A PARAMETRIX METHOD IN INTEGRAL GEOMETRY

By

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Abstract. The objective of reconstructive integral geometry is to recover a function from its integrals over a set of subvarieties. A parametrix is a method of reconstruction of a function from its integral data up to a smoothing operator. In the simplest case, a parametrix recovers a function with a jump singularity along a curve (surface) up to a continuous function, which can be quite informative in medical imaging. We provide an explicit construction for a wide class of acquisition geometries. The case of photo-acoustic geometry is of special interest.

1 Introduction

Let \((X, g)\) be a Riemannian manifold and \(\Sigma\) be a family of smooth submanifolds \(\sigma \subset X\). For a function \(f\) defined in \(X\) with compact support, the family of integrals

\[
g(\sigma) = \int_{\sigma} f \, d_g S, \quad \sigma \in \Sigma
\]

defines function on \(\Sigma\). The family \(\Sigma\) is called the acquisition geometry of the integral transform \(R_{\Sigma} f \doteq g\). An analytic inversion formula \(g \mapsto f\) is known only for special types of acquisition geometries \(\Sigma\); see the survey in [11]. Here, we construct a parametrix for a class of weighted integral transforms \(R_{\Sigma}\) for which analytic reconstruction is not known (Sections 5–8).

A parametrix recovers not only the wave front of a function \(f\) but also the profile of its singularity. A parametrix for a class of integral transforms was constructed earlier by Beylkin [1] in terms of Fourier integral operators. Pestov and Uhlmann [12] gave a construction of a parametrix for the geodesic integral transform on two-dimensional simple Riemannian manifolds.

In Section 9, we apply our construction for photo-acoustic (thermo-acoustic) acquisition geometry. This topic was studied in papers of Popov and Sushko [13], Kunyansky [8], Xu-Wang [15], Natterer [9], and in [11]. Our method is based on the Calderón-Zygmund theory of singular integral operators adapted in Section 10.
2 Parametrices in Sobolev spaces

Let $X$ and $Y$ be compact manifolds with boundaries of class $C^\kappa$, where $\kappa$ is a natural number. The Sobolev spaces $H^\alpha(X)$ and $H^\alpha(Y)$ are well-defined for every real $\alpha$, $|\alpha| < \kappa$; see, e.g., [14]. We say that a densely defined operator $A : L_2(X) \to L_2(Y)$ has Sobolev order $d \in \mathbb{R}$ if it generates a bounded operator $A_\alpha : H^\alpha(X) \to H^{\alpha-d}(Y)$ for every $\alpha$, $|\alpha| < \kappa$, $|\alpha - d| < \kappa$, which is the restriction of $A$ for positive $\alpha$ and a closure of $A$ for negative $\alpha$. If $d$ is negative, $A$ is called a $d$-smoothing operator. An operator $P : L_2(Y) \to L_2(X)$ is said to be an $s$-parametrix for $A$ if $0 < s \leq \kappa$ and $PA = \text{Id} + R$, where the remainder $R$ is a $s$-smoothing operator. If $P_1$ is a 1-parametrix and $R_1$ is a remainder, a $k$-parametrix $P_k$ can be found for any natural number $k$ recursively by $P_k = P_{k-1} - R_{k-1}P_1$, $R_k = -R_{k-1}R_1$ for $k = 2, \ldots, \kappa$. Every 1-smoothing operator is compact in $L_2(X)$; hence $P_1A$ is a Fredholm operator, and the image of $A$ is closed. An $s$-parametrix $P_s$ recovers the singularity of an arbitrary function $f \in H^\alpha(X)$ from $Af$ up to a function $h = R_s f \in H^{\alpha+s}(X)$. In particular, if $f = \delta_y$ is the delta-function at a point $y \in X$ and $s > n$, then $h = R_s \delta_y$ is continuous. In fact, $\delta_y \in H^\alpha(X)$ for every $\alpha < -n/2$, which implies $h \in H^{\alpha+s}(X)$. The space $H^{\alpha+s}(X)$ is contained in $C(X)$ if $\alpha > n/2 - s$; hence $h$ is continuous. The equation $P_s A \delta_y = \delta_y + h$ shows that every delta function can be recognized from data of $A \delta_y$ by means of an $s$-parametrix $P_s$.

3 Generating functions and integrals

Let $X$ and $\Sigma$ be smooth $n$-dimensional manifolds, and $\Phi : X \times \Sigma \to \mathbb{R}$ a $C^2$-smooth real function such that $d\Phi \neq 0$ on the set $Z = \Phi^{-1}(0)$. Let $p : Z \to X$, $\pi : Z \to \Sigma$ be the natural projections. Suppose that

\[ \det(d_{x,t}d_{\sigma,\tau}(t\tau\Phi(x, \sigma))) \neq 0, \]

where $d_{x,t}$, $d_{\sigma,\tau}$ are exterior differentials in the manifolds $X \times \mathbb{R}$ and $\Sigma \times \mathbb{R}$, respectively.

**Proposition 3.1.** Property (2) holds if and only if $\pi$ has rank $n$ and $p^* : N^*(Z) \to T^*(X)$ is a local diffeomorphism, where $N^*(Z)$ denotes the conormal bundle of $Z$ and $p^*(x, \sigma; \zeta, s) = (x, \zeta) \in T^*(X)$.

For a proof, see [10, Proposition 1.1].

It follows that for each $\sigma \in \Sigma$, the set $Z(\sigma) = \pi^{-1}(\sigma) = \{x : \Phi(x, \sigma) = 0\}$ is a $C^1$-hypersurface in $X$; and for every point $x \in X$ and tangent hyperplane $h \subset T_x(X)$, there exists a locally unique hypersurface $Z(\sigma)$ through $x$ tangent to $h$. 

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The function $\Phi$ is called **generating** for the acquisition geometry $\{Z(\sigma) : \sigma \in \Sigma\}$. Let $dV$ be a volume form on $X$ and $\rho = \rho(x, \sigma)$ a continuous function on $Z$. We define a weighted integral transform of the continuous function $f$ with compact support in $X$ by

$$M_r f(\sigma) = \int_X \delta(\Phi(x, \sigma))r(x, \sigma)f(x)dV.$$  \hspace{1cm} (3)

The limit exists since $d_x \Phi \neq 0$. We can write this integral in the form

$$M_r f(\sigma) = \int_{Z(\sigma)} f(x)r(x, \sigma)q(x, \sigma),$$  \hspace{1cm} (4)

where $q = dV/d\theta$ denotes an arbitrary $(n-1)$-differential form $q$ such that $d\Phi \wedge q = dV$. An orientation is defined in a hypersurface $Z(\sigma)$ by means of the form $d_x \Phi$, and the integral over $Z(\sigma)$ is well-defined.

