that for any oscillatory set Z that admits a strict separator, there exists a strictly positive mass distribution on Z that generates levitation in H. If Z is regular, the set of such mass distributions form a convex cone of dimension $\left|\begin{smallmatrix} m-2+n \\ n \end{smallmatrix}\right|$. Also layers bounded by close level sets of the corresponding oscillatory polynomial p generate levitation in a cavity. The key notion is a strict separator of an oscillatory polynomial.

Theorem 11. *For an arbitrary oscillatory polynomial p in E^3 with a compact zero set and any strict separator q, the distribution of mass in E^3 with density $|q|\,\delta(p)$ generates the zero gravitation field in the hyperbolic cavity* H *of p.*

Proof. For any $\omega \in \mathrm{S}\setminus\mathrm{S}'$, the polynomial $p_\omega(t) = p(a + t\omega)$ has m real zeros $t_k(\omega)$, $k = \pm 1, \pm 2, ..., \pm \mu$ numerated as in theorem 1. By the residue theorem we have:

$$\sum \frac{q(a + t_k\omega)}{p_t'(a + t_k\omega)} = \sum \mathrm{res}_{t_k} \frac{q(a + t\omega)}{p_t'(a + t\omega)} \mathrm{d}t = -\,\mathrm{res}_\infty \frac{q}{p} \mathrm{d}t = 0. \tag{17}$$

According to the numeration a zero t_k is positive or negative together with k. Suppose that $p < 0$ and $q > 0$ in H. Signs of $p_t'(a + t_k\omega)$, $k = -\mu, ..., -1, 1, ..., \mu$ alternate and $kp_t'(a + t_k\omega) > 0$ for any odd k whereas $kp_t'(a + t_k\omega) < 0$ with even k. The sign of $q(a + t_k\omega)$ alternates in the different way: $q(a + t_k\omega) > 0$ for odd k and $q(a + t_k\omega) < 0$ for even k. Therefore we have:

$$\frac{q(a + t_k\omega)}{p_t'(a + t_k\omega)} > 0 \text{ for } k > 0; \quad \frac{q(a + t_k\omega)}{p_t'(a + t_k\omega)} < 0 \text{ for } k < 0. \tag{18}$$

The sum of these fractions vanishes by (17). By Newton's law the gravitation field decreases at the same rate as a beam of straight rays diverges. Following Newton's geometrical method we consider a small solid angle $\mathrm{C}(a, \omega) \subset E$ with vertex at a of spherical measure $\mathrm{d}\omega$. The mass of a piece of $\mathrm{Z} \cap \mathrm{C}(a, \omega)$ at a point $x = a + t_k\omega$ is equal to

$$m_k \doteqdot |t_k|^2 \frac{|q(x)|}{|\langle \omega, \nabla p(x)\rangle|} \Omega = |t_k|^2 \left| \frac{q(x)}{p_t'(x)} \right| \Omega.$$

Its contribution to the field at a is equal to $F_k \doteqdot m_k |t_k|^{-2} \mathrm{d}\omega$ for $k > 0$ and $F_k \doteqdot -|t_k|^{-2} m_k \Omega$ for $k < 0$, since these points are on opposite site of a. By (18) the total of these contributions equals the sum (17) times Ω, hence cancels. Theorem 11 follows since this conclusion holds for almost all ω.

Remark 1. For the polynomial $p = |x|^2 - 1$, the above construction gives $q_0 = 2$. Theorem 11 guarantees levitation in a sphere generated by a uniform distribution of mass on the sphere. This is Newton's attraction theorem. If Z is a compact regular oscillatory set of degree m and q is a strict separator, then any polynomial $\tilde{q}$ of degree $m - 2$ that is sufficiently close to q is a strict separator. By theorem 11 for any such $\tilde{q}$, $\tilde{q}\mathrm{d}\xi/\mathrm{d}p$ is a volume form in Z. Any mass distribution in Z that is proportional to this form admits levitation in H.

Remark 2. Theorem 11 is generalized for arbitrary linear space $\mathbb{R}^n$, if the corresponding 'gravitation' force generated by a delta-like mass at the origin has the potential $U = \sigma(\omega) r^{-n+1}$, where $\sigma(\omega)$ is an arbitrary even function of $\omega \in \mathrm{S}$.

Corollary 12. *Under conditions of theorem 11, if $p < 0$ in* H, *the distribution of mass with density $|q|\,\mathrm{d}x$ in the layer $L \doteqdot \{x\colon a \leqslant p(x) \leqslant b\}$ generates the zero gravitation field in* H *for arbitrary a, b such that $q \neq 0$ in L.*

Proof. By theorem 11, the density $|q|\,\delta(p-\lambda)$ generates the zero gravity in H for any λ such that q separates $p-\lambda$. By Fubini's the same is true for the density

$$\int_a^b |q|\,\delta(p-\lambda)\mathrm{d}\lambda = |q|\,\mathrm{d}x$$

supported by the layer $\{a \leqslant p \leqslant b\}$ if q separates $p-\lambda$ for $a \leqslant \lambda \leqslant b$. □

A layer generated by a hypotrochoid is shown in figure 6.

Example 7. A surface of normals of the system of crystal optics is given by the equation $p(\xi) = 0$ where:

$$p(\xi) = \left(\sigma_1\xi_1^2 + \sigma_2\xi_2^2 + \sigma_3\xi_3^2\right)|\xi|^2$$
$$- (\sigma_3+\sigma_2)\sigma_1\xi_1^2 - (\sigma_1+\sigma_3)\sigma_2\xi_2^2 - (\sigma_1+\sigma_2)\sigma_3\xi_3^2 + \sigma_1\sigma_2\sigma_3$$

is an even elliptic oscillatory polynomial. The polynomial $q = \mathbf{e}_0(p) - 4p$ is a strict separator.

9. Non strict case

Theorem 13. *Let p be an elliptic oscillatory polynomial of degree m and q be a separator. The gravitation field generated by mass distribution in* Z *with the density $|q|\,\delta(p)$ is constant in the hyperbolic cavity and equals*

$$F = -\int_{\mathrm{P}^{n-1}} \frac{q_{m-1}(\omega)}{p_m(\omega)} \cdot \omega\ \Omega, \tag{19}$$

where integration van be taken over an arbitrary unit hemisphere.

Note that the integrand $q_{m-1}(\omega)/p_m(\omega)\cdot\omega$ is an even vector function of ω.

Proof. Let again $\mathrm{C}(a,\omega)$ be a small solid angle as in theorem 11. By (17) the gravitation field generated by points on $\mathrm{Z}\cap\mathrm{C}(a,\omega)$ upon a point $a \in \mathrm{H}$ equals the vector

$$\sum_{k=1}^m \frac{q(x)}{p_t'(x)}\cdot\omega\ \Omega.$$

where $x = a + t_k\omega$. By the residue theorem, the sum of scalars is equal to

$$\sum_{k=1}^m \frac{q(x)}{p_t'(x)} = -\mathrm{res}_\infty \frac{q}{p}\mathrm{d}t = -\frac{q_{m-1}(\omega)}{p_m(\omega)}.$$

Integrating over any unit hemisphere, we get (19). □

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