A PARAMETRIX METHOD IN INTEGRAL GEOMETRY

By

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Abstract. The objective of reconstructive integral geometry is to recover a function from its integrals over a set of subvarieties. A parametrix is a method of reconstruction of a function from its integral data up to a smoothing operator. In the simplest case, a parametrix recovers a function with a jump singularity along a curve (surface) up to a continuous function, which can be quite informative in medical imaging. We provide an explicit construction for a wide class of acquisition geometries. The case of photo-acoustic geometry is of special interest.

1 Introduction

Let (X, g) be a Riemannian manifold and Σ be a family of smooth submanifolds $\sigma \subset X$. For a function f defined in X with compact support, the family of integrals

$$g(\sigma) = \int_\sigma f \,\mathrm{d}_{\mathrm{g}} S, \quad \sigma \in \Sigma \tag{1}$$

defines function on Σ. The family Σ is called the acquisition geometry of the integral transform $\mathrm{R}_\Sigma f \doteq g$. An analytic inversion formula $g \mapsto f$ is known only for special types of acquisition geometries Σ; see the survey in [11]. Here, we construct a parametrix for a class of weighted integral transforms R_Σ for which analytic reconstruction is not known (Sections 5–8).

A parametrix recovers not only the wave front of a function f but also the profile of its singularity. A parametrix for a class of integral transforms was constructed earlier by Beylkin [1] in terms of Fourier integral operators. Pestov and Uhlmann [12] gave a construction of a parametrix for the geodesic integral transform on two-dimensional simple Riemannian manifolds.

In Section 9, we apply our construction for photo-acoustic (thermo-acoustic) acquisition geometry. This topic was studied in papers of Popov and Sushko [13], Kunyansky [8], Xu-Wang [15], Natterer [9], and in [11]. Our method is based on the Calderón-Zygmund theory of singular integral operators adapted in Section 10.

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2 Parametrices in Sobolev spaces

Let X and Y be compact manifolds with boundaries of class C^κ, where κ is a natural number. The Sobolev spaces $H^\alpha(X)$ and $H^\alpha(Y)$ are well-defined for every real α, $|\alpha| < \kappa$; see, e.g., [14]. We say that a densely defined operator $A : L_2(X) \to L_2(Y)$ has Sobolev order $d \in \mathbb{R}$ if it generates a bounded operator $A_\alpha : H^\alpha(X) \to H^{\alpha-d}(Y)$ for every α, $|\alpha| < \kappa$, $|\alpha - d| < \kappa$, which is the restriction of A for positive α and a closure of A for negative α. If d is negative, A is called a **d-smoothing operator**. An operator $P : L_2(Y) \to L_2(X)$ is said to be an **s-parametrix for** A if $0 < s \le \kappa$ and $PA = \mathrm{Id}+R$, where the remainder R is a s-smoothing operator. If P_1 is a 1-parametrix and R_1 is a remainder, a k-parametrix P_k can be found for any natural number k recursively by $P_k = P_{k-1} - R_{k-1}P_1$, $R_k = -R_{k-1}R_1$ for $k = 2, \ldots, \kappa$. Every 1-smoothing operator is compact in $L_2(X)$; hence P_1A is a Fredholm operator, and the image of A is closed. An s-parametrix P_s recovers the singularity of an arbitrary function $f \in H^\alpha(X)$ from Af up to a function $h = R_sf \in H^{\alpha+s}(X)$. In particular, if $f = \delta_y$ is the delta-function at a point $y \in X$ and $s > n$, then $h = R_s\delta_y$ is continuous. In fact, $\delta_y \in H^\alpha(X)$ for every $\alpha < -n/2$, which implies $h \in H^{\alpha+s}(X)$. The space $H^{\alpha+s}(X)$ is contained in $C(X)$ if $\alpha > n/2 - s$; hence h is continuous. The equation $P_sA\delta_y = \delta_y + h$ shows that every delta function can be recognized from data of $A\delta_y$ by means of an s-parametrix P_s.