Choose a volume form $d\Sigma$ on $\Sigma$ and interchange the roles of $X$ and $\Sigma$, keeping the same generating function $\Phi$. The corresponding integral transform $M_r^*$ is called the **back projection** operator. Note that condition (2) is symmetric, and Proposition 3.2 holds also for the operator $M_r^*$.

For a closed set $K \subset X$ and a real $\alpha$, we denote by $H^\alpha_0(K(X))$ the subspace of $H^\alpha(X)$ consisting of distributions with support in $K$. The subspace $H^\alpha_0(\Sigma)$ of $H^\alpha(\Sigma)$ is defined similarly.

**Proposition 3.2.** If $\Phi$ is a smooth generating function satisfying (2) and $r$ is a smooth function, then for any compact set $K \subset X$ with smooth boundary, any real $\alpha$, and any smooth function $\phi$ with compact support in $\Sigma$,

$$\|\phi M_r f\|^{\alpha(n-1)/2} \leq C \|f\|^\alpha, \ f \in H^\alpha_0(K(X)),$$

where $C$ is a constant which does not depend on $f$. If the map $p$ is proper, then the operator $M_r : H^\alpha_0(K(X)) \to H^\alpha_0(\Sigma)$ is densely defined, where $L = \pi p^{-1}(K)$, and has Sobolev order $(1-n)/2$.

**Proof.** We can write the transform as a Fourier integral operator:

$$M_r f(\sigma) = \int_K \int_{\mathbb{R}} \exp(2\pi i \tau \Phi(x, \sigma))r(x, \sigma)f(x)d\tau dV.$$  \hspace{1cm} (4)

The critical set of the phase function $\tau \Phi(x, \sigma)$ is the hypersurface $F(\sigma)$, and the condition $d_x \Phi \neq 0$ implies that the phase function is non-degenerate. The corresponding conic Lagrange variety is

$$L = \{(x, \sigma; \zeta, s) \in T^*(X \times \Sigma) : \Phi(x, \sigma) = 0, \ s = \lambda d_\sigma \Phi, \ \zeta = \lambda d_x \Phi, \ \lambda \neq 0\}.$$
The rank of the matrix $\partial(x, \xi)/\partial(\sigma, s)$ equals $2n$ at every point of $L$. This follows from (2); for details, see [10, Lemma 3.2]. Therefore, projections of $L$ to $T^*(X)$ and to $T^*(\Sigma)$ are submersions, that is, $L$ is locally the graph of a canonical transformation. The symbol $a(x, \sigma; \xi, s) = 1$ is a homogeneous function of $\xi, s$ of order 0. The order $m$ of the Fourier integral operator $M_r$ satisfies

$$m + \text{dim} X \times \Sigma/4 - N/2 = 0,$$

where $\text{dim} X \times \Sigma = 2n$ and $N = 1$ is the number of variables $\tau$. This yields $m = (1 - n)/2$, which means that the functional

$$\psi \mapsto \int_X \int_{\Sigma} \int_{\mathbb{R}} \exp(2\pi i \tau \Phi(x, \sigma)) r(x, \sigma) \psi(x, \sigma) d\tau dV d\Sigma$$

defined for smooth test densities $\psi$, is a distribution of the class $I^{(1-n)/2}(X \times \Sigma, L)$ in the sense of Hörmander. By [6, Corollary 25.3.2], the operator $\phi M_r$ defines a continuous map $H^a_F(X) \to H^{a+(n-1)/2}_F(\Sigma)$ for every real $a$, where $F = \text{supp} \phi$.

If $p$ is proper, the set $L = \pi(p^{-1}(K))$ is compact, and we can choose a cut-off function such that $\phi = 1$ in $L$. Then $\phi M_r f = M_r f$, and the second statement follows.

We say that a generating function $\Phi$ is resolved if $\Sigma = \mathbb{R} \times \Omega$, where $\Omega$ is the unit sphere in euclidean space $E^n$ and $\Phi(x, \sigma) = \theta(x, \omega) - \lambda$, $\sigma = (\lambda, \omega)$, $\lambda \in \mathbb{R}$, $\omega \in \Omega$ for a function $\theta \in C^2(X \times \Omega)$. If $\Phi$ is resolved, the map $p : Z \to X$ is proper since $\theta$ is continuous. It follows that $M_r f$ has compact support in $\Sigma$ if $f$ does.

**Definition.** We call a generating function $\Phi$ regular if it is resolved, satisfies (2), and the equations

$$\theta(x, \omega) = \theta(y, \omega), \quad d_\omega \theta(x, \omega) = d_\omega \theta(y, \omega)$$

are satisfied simultaneously for no $x \neq y \in X, \omega \in \Omega$ (that is to say, the family \{ $Z(\sigma) : \sigma \in \Sigma$ \} has no conjugate points.)

### 4 Principal value integrals

Let $f$ be a smooth real function on a manifold $X$ having only simple zeros, i.e., $df(x) \neq 0$ whenever $f(x) = 0$. For a natural number $n$, we consider the functional

$$I_n(a) = \int_X \frac{a}{(f - i\varepsilon)^n} = \lim_{\varepsilon \searrow 0} \int_X \frac{a}{(f - i\varepsilon)^n},$$
defined for test densities $a$ in $X$. For a real density $a$, the functional
\[ \int_X \frac{a}{f^n} = \text{Re} I_n(a) \]
is called a **principal value integral**.

**Proposition 4.1.** For every smooth function $f$ having only simple zeros, the limit in (6) exists for every test density $a$. The functional $I_n$ is a generalized function in $X$.

**Proof.** For an arbitrary smooth function $g$, tangent field $t$, and test density $a$ in $X$,
\[ d(g \wedge (t \downarrow a)) = dg \wedge (t \downarrow a) + g \, d(t \downarrow a), \]
where the symbol $\downarrow$ denotes the inner product of a field and a form. If $a$ has compact support, the integral of the left hand side over $X$ vanishes, and
\[ \int t(g) \, a = \int (t \downarrow dg) \wedge a = \int dg \wedge (t \downarrow a) = - \int g \, d(t \downarrow a). \]

To prove the statement, choose a tangent field $t_1$ and a smooth function $t_0$ in $X$ such that $t_1(f) + t_0 = 1$, and apply induction in $n \geq 1$. In the case $n = 1$, we integrate by parts in (6) and apply (7) to obtain
\[ I_1(a) = \int \frac{(t_1(f) + t_0)a}{f - i0} = t_1(\log(f - i0))a + \int t_0a \]
\[ = - \int \log(f - i0)d(t_1 \downarrow a) + \int t_0a, \]
where $\log(f - i0)$ is a locally integrable function.

For the case $n > 1$, we can write
\[ I_n(a) = \int \frac{(t_1(f) + t_0)a}{(f - i0)^n} = \frac{1}{1 - n} \int t_1[\frac{1}{(f - i0)^{n-1}}]a + I_{n-1}(t_0a) \]
\[ = \frac{1}{n - 1} \int \frac{d(t_1 \downarrow a)}{(f - i0)^{n-1}} + I_{n-1}(t_0a) = I_{n-1}(\frac{d(t_1 \downarrow a)}{n - 1} + t_0a). \]
Here, the form $d(t_1 \downarrow a)$ is again a smooth density with compact support in $X$. \(\square\)

5 Filtered back projection operator

**Theorem 5.1.** Let $X$ be an open set in euclidean space $E^n$, $\Phi = \theta - \lambda$ a smooth regular generating function on $X \times \mathbb{R} \times \Omega$ of class $C^\kappa$, and $\rho \in C^\kappa(X \times \Omega)$, where $\kappa > n + 1$. Define an operator by the principal value integral
\[ Q_x g(x) = \pi_n(n - 1)! \int g(\lambda, \omega) d\sigma(\lambda) d\Omega \]

\[ \int \frac{\rho(x, \omega)g(\lambda, \omega)d\lambda d\Omega}{(\theta(x, \omega) - \lambda)^n} \]
for even $n$ and by

$$Q_{\rho}g(x) = \pi_n \int_{\Omega} \rho(x, \omega) g^{(n-1)}(\theta(x, \omega), \omega) d\omega$$

for odd $n \geq 3$, where $g$ is a function defined on $\mathbb{R} \times \Omega$, $g^{(n-1)} = (\partial/\partial \lambda)^{n-1} g$, and $\pi_n = -(2\pi)^{-n}$ for even $n$, $\pi_n = 2(2\pi)^{1-n}$ for odd $n$. Then

$$Q_{\rho}M_r = D_{r,\rho} \text{Id} + A_{r,\rho},$$

where

$$D_{r,\rho}(x) = \frac{1}{|S^{n-1}|} \int_{\Omega} \frac{\rho(x, \omega) r(x, \omega) d\omega}{|\nabla_x \theta(x, \omega)|^n}.$$  

$A_{\rho}$ is a singular integral operator of Sobolev order 0 with the kernel $-(n-1)! \text{Re } \Theta$ if $n$ even and $1/2\pi(n-1)! \text{Im } \Theta$ if $n$ odd. Here, the singular integral

$$\Theta(x, y) = \int_{\Omega} \frac{\rho(x, \omega) r(y, \omega) d\omega}{(\theta(y, \omega) - \theta(x, \omega) - i0)^n}$$

is defined for $x \neq y \in X$ by the method of Section 4.

**Lemma 5.2.** The composition $Q_{\rho}M_r$ extends to a continuous operator $L_2(X)_{\text{comp}} \rightarrow L_2(X)_{\text{loc}}$.

**Proof.** For even $n$, integrating by parts in (8) yields

$$Q_{\rho}g(x) = \pi_n (n-1)! \int_{\Omega} \rho(x, \omega) \int_{\mathbb{R}} (\theta(x, \omega) - \lambda)^{-n} g(\lambda, \omega) d\lambda d\omega,$$

that is,

$$Q_{\rho} = \pi_n (n-1)! M_{\rho}^*(\Lambda_n \times \text{Id}),$$

where $\Lambda_n$ is the convolution operator in $\mathbb{R}$ with the principal value kernel $\lambda^{-n}$ acting on the $\lambda$ variable and

$$M_{\rho}^* g(x) = \int_{\Omega} \rho(x, \omega) g(\theta(x, \omega), \omega) d\omega$$

is a weighted back projection operator. By Proposition 3.2, the operator $M_r$ is bounded in the spaces $H^0_k(X) \rightarrow H^{(n-1)/2}_L(\Sigma)$, where $L = \pi(p^{-1}(K))$ is a compact in $\Sigma$ since $p$ is proper. The convolution operator has a factorization $\Lambda_n = C_n(\partial/\partial \lambda)^{n-1} H$, where $H$ is a Hilbert operator and $C_n$ is a constant. It follows that for arbitrary $\alpha \in \mathbb{R}$, $\Lambda_n$ defines a bounded map $H^\alpha_\mathbb{R} \rightarrow H^{a-n+1}_\mathbb{R}$. Taking $\alpha = n-1$, we conclude that $\Lambda_n \times \text{Id} : H^{(n-1)/2}_\mathbb{R} \rightarrow H^{-(n-1)/2}_\mathbb{R}$ and $H^{-(n-1)/2}_\mathbb{R} \times \Omega \rightarrow H^{0-n+1}_\mathbb{R} \times \Omega$ is also bounded. By [10, Proposition 3.1], $M_{\rho}^*$ is continuous as an operator $H^{-(n-1)/2}_\mathbb{R} \times \Omega \rightarrow H^0_\mathbb{R} \times \Omega$. 


Finally, $M^*_\rho(\Lambda_n \times \text{Id})M_r$ is continuous as an operator $L_2(X)_{\text{comp}} \to L_2(X)_{\text{loc}}$, and the statement follows.

In the case of odd $n$, there exists a similar factorization with $\Lambda_n = C_n(\partial/\partial\lambda)^{n-1}$, which leads to the same conclusion.

\textbf{Lemma 5.3.} For even $n$, arbitrary $x \in X$, $\omega \in \Omega$, and small $\varepsilon$, 

\begin{equation}
    d_{n,\varepsilon}(x, \omega) \doteq \frac{(n-1)!}{(2\pi i)^{n-1}} \Re \int_X \frac{\rho(x, \omega)r(y, \omega)e_\varepsilon(s)dV(y)}{(\theta(y, \omega) - \theta(x, \omega) - i0)^n} = \frac{1}{|S^{n-1}|} \frac{\rho(x, \omega)r(x, \omega)}{|\nabla\theta(x, \omega)|^n} + o(1),
\end{equation}

where $o(1) \to 0$ as $\varepsilon \to 0$. For odd $n$,

\begin{equation}
    d_{n,\varepsilon}(x, \omega) \doteq \frac{(n-1)!}{(2\pi i)^{n-1}} \Im \int_X \frac{\rho(x, \omega)r(y, \omega)e_\varepsilon(s)dV(y)}{(\theta(y, \omega) - \theta(x, \omega) - i0)^n} = \frac{2\rho(x, \omega)r(x, \omega)}{2|S^{n-1}|} + O(\varepsilon),
\end{equation}

where $s = y - x$.

\textbf{Proof.} The proof follows along the lines of [11, Lemma 3.3].