3 Generating functions and integrals

Let X and Σ be smooth n-dimensional manifolds, and $\Phi : X \times \Sigma \to \mathbb{R}$ a C^2-smooth real function such that $\mathrm{d}\Phi \neq 0$ on the set $Z = \Phi^{-1}(0)$. Let $p : Z \to X$, $\pi : Z \to \Sigma$ be the natural projections. Suppose that

$$\det(\mathrm{d}_{x,t}\mathrm{d}_{\sigma,\tau}(t\tau\Phi(x,\sigma))) \neq 0, \tag{2}$$

where $\mathrm{d}_{x,t}$, $\mathrm{d}_{\sigma,\tau}$ are exterior differentials in the manifolds $X \times \mathbb{R}$ and $\Sigma \times \mathbb{R}$, respectively.

Proposition 3.1. *Property* (2) *holds if and only if π has rank n and $p^* : N^*(Z) \to T^*(X)$ is a local diffeomorphism, where $N^*(Z)$ denotes the conormal bundle of Z and $p^*(x, \sigma; \xi, s) = (x, \xi) \in T^*(X)$.*

For a proof, see [10, Proposition 1.1].

It follows that for each $\sigma \in \Sigma$, the set $Z(\sigma) = \pi^{-1}(\sigma) = \{x : \Phi(x,\sigma) = 0\}$ is a C^1-hypersurface in X; and for every point $x \in X$ and tangent hyperplane $h \subset T_x(X)$, there exists a locally unique hypersurface $Z(\sigma)$ through x tangent to h.

The function Φ is called **generating** for the acquisition geometry $\{Z(\sigma) : \sigma \in \Sigma\}$. Let $\mathrm{d}V$ be a volume form on X and $\rho = \rho(x, \sigma)$ a continuous function on Z. We define a weighted integral transform of the continuous function f with compact support in X by

(3) $$M_r f(\sigma) = \int_X \delta(\Phi(x, \sigma)) r(x, \sigma) f(x) \mathrm{d}V.$$

The limit exists since $\mathrm{d}_x\Phi \neq 0$. We can write this integral in the form

(4) $$M_r f(\sigma) = \int_{Z(\sigma)} f(x) r(x, \sigma) q(x, \sigma),$$

where $q = \mathrm{d}V/\mathrm{d}\theta$ denotes an arbitrary $(n-1)$-differential form q such that $\mathrm{d}\Phi \wedge q = \mathrm{d}V$. An orientation is defined in a hypersurface $Z(\sigma)$ by means of the form $\mathrm{d}_x\Phi$, and the integral over $Z(\sigma)$ is well-defined.

Choose a volume form $\mathrm{d}\Sigma$ on Σ and interchange the roles of X and Σ, keeping the same generating function Φ. The corresponding integral transform M_r^* is called the **back projection** operator. Note that condition (2) is symmetric, and Proposition 3.2 holds also for the operator M_r^*.

For a closed set $K \subset X$ and a real α, we denote by $H_K^\alpha(X)$ the subspace of $H^\alpha(X)$ consisting of distributions with support in K. The subspace $H_L^\alpha(\Sigma)$ of $H^\alpha(\Sigma)$ is defined similarly.

Proposition 3.2. *If Φ is a smooth generating function satisfying* (2) *and r is a smooth function, then for any compact set $K \subset X$ with smooth boundary, any real α, and any smooth function ϕ with compact support in Σ,*

$$\|\phi M_r f\|^{\alpha+(n-1)/2} \leq C \|f\|^\alpha, \quad f \in H_K^\alpha(X),$$

where C is a constant which does not depend on f. If the map p is proper, then the operator $M_r : H_K^0(X) \to H_L^0(\Sigma)$ is densely defined, where $L = \pi p^{-1}(K)$, and has Sobolev order $(1-n)/2$.