\textbf{Proof of Theorem 5.1.} We have

$$d_n(x) = \lim_{\varepsilon \to 0} \int_\Omega d_{n,\varepsilon}(x, \omega) \, d\Omega.$$ 

Taking the limit and integrating (12) over $\Omega$ we obtain, for even $n$,

$$d_n(x) = \frac{1}{|S^{n-1}|} \int_\Omega \frac{\rho(x, \omega)r(x, \omega) \, d\Omega}{|\nabla\theta(x, \omega)|^n} = D_{r,\rho}(x),$$

which implies (10). For odd $n$, we again obtain $d_n = D_{r,\rho}$, and (10) follows.

Fixing a point $x \in X$ and setting $f_\varepsilon(y) = e_\varepsilon(y - x)f(y)$ for a $C^n$-function $f$ in $X$ gives

$$Q_\rho M_r(f_\varepsilon)(x) = c_n \int \frac{\rho M_r(f_\varepsilon) \, d\lambda \, d\Omega}{(\theta - \lambda)^n} = c_n \Re \int_X \frac{\rho(x, \omega)r(y, \omega)e_\varepsilon(y - x)f(y)dV(y)}{(\theta(y, \omega) - \theta(x, \omega) - i0)^n},$$

where $c_n = \pi_n(n-1)!$. By Lemma 5.3, the right hand side tends to $D_{r,\rho}(x)f(x)$ as $\varepsilon \to 0$. The operator $f \mapsto D_{r,\rho}f$ acting in $L_2(X)_{\text{comp}}$ is obviously bounded; and by Lemma 5.2, the residue $A_{r,\rho} = Q_\rho M_r - D_{r,\rho}\text{Id}$ is the off-diagonal kernel of $Q_\rho M_r$ and is a bounded operator $L_2(X)_{\text{comp}}(X) \to L_2(X)_{\text{loc}}$. Take an arbitrary function
$f \in L_2(X)_{\text{comp}}$ that vanishes in a neighborhood of $x$ and calculate

$$
Q_{\rho}M_{\rho}f(x) = c_n \int X \frac{M_{\rho}f(\lambda, \omega) d\lambda \rho(x, \omega)d\Omega}{(\theta(x, \omega) - \lambda)^n}
$$

$$
= c_n \int \Omega \rho(x, \omega)d\Omega \int_{\mathbb{R}} \frac{d\lambda}{\theta(x, \omega) - \lambda)^n} \int_{Z(\lambda, \omega)} r(y, \omega)f(y)q
$$

$$
= c_n \int X \left( \int \Omega \frac{\rho(x, \omega)r(y, \omega)d\Omega}{(\theta(x, \omega) - \theta(y, \omega))^n} \right) f(y)d\theta \wedge q
$$

$$
= c_n \int X \Re \Theta(x, y)f(y)dV,
$$

where we have applied (4) and $\Theta$ is as in (11). The relation $d\lambda = d\theta$ holds in $Z$, and the equation $d\theta \wedge q = dV$ is satisfied in $X$, by definition. Thus the function $-c_n \Re \Theta$ is the off-diagonal kernel of the operator $Q_{\rho}M_{\rho}$. A similar equation holds for odd $n$ with the kernel $\Im \Theta$.

\section{Off-diagonal kernel}

\textbf{Proposition 6.1.} Under the assumptions of Theorem 5.1, the operator $A_{\rho, \theta} = Q_{\rho}M_{\rho} - D_{\rho, \theta} \Id$ is a singular integral operator of Sobolev order 0 with leading term

$$
a_0(x, s) = \begin{cases} 
\Re q_0(x, s) & \text{for even } n, \\
-\Im q_0(x, s)/\pi & \text{for odd } n,
\end{cases}
$$

where

$$
q_0(x, s) = -(n-1)! \int \Omega \frac{\rho(x, \omega) r(x, \omega) d\Omega}{(\nabla \theta(x, \omega), s) - i0)^n}, \quad s = y - x.
$$

\textbf{Proof.} We show that the kernel $\Theta$ defines an operator of Sobolev order 0. Applying the Lagrange formula to $\theta(y, \omega)$ at $y = x$ and estimating the remainder, we get

$$
\theta(y, \omega) - \theta(x, \omega) = (\nabla \theta(x, \omega), s) + \xi(x, s, \omega),
$$

where

$$
\max_{|l|+|j| \leq \kappa-2} \max_{x \in K} \max_{\omega} \max |s|^{|l|} |D_s D_x^j \xi(x, s, \omega)| \leq C_K |s|^2.
$$

For $F(p) = (1 - p)^{-n}$, the Lagrange formula yields

$$
(1 + p)^{-n} = 1 - np \int_0^1 (1 + tp)^{-n-1} dt = 1 - np + \frac{n(n + 1)}{2} p^2 \int_0^1 (1 + tp)^{-n-2} dt.
$$
Taking \( p = \xi(x, s)(\langle \nabla \theta(x, \omega), s \rangle - i0)^{-1} \) and multiplying by

\[
(\langle \nabla \theta(x, \omega), s \rangle - i0)^{-n}
\]
yields

\[
1 \quad \frac{1}{(\langle \nabla \theta(x, \omega), s \rangle - i0)^n} = \frac{1}{(\langle \nabla \theta(x, \omega), s \rangle - i0)^n - n \int_0^1 \xi dt} \quad (\langle \nabla \theta(x, \omega), s \rangle + t\xi - i0)^{n+1}.
\]

Integrating against the density \( \rho(x, \omega)r(y, \omega)d\Omega \) yields

\[
\Theta(x, y) = \int \frac{\rho(x, \omega)r(y, \omega)d\Omega}{(\theta(y, \omega) - \theta(x, \omega) - i0)^n} = q_0(x, s) + r_1(x, s),
\]

where the first term on the right hand side is as in (13) and

\[
r_1(x, s) = -n \int_0^1 dt \int \frac{\xi(x, s, \omega)\rho(x, \omega)r(x, \omega)d\Omega}{(\langle \nabla \theta(x, \omega), s \rangle + t\xi - i0)^{n+1}} + \int \frac{\rho(x, \omega)r(y, \omega) - r(x, \omega)d\Omega}{(\theta(y, \omega) - \theta(x, \omega) - i0)^n}.
\]

For each \( i = 1, 2, \ldots, n \), we can find a smooth function \( t_0i \) and a smooth tangent field \( t_1i \) in \( \Omega \) such that \( t_0i \nabla x\theta + t_1i(\nabla x\theta) = e_i \), where the gradient \( \nabla x\theta \) is viewed as a column vector and \( e_i \) is the \( i \)-th column of the unit \( n \times n \)-matrix. For a local construction, we use columns of the inverse to the matrix

\[
\begin{pmatrix}
\theta_1 & \partial_\omega_1 \theta_1 & \cdots & \partial_\omega_n \theta_1 \\
\theta_2 & \partial_\omega_1 \theta_2 & \cdots & \partial_\omega_n \theta_2 \\
\vdots & \vdots & \cdots & \vdots \\
\theta_n & \partial_\omega_1 \theta_n & \cdots & \partial_\omega_n \theta_n
\end{pmatrix},
\]

where \( \theta_k = \partial \theta/\partial x_k, k = 1, \ldots, n \). By (2), this matrix is invertible for any local coordinate system \( \omega_1, \ldots, \omega_{n-1} \) in \( \Omega \). We extend the functions \( t_0i \) and fields \( t_{1i} \) to the whole sphere by means of a partition of unity. Consider the differential operator

\[
t_0 + t_1, \quad t_0 = |s|^{-2} \sum_{i=1}^n s_i t_{0i}, \quad t_1 = |s|^{-2} \sum_{i=1}^n s_i t_{1i},
\]

where \( t_0 \) is a function and \( t_1 \) is a tangent field on the sphere with coefficient depending on \( s \). Consider the function \( v = v(x, s) = \langle \nabla \theta(x, \omega), s \rangle - i0 \). We have

\[
t_0v + t_1(v) = |s|^{-2} \sum_{i=1}^n s_i^2 = 1.
\]
Therefore \( \tau_{-k}(v^{-k}) = (t_0 v + k^{-1} t_1(v))v^{-k-1} = v^{-k-1} \) for an arbitrary integer \( k \neq 0 \), where \( \tau_{-k} = t_0 + k^{-1} t_1 \) and \( t_0 + t_1(\log v) = v^{-1} \). Integration by parts yields

\[
q_0 = \int_{\Omega} \rho v^{-n} d\Omega = \int_{\Omega} \tau_{1-n}(v^{-1})\rho d\Omega = \frac{1}{1 - n} \int_{\Omega} v^{1-n} \tau_{1-n}^*(\rho) d\Omega,
\]

where \( \tau_k^* \) is the adjoint differential operator to \( \tau_k \) which is a homogeneous function of \( s \) of degree \(-1\). Integrating by parts \( n \) times, we obtain

\[
q_0(s) = \cdots = b_n \int_{\Omega} v^{-1} \tau_{1}^* (\cdots \tau_{1-n}^*(\rho)) d\Omega
\]

(17)

\[
= b_n \int_{\Omega} [\log v t_1^* + t_0^*] \tau_{1}^* (\cdots \tau_{1-n}^*(\rho)) d\Omega,
\]

where \( b_n = (-1)^{n-1} / (n-1)! \). The function \( t_1^* \tau_{1-n}^* (\cdots \tau_{1-n}^*(\rho)) \) as well as the function \( t_0^* \tau_{1-n}^* (\cdots \tau_{1-n}^*(\rho)) \) has homogeneous coefficients of degree \(-n\) with respect to \( s \). We check that the integral (17) is a homogeneous function of degree \(-n\). Indeed,

\[
\log v = \log(v/|s|) + \log |s|,
\]

where the first term in the right hand side is homogeneous of degree 0 and the second term vanishes since

\[
\int_{\Omega} t_1^* \tau_{1-n}^* (\cdots \tau_{1-n}^*(\rho)) \log |s| d\Omega = \int_{\Omega} \tau_{1-n}^* (\cdots \tau_{1-n}^*(\rho)) t_1(\log |s|) d\Omega = 0
\]

and \( \log |s| \) does not depend on \( \omega \). This proves that the kernel \( q_0 \) is homogeneous in \( s \) of degree \(-n\). To get a bounded kernel in the right hand side of (17), we integrate by parts one more time and obtain an integral with bounded kernel \( v \log v \).

Integrating by parts again, we get, for an arbitrary natural number \( \kappa \geq n + 1 \),

\[
\max_{i+j \leq \kappa-1} \max_{x \in K} |s|^{n+i} |D_x^i D_s^j q_0(x, s)| \leq C_K, \ s \neq 0
\]

for an arbitrary compact set \( K \subset X \). Similar arguments applied to (16) show that for sufficiently small \( \varepsilon \) and \( |s| < \varepsilon \),

\[
\max_{i+j \leq \kappa-1} \max_{x \in K} |s|^{n+i} |D_x^i D_s^j r_1(x, s)| \leq C_K |s|, \ s \neq 0.
\]

(18)

Therefore, the right hand side of (11) has the structure of equations (33)–(34) below with principal term (13).

To show that the kernel \( \text{Re}(i^n q_0) \) satisfies (32), we invoke the following fact, whose proof is given in [11, Lemma 4.3].

**Proposition 6.2.** Let \( v \in \mathbb{R}^n \) and \( a \in \mathbb{R} \) be such that \( |a| < |v| \). Then for even \( n \geq 2 \),

\[
\text{Re} \int_{\Omega} \frac{i^n d\Omega}{(\langle s, v \rangle - a - i0)^n} = 0,
\]

where \( d\Omega \) is the euclidean volume form on the unit sphere \( \Omega \) in \( \mathbb{R}^n \).
This yields
\[ \text{Re} \int_{\Omega} \frac{i^n d\Omega}{(\langle \nabla \theta(x, \omega), s \rangle - i0)^n} = 0 \]
for all \( x \in X \). Integrating over \( \Omega \) and changing the order of integrals yields
\[ \text{Re} \int_{\Omega} i^n q_0(x, s) d\Omega = 0, \]
which completes the proof of Proposition 6.1.

\[ \square \]

7 Vanishing of the off-diagonal kernel and parametrix

Assume that \( \Omega \) is oriented by the volume form \( d\Omega \). A key point of our construction is the following proposition.

**Proposition 7.1.** The integral (13) vanishes if \( \eta = r \rho \) satisfies
\[ \eta(x, \omega) d\Omega = \frac{1}{(n-1)!} \nabla \theta \wedge (d_\omega \nabla \theta)^{n-1}. \]

**Proof.** Fix \( x \in X \) and consider the hypersurface
\[ H = \text{Im} \{ \nabla \theta(x, \cdot) : \Omega \to E^n \}. \]

Equation (19) can be written in the form \( \eta(x, \omega) d\Omega = \nabla \theta \wedge dh \), where the differential form \( dh = 1/(n-1)! (d_\omega \nabla \theta)^{n-1} \) is the euclidean area form of \( H \) expressed in coordinates \( \omega \). Define \( z : H \to S^{n-1} \) by \( z(h) = h/|h| \). Then
\[ \eta(x, \omega) d\Omega = \nabla \theta \wedge dh = |\nabla \theta|^n dz, \]
where \( dz \) is the area form of \( S^{n-1} \). This yields
\[ \int_{\Omega} \frac{\eta(x, \omega) d\Omega}{(\langle \nabla \theta(x, \omega), s \rangle - i0)^n} = \int_{\Omega} \frac{dz(\omega)}{(\langle z, s \rangle - i0)^n}, \]
where \( x \in X \), \( s \neq 0 \), and
\[ z = z(\omega) = \frac{\nabla \theta(x, \omega)}{|\nabla \theta(x, \omega)|}. \]

Consider the map \( \zeta : \Omega \to S^{n-1}, \omega \mapsto z(\omega) \), and choose an orientation of \( \Omega \). The invariant \( \text{deg} \ \zeta \) is well-defined and does not vanish because of (2). Replacing the variables \( \omega \) by \( z \in S^{n-1} \) on the right hand side of (21), we obtain
\[ Z_n = \text{deg} \ \zeta \int_{\Omega} \frac{dz}{(\langle z, s \rangle - i0)^n}. \]

According to [11, Proposition 4.3], \( \text{Re} i^n Z_n = 0 \) for all \( n \geq 2 \). \[ \square \]
Remark. The map $z$ is proper and, by (2), is locally bijective. Choosing an orientation of $\Omega$ such that $\eta > 0$, we have $\deg \zeta > 0$ and $\deg \zeta = 1$ if $n > 2$.