Proof. We can write the transform as a Fourier integral operator:

$$M_r f(\sigma) = \int_K \int_{\mathbb{R}} \exp(2\pi\imath\tau\Phi(x, \sigma)) r(x, \sigma) f(x) \mathrm{d}\tau \mathrm{d}V.$$

The critical set of the phase function $\tau\Phi(x, \sigma)$ is the hypersurface $F(\sigma)$, and the condition $\mathrm{d}_x\Phi \neq 0$ implies that the phase function is non-degenerate. The corresponding conic Lagrange variety is

$$\mathrm{L} = \{(x, \sigma; \xi, s) \in T^*(X \times \Sigma) : \Phi(x, \sigma) = 0, \; s = \lambda \mathrm{d}_\sigma\Phi, \; \xi = \lambda \mathrm{d}_x\Phi, \; \lambda \neq 0\}.$$

The rank of the matrix $\partial(x, \xi)/\partial(\sigma, s)$ equals $2n$ at every point of L. This follows from (2); for details, see [10, Lemma 3.2]. Therefore, projections of L to $T^*(X)$ and to $T^*(\Sigma)$ are submersions, that is, L is locally the graph of a canonical transformation. The symbol $a(x, \sigma; \xi, s) = 1$ is a homogeneous function of ξ, s of order 0. The order m of the Fourier integral operator M_r satisfies

$$m + \dim X \times \Sigma/4 - N/2 = 0,$$

where $\dim X \times \Sigma = 2n$ and $N = 1$ is the number of variables τ. This yields $m = (1 - n)/2$, which means that the functional

$$\psi \mapsto \int_\Sigma \int_X \int_{\mathbb{R}} \exp(2\pi\imath\tau\Phi(x, \sigma)) r(x, \sigma) \psi(x, \sigma) \mathrm{d}\tau \mathrm{d}V \, \mathrm{d}\Sigma$$

defined for smooth test densities ψ, is a distribution of the class $I^{(1-n)/2}(X \times \Sigma, \mathrm{L})$ in the sense of Hörmander. By [6, Corollary 25.3.2], the operator ϕM_r defines a continuous map $H^\alpha_K(X) \to H^{\alpha+(n-1)/2}_F(\Sigma)$ for every real α, where $F = \operatorname{supp} \phi$.

If p is proper, the set $L = \pi(p^{-1}(K))$ is compact, and we can choose a cut-off function such that $\phi = 1$ in L. Then $\phi M_r f = M_r f$, and the second statement follows. □

We say that a generating function Φ is **resolved** if $\Sigma = \mathbb{R} \times \Omega$, where Ω is the unit sphere in euclidean space E^n and $\Phi(x, \sigma) = \theta(x, \omega) - \lambda$, $\sigma = (\lambda, \omega)$, $\lambda \in \mathbb{R}$, $\omega \in \Omega$ for a function $\theta \in C^2(X \times \Omega)$. If Φ is resolved, the map $p : Z \to X$ is proper since θ is continuous. It follows that $M_r f$ has compact support in Σ if f does.

Definition. We call a generating function Φ **regular** if it is resolved, satisfies (2), and the equations

(5) $$\theta(x, \omega) = \theta(y, \omega), \quad \mathrm{d}_\omega\theta(x, \omega) = \mathrm{d}_\omega\theta(y, \omega)$$

are satisfied simultaneously for no $x \neq y \in X$, $\omega \in \Omega$ (that is to say, the family $\{Z(\sigma) : \sigma \in \Sigma\}$ has no conjugate points.)

4 Principal value integrals

Let f be a smooth real function on a manifold X having only simple zeros, i.e., $\mathrm{d}f(x) \neq 0$ whenever $f(x) = 0$. For a natural number n, we consider the functional

(6) $$I_n(a) = \int_X \frac{a}{(f - i0)^n} = \lim_{\varepsilon \searrow 0} \int_X \frac{a}{(f - i\varepsilon)^n},$$

defined for test densities a in X. For a real density a, the functional

$$\int_X \frac{a}{f^n} \doteq \operatorname{Re} I_n(a)$$

is called a **principal value integral**.