Corollary 7.2. If $\Phi$ is a generating function as in Theorem 5.1 and $r_\rho = \eta$ is as in (19), then the operator $P_1 = D_{r_\rho}^{-1}Q_\rho$, where $Q_\rho$ is as in (8)--(9) and $D_{r_\rho}$ is as in (10), can be written in the form

$$P_1g(x) = \frac{\pi_n}{\deg \zeta} \int_{\Omega} \int_{\mathbb{R}} g^{(n-1)}(\lambda, \omega) d\lambda d\omega |\nabla \theta(x, \omega)|^n d\Omega(\omega)$$

for even $n$ and in the form

$$P_1g(x) = \frac{\pi_n}{\deg \zeta} \int_{\Omega} g^{(n-1)}(\theta(x, \omega), \omega) |\nabla \theta(x, \omega)|^n d\Omega(\omega)$$

for odd $n$, where $z$ is defined by (22).

Proof. By (10) and (20), we have

$$D_{r_\rho}(x) = \frac{1}{|S^{n-1}|} \int_{\Omega} \frac{\eta(x, \omega) d\Omega}{|\nabla \theta(x, \omega)|^n} = \frac{1}{|S^{n-1}|} \int_{\Omega} dz = \deg \zeta,$$

so (23) and (24) follow from (8) and (9).

In the next section, we show that $P_1$ is a 1-parametrix.

8 Calculation of a remainder

Theorem 8.1. Let $X$ be an open set in euclidean space $E^n$, $\Phi = \theta - \lambda$ a regular generating function in $X \times \mathbb{R} \times \Omega$ of class $C^\kappa$, where $\kappa > n + 4$, and $r_\rho = \eta \in C^\kappa(X \times \Omega)$ as in (19). Then the operator $P_1$ is a 1-parametrix for $M_r$, and the remainder $R_1 = P_1M - \text{Id}$ is a 1-smoothing integral operator with leading term

$$b_1(x, s) = \frac{1}{D_{r_\rho}(x)} \begin{cases} n \text{ Re } q_1(x, s) & \text{for even } n \\ -n \text{ Im } q_1(x, s)/\pi & \text{for odd } n, \end{cases}$$

where

$$q_1(x, s) = \int \frac{\mu(x, s, \omega) \eta(x, \omega) d\Omega}{(|\nabla \theta(x, \omega), s| - i0)^{n+1}}, \quad \mu(x, s, \omega) = \frac{1}{2} \langle \nabla_x^2 \theta(x, \omega), s^2 \rangle.$$

Proof. By Propositions 6.1 and 7.1, for even $n$,

$$P_1M_r f(x) - f(x) = D_{r_\rho}^{-1}(x) \int \text{Re } \Theta(x, y) f(y) dV = D_{r_\rho}^{-1}(x) \int \text{Re } r_1(x, y) f(y) dV,$$
and the kernel $r_1$ defined in (16) satisfies (18); the same conclusion holds for odd $n$. Therefore, the function $D_{1,\rho}^{-1} \text{Re} r_1$ is the kernel of $R_1$ and $q_0$ vanishes.

We can specify the structure of $r_1$. First, applying the Lagrange formula for $\xi$, we have $\tilde{\xi}(x, s, \omega) = \mu(x, s, \omega) + \sigma(x, s, \omega)$, where the remainder $\sigma$ satisfies an inequality like (14), with the power $|s|^3$ instead of $|s|^2$. Set

\[
p = \frac{\mu + \sigma}{(\nabla \theta(x, \omega), s) - i0}
\]

in the right hand side of (15) to obtain $r_1(x, y) = -nq_1(x) + r_2(x, s)$, where $q_1$ is as in (27) and the remainder $r_2$ admits an estimation like (18) with the factor $|s|^2$ instead of $|s|$ on the right hand side. The kernel $q_1$ is homogeneous of degree $1 - n$, which implies (26). By Proposition 10.4, $r_1$ is a 1-smoothing operator. □

Remarks. 1. Beylkin [1] has constructed a parametrix for $M_r$ in terms of Fourier integral operators. His construction depends on the assumption that $\tilde{\theta}(x, -\omega) = -\tilde{\theta}(x, \omega)$, which is not satisfied in the case of photo-acoustic acquisition geometry.

2. Higher parametrices $P_k$ and remainders $R_k$, $k = 2, 3, 4, \ldots$, can be calculated as in Section 2 by means of the Lagrange formula like (15) with more terms.

Proposition 8.2. Suppose that $\tilde{\theta}(x, \xi) \equiv |\xi|\theta(x, \xi/|\xi|)$ is a linear function of $\xi \in \mathbb{R}^n$ and that the function $\theta(y, \omega) - \theta(x, \omega)$ has at least one root $\omega \in \Omega$ for each $x \neq y \in X$ and $r = 1$. Then the parametrix $P_1$ coincides with the left inverse operator constructed in [11], namely,

\[
P_1 g(x) = \frac{\pi_n}{D_1(x)} \int_{\Omega} \int_{\mathbb{R}} g^{(n-1)}(\lambda, \omega) \frac{d\lambda d\omega}{\theta(x, \omega) - \lambda}
\]

for even $n$, and

\[
P_1 g(x) = \frac{\pi_n}{D_1(x)} \int_{\Omega} \frac{g^{(n-1)}(\lambda, \omega)}{|\lambda - \theta(x, \omega)|} d\omega
\]

for odd $n$.

Proof. We have

\[(n - 1)!\eta d\Omega \equiv \nabla \theta \wedge (d_\omega \nabla \theta)^{\wedge n-1} = |\xi|^{-n}(d_\xi \nabla \tilde{\theta})^{\wedge n}/d|\xi|,
\]

where $\omega = \xi/|\xi|$. The quotient $\rho \equiv (d_\xi \nabla \tilde{\theta})^{\wedge n}/d_\xi \tilde{\theta}$ does not depend on $\xi$ since $\tilde{\theta}$ is a linear function of $\xi$. Therefore, the right hand side equals

\[
\rho(d_\xi \tilde{\theta})^{\wedge n}/d|\xi| = \rho d\Omega,
\]

that is, $\eta = \rho$; hence $\eta$ does not depend on $\omega$. Thus, the factor $\rho$ in (8) cancels the same factor in the (10) for $D_{1,\rho}$. By [11, Theorem 3.1], it follows that (8) and (9)
with $\rho = 1$ and the factor $D_{\Gamma_1,\rho}^{-1}$ define a left inverse operator $L$ to $M_1$. Therefore, the parametrix $P_1$ coincides with $L$. □

9 Photo-acoustic acquisition geometries

Let $\Gamma$ be a hypersurface in euclidean space $E^n$, and let

$$ R_\Gamma f(r, \xi) = \int_{|x-\xi|=r} f dS, \quad \xi \in \Gamma, \quad r > 0, $$

be the corresponding spherical integral transform of a function $f$ on $E^n$. The problem of inverting this transform has been studied for at least a decade in view of applications to photo-acoustic tomography. A special construction of a parametrix was proposed by Popov and Sushko [13] based on a reduction to the Radon transform. For an arbitrary spherical central surface $\Gamma$, Finch et. al. [4], [3] found exact reconstruction formulas. Other reconstruction formulas were given by Kunyansky [8] and Xu and Wang [15]. In [9], explicit reconstruction formulas were found for an arbitrary ellipsoid $\Gamma \subset E^3$, and in [11], a reconstruction is given for ellipsoids $\Gamma$ in a space of arbitrary dimension. Natterer [9] showed that the same formula holds for an arbitrary convex surface $\Gamma$ in $E^3$ up to an explicitly calculated remainder, and Haltmeier [5] did the same for $E^2$. It can be checked that the remainder does not vanish in the general case (in contrast with [7]). We prove below that the formulas in (29)-(30) below provide a 1-parametrix in arbitrary dimension. In three dimensional space, this coincides with the main term of Natterer’s formula up to a 1-smoothing operator, and the remainders in Natterer’s and Haltmeier’s formulas are also 1-smoothing operators. An exact reconstruction formula for the operator $R_\Gamma$ is still known only for a narrow class of algebraic curves and surfaces [11].

**Proposition 9.1.** Let $X$ be a compact convex domain in $E^n$ with a boundary $\Gamma$ parametrized by a smooth map of rank $n - 1 \, x = \xi(\omega), \, \omega \in \Omega$, where $\Omega = S^{n-1}$ is the unit sphere in euclidean space $E^n$. Then the generating function $\Phi(x; \lambda, \omega) = |x - \xi(\omega)| - \lambda$ defined in $X \times \mathbb{R}_+ \times \Omega$ is regular.

**Proof.** We have

$$ \eta d\Omega = \frac{1}{(n-1)!} \left( \frac{-1}{|x-\xi|} \right)^n (x-\xi) \wedge (d\xi)^{(n-1)} \neq 0, $$

since the map $\xi$ has rank $n - 1$. This proves (2).

To prove (5), we consider the function

$$ \theta(y, \omega) - \theta(x, \omega) = |y - \xi(\omega)| - |x - \xi(\omega)| $$
and show that for \( x, y \in X, x \neq y \), each zero \( \omega \) of this function is simple. Indeed, were \( \omega \) a nonsimple zero, then we would have

\[
0 = \theta(y, \omega) - \theta(x, \omega) = d_\omega(\theta(y, \omega) - \theta(x, \omega)) = \frac{\langle x - y, d\xi \rangle}{|x - \xi(\omega)|^2}
\]

The second equation implies that the vector \( x - y \neq 0 \) is orthogonal to the tangent hyperplane \( T \) of \( \Gamma \) at \( \xi(\omega) \). By the first equation, \( x \) and \( y \) are the same distance to \( \xi(\omega) \) and are therefore symmetric with respect to \( T \). This is impossible, since \( X \) is convex and hence lies on one side of \( T \). \( \square \)

**Corollary 9.2.** For even \( n \),

\[
f(x) - S_1f(x) = \pi_n \int_\Omega \int \mathbb{R}_\Gamma f^{(n-1)}(\lambda, \omega) d\lambda \frac{dz}{|x - \xi| - \lambda},
\]

and for odd \( n \),

\[
f(x) - S_1f(x) = \pi_n \int_\Omega \mathbb{R}_\Gamma f^{(n-1)}(|x - \xi|, \xi) dz,
\]

where

\[
z = \frac{\xi - x}{|\xi - x|}, \quad dz = \frac{\cos \psi}{|x - \xi|^{n-1}} d\xi
\]

and \( S_1 \) is a 1-smoothing operator in \( X \).

**Proof.** We have \( |\nabla \theta| = 1 \), \( \nabla \theta(x, \omega) = z \); hence \( Rf = M_1f \), and we can apply Corollary 7.2. By (25), we have \( D_{1,\rho} = \deg z \) and \( \deg z = 1 \) for all \( x \in X \) since the map \( z : \Omega \to S^{n-1} \) is bijective. This yields \( D_{1,\rho}(x) = 1 \). \( \square \)

For the case \( n = 3 \), we obtain

\[
f(x) = \pi_3 \int_\Gamma \mathbb{R}_\Gamma f^{(2)}(|x - \xi|, \xi) \frac{\cos \psi}{|x - \xi|^2} d\xi + S_1f(x),
\]

where \( \mathbb{R}_\Gamma f^{(2)} = (\partial/\partial r)^2 \mathbb{R}_\Gamma f \). The first term coincides with that of the reconstruction [9], which is exact up to a 1-smoothing operator \( S_1 \).

### 10 Singular integral operators

Let \( E^n \) be euclidean space of dimension \( n \geq 1 \), and let \( a(x, s) \) be a locally bounded function on \( E^n \times (E^n \setminus \{0\}) \). Consider the integral transform \( A \) defined by

\[
Af(x) = \lim_{\varepsilon \to 0} \int_{|s| > \varepsilon} a(x, s)f(x + s) ds
\]

for functions \( f \in L_2(E^n)_{\text{comp}} \). Let \( \Omega \) be the unit sphere in \( E^n \).
Theorem 10.1. Let $a : E^n \times (E^n \setminus \{0\}) \to \mathbb{R}$ be a positively homogeneous function of degree $-n$ in the variable $s$ for each $x$, locally bounded on $E^n \times \Omega$ satisfying

$$\int_{\Omega} a(x, s) d\Omega(s) = 0, \ x \in E^n,$$

where $d\Omega$ is the euclidean volume form on $\Omega$. Then (31) defines a continuous operator $A : L^{2}_{\text{comp}} \to L^{2}_{\text{loc}}$.