Proposition 4.1. *For every smooth function f having only simple zeros, the limit in* (6) *exists for every test density a. The functional I_n is a generalized function in X.*

Proof. For an arbitrary smooth function g, tangent field t, and test density a in X,

$$\mathrm{d}(g \wedge (t \llcorner a)) = \mathrm{d}g \wedge (t \llcorner a) + g\, \mathrm{d}(t \llcorner a),$$

where the symbol $\llcorner$ denotes the inner product of a field and a form. If a has compact support, the integral of the left hand side over X vanishes, and

$$\int t(g)\, a = \int (t \llcorner \mathrm{d}g) \wedge a = \int \mathrm{d}g \wedge (t \llcorner a) = -\int g\, \mathrm{d}(t \llcorner a). \tag{7}$$

To prove the statement, choose a tangent field t_1 and a smooth function t_0 in X such that $t_1(f) + t_0 f = 1$, and apply induction in $n \geq 1$. In the case $n = 1$, we integrate by parts in (6) and apply (7) to obtain

$$\begin{aligned} I_1(a) &= \int \frac{(t_1(f) + t_0 f)a}{f - i0} = \int t_1(\log(f - i0))a + \int t_0 a \\ &= -\int \log(f - i0)\mathrm{d}(t_1 \llcorner a) + \int t_0 a, \end{aligned}$$

where $\log(f - i0)$ is a locally integrable function.

For the case $n > 1$, we can write

$$\begin{aligned} I_n(a) &= \int \frac{(t_1(f) + t_0 f)a}{(f - i0)^n} = \frac{1}{1-n} \int t_1[\frac{1}{(f - i0)^{n-1}}]a + I_{n-1}(t_0 a) \\ &= \frac{1}{n-1} \int \frac{\mathrm{d}(t_1 \llcorner a)}{(f - i0)^{n-1}} + I_{n-1}(t_0 a) = I_{n-1}(\frac{\mathrm{d}(t_1 \llcorner a)}{n-1} + t_0 a). \end{aligned}$$

Here, the form $\mathrm{d}(t_1 \llcorner a)$ is again a smooth density with compact support in X. □

5 Filtered back projection operator

Theorem 5.1. *Let X be an open set in euclidean space E^n, $\Phi = \theta - \lambda$ a smooth regular generating function on $X \times \mathbb{R} \times \Omega$ of class C^κ, and $\rho \in C^\kappa(X \times \Omega)$, where $\kappa > n + 1$. Define an operator by the principal value integral*

$$Q_\rho g(x) \doteq \pi_n (n-1)! \int_\Sigma \frac{\rho(x,\omega) g(\lambda,\omega) \mathrm{d}\lambda\, \mathrm{d}\Omega}{(\theta(x,\omega) - \lambda)^n} \tag{8}$$

for even n and by

$$Q_\rho g(x) \doteq \pi_n \int_\Omega \rho(x,\omega) g^{(n-1)}(\theta(x,\omega),\omega)\mathrm{d}\Omega \tag{9}$$

for odd $n \geq 3$, where g is a function defined on $\mathbb{R} \times \Omega$, $g^{(n-1)} = (\partial/\partial\lambda)^{n-1} g$, and $\pi_n = -(2\pi i)^{-n}$ for even n, $\pi_n = 2(2\pi i)^{1-n}$ for odd n. Then

$$Q_\rho M_r = D_{r,\rho}\mathrm{Id} + A_{r,\rho},$$

where

$$D_{r,\rho}(x) = \frac{1}{|\mathrm{S}^{n-1}|}\int_\Omega \frac{\rho(x,\omega) r(x,\omega)\mathrm{d}\Omega}{|\nabla_x \theta(x,\omega)|^n}. \tag{10}$$

A_ρ is a singular integral operator of Sobolev order 0 with the kernel $-(n-1)!\,\mathrm{Re}\,\Theta$ if n even and $1/2\pi(n-1)!\,\mathrm{Im}\,\Theta$ if n odd. Here, the singular integral

$$\Theta(x,y) = \int_\Omega \frac{\rho(x,\omega) r(y,\omega)\mathrm{d}\Omega}{(\theta(y,\omega)-\theta(x,\omega)-i0)^n} \tag{11}$$

is defined for $x \neq y \in X$ by the method of Section 4.