This is a simplified version of the Calderón-Zygmund Theorem [2].

Lemma 10.2. Let $k \in \mathbb{Z}$ and $A_k : H^k \to H^{k-d}$ and $A_{k+1} : H^{k+1} \to H^{k+1-d}$ be bounded linear operators such that $A_{k+1}$ is the restriction of $A_k$. Then for $k < \alpha < k + 1$, the restriction of $A_k$ to $H^\alpha$ defines a bounded operator $A_\alpha : H^\alpha \to H^{\alpha-d}$ such that $\|A_\alpha\| \leq C \|A_k\|^{\alpha-k} \|A_{k+1}\|^{k+1-\alpha}$, where $C$ depends only on $X$.

Proof. Since $H^{k+1}$ is dense in $H^k$, the restrictions $A_{k+1}$ and $A_\alpha$ are uniquely defined. The lemma follows from the fact that any Sobolev space $H^\alpha$ is a complex interpolation of spaces $H^\beta$ and $H^{\beta+1}$, where $\beta < \alpha < \beta + 1$ and $\varepsilon = \alpha - \beta$ is the exponent of interpolation; see [14].

Let $D^i = (\partial/\partial x_1)^{i_1} \cdots (\partial/\partial x_n)^{i_n}$ and $D^i = \cdots$, where $i = (i_1, \ldots, i_n)$ is a multi-index.

Theorem 10.3. Let $\kappa$ be a natural number and

$$a(x, s) = a_0(x, s) + r_1(x, s)$$

be a kernel supported in $X \times (E^n \setminus \{0\})$ of class $C^\kappa$, where for each $x$, $a_0(x, s)$ is a homogeneous function of $s$ of degree $-n$ satisfying (32) and $r_1$ satisfies

$$\max_{i+j \leq \kappa} \max_{x \in X} |s|^{i+n} |D^i D^j r_1(x, s)| \leq C |s|,$$

for some constant $C$. Then for each compact set $X \subset E^n$ with boundary of class $C^\kappa$, $A$ defines a bounded operator $L^2(X) \to L^2(X)$ of Sobolev order 0.

This operator is called a singular integral operator of class $C^\kappa$ with principal term $a_0$.

Proof. We abbreviate $L^2 = L^2(X)$ and $H^\alpha = H^\alpha(X)$ for $\alpha \in \mathbb{R}$. Assume that $f \in H^\alpha$ for some natural number $\alpha \leq \kappa$ and apply a partial derivative $D^j$, $|j| \leq \alpha$, to $Af$ to obtain

$$D^j Af(x) = \int D^j a(x, s) f(x + s) ds + \int a(x, s) D^j f(x + s) ds.$$
The kernel $D^j_xa$ is a singular integral operator of class $C^{\kappa-a}$, and the first term is contained in $L_2(X)$ by the Calderón-Zygmund Theorem. The same is true for the second term since $D^jAf \in L_2$. This yields $D^jAf \in L_2$; hence $Af \in H^\alpha$.

If $\alpha \geq -\kappa$ is a negative integer, we use decreasing induction in $\alpha$. An arbitrary function $f \in H^\alpha$ can be written in the form $f = D^jg_j$, where $g_j \in L_2$ and summation in $j$, $|j| = -\alpha$ is assumed. Reading (35) from right to left yields

$$Af(x) = \int a(x, s)D^jg_j(x + s)ds = D^jAg_j(x) - \int D^j_xa(x, s)g_j(x + s)ds.$$  

The first term belongs to $H^\alpha$ since $Ag_j \in L_2$. The second term belongs to $L_2(X) \subset H^\alpha$ by the Calderón-Zygmund theorem, which implies $Af \in H^\alpha$.

If $\alpha$ is not an integer, we apply Lemma 10.2 to $A$.

**Proposition 10.4.** If $a_1$ is a function in $X \times (E^n \setminus \{0\})$ of class $C^\kappa$ which is homogeneous in $s$ of degree $l - n$ for some natural number $l < \kappa$, and $r_{l+1}$ satisfies (34) with right hand side $C|s|^{\alpha+1}$, then the integral transform $A$ with kernel $a = a_1 + r_{l+1}$ is an $l$-smoothing operator of class $C^{\kappa-1}$.

**Proof.** First claim that the function $a_0 \doteq D^j_xa_1$ satisfies (32) for every multiindex $j$, $|j| = l$. Indeed, consider the differential form $\nu = a_0d\Omega = D^j_xa_1d\Omega$ of degree $n - 1$ in $E^n \setminus \{0\}$ ($x$ is fixed). This form is closed since $a_0$ is homogeneous of degree $-n$. We can write $D^j_x = (\partial/\partial s_k)D^j_x$ for some $k$ and $i$, and have $a_0 = \partial b/\partial s_k$, $b = D^j_xa_1$. Suppose that $k > 1$, and consider the hyperplanes $H_{\pm,0} = \{s_1 = \pm1, 0\}$. The central projection $\pi(s) = s/|s|$ in $E^n \setminus \{0\}$ maps the set $H_+ \cup H_- \to \Omega \setminus H_0$. By Stokes’ Theorem, $\int_{\Omega \setminus H_0} \nu = \int_{H_+ \cup H_-} \nu$, since the form $\nu$ is closed and decreases sufficiently fast at $\infty$. The left hand side equals $\int_{\Omega} \nu$, while the right hand side vanishes since $\nu = \pm d(bds_2 \wedge \cdots \wedge \hat{d}s_k \wedge \cdots \wedge ds_n)$ in $H_{\pm}$. This establishes the claim.

Now, for simplicity, suppose that $l = 1$. Substituting $D^j_xf(x + s) = D^j_xf(x + s)$ for $|j| = 1$ in the second term of (35) and integrating by parts, we arrive at

$$D^jAf(x) = \int D^j_xa(x, s)f(x + s)ds - \int D^j_xa(x, s)f(x + s)ds.$$  

The kernel $D^j_xa = D^j_xa_1 + D^j_xr_2$ satisfies (34) and hence by the Calderón-Zygmund Theorem, defines an operator of order 0. The kernel $D^j_xa = D^j_xa_1 + D^j_xr_2$ is of the form of (33), where $a_0 \doteq D^j_xa_1$ is homogeneous of degree $-n$ in $s$, and belongs to class $C^{\kappa-1}$. The function $r_1 = D^j_xr_2$ satisfies (34) with $\kappa - 1$ instead of $\kappa$. From the claim and the Calderón-Zygmund Theorem, we conclude that the second term of (36) also defines an operator of order 0. The same is true for operators $D^j_xA$, $|j| = 1$, which implies that $A$ is a 1-smoothing operator.
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