Lemma 5.2. *The composition $Q_\rho M_r$ extends to a continuous operator $L_2(X)_{\mathrm{comp}} \to L_2(X)_{\mathrm{loc}}$.*

Proof. For even n, integrating by parts in (8) yields

$$Q_\rho g(x) = \pi_n (n-1)! \int_\Omega \rho(x,\omega) \int_\mathbb{R} (\theta(x,\omega)-\lambda)^{-n} g(\lambda,\omega)\mathrm{d}\lambda \mathrm{d}\omega,$$

that is,

$$Q_\rho = \pi_n (n-1)! M_\rho^* (\Lambda_n \times \mathrm{Id}),$$

where Λ_n is the convolution operator in $\mathbb{R}$ with the principal value kernel λ^{-n} acting on the λ variable and

$$M_\rho^* g(x) = \int_\Omega \rho(x,\omega) g(\theta(x,\omega),\omega)\mathrm{d}\omega$$

is a weighted back projection operator. By Proposition 3.2, the operator M_r is bounded in the spaces $H_K^0(X) \to H_L^{(n-1)/2}(\Sigma)$, where $L = \pi(p^{-1}(K))$ is a compact in Σ since p is proper. The convolution operator has a factorization $\Lambda_n = C_n(\partial/\partial\lambda)^{n-1}\mathrm{H}$, where H is a Hilbert operator and C_n is a constant. It follows that for arbitrary $\alpha \in \mathbb{R}$, Λ_n defines a bounded map $H^\alpha(\mathbb{R}) \to H^{\alpha-n+1}(\mathbb{R})$. Taking $\alpha = n-1$, we conclude that $\Lambda_n \times \mathrm{Id} : H^{(n-1)/2}_{\mathbb{R}\times\Omega}(\Sigma) \to H^{-(n-1)/2}_{\mathbb{R}\times\Omega}(\Sigma)$ is also bounded. By [10, Proposition 3.1], M_ρ^* is continuous as an operator $H^{-(n-1)/2}_{\mathbb{R}\times\Omega}(\Sigma) \to H^0_{\mathrm{loc}}(X)$.

Finally, $M_\rho^*(\Lambda_n \times \mathrm{Id})M_r$ is continuous as an operator $L_2(X)_{\mathrm{comp}} \to L_2(X)_{\mathrm{loc}}$, and the statement follows.

In the case of odd n, there exists a similar factorization with $\Lambda_n = C_n(\partial/\partial\lambda)^{n-1}$, which leads to the same conclusion. □

Lemma 5.3. *For even n, arbitrary $x \in X$, $\omega \in \Omega$, and small ε,*

$$\begin{aligned} d_{n,\varepsilon}(x,\omega) &\doteq -\frac{(n-1)!}{(2\pi i)^{n-1}} \operatorname{Re} \int_X \frac{\rho(x,\omega)r(y,\omega)e_\varepsilon(s)\mathrm{d}V(y)}{(\theta(y,\omega)-\theta(x,\omega)-i0)^n} \\ &= \frac{1}{|\mathrm{S}^{n-1}|}\frac{\rho(x,\omega)r(x,\omega)}{|\nabla\theta(x,\omega)|^n} + o(1), \end{aligned} \tag{12}$$

where $o(1) \to 0$ as $\varepsilon \to 0$. For odd n,

$$\begin{aligned} d_{n,\varepsilon}(x,\omega) &\doteq \frac{(n-1)!}{(2\pi i)^{n-1}} \operatorname{Im} \int_X \frac{\rho(x,\omega)r(y,\omega)e_\varepsilon(s)\mathrm{d}V}{(\theta(y,\omega)-\theta(x,\omega)-i0)^n} \\ &= \frac{2\rho(x,\omega)r(x,\omega)}{|\mathrm{S}^{n-1}||\nabla\theta(x,\omega)|^n} + O(\varepsilon), \end{aligned}$$

where $s = y - x$.

Proof. The proof follows along the lines of [11, Lemma 3.3]. □

Proof of Theorem 5.1. We have

$$d_n(x) = \lim_{\varepsilon\to 0}\int_\Omega d_{n,\varepsilon}(x,\omega)\,\mathrm{d}\Omega.$$

Taking the limit and integrating (12) over Ω we obtain, for even n,

$$d_n(x) = \frac{1}{\left|\mathrm{S}^{n-1}\right|}\int_\Omega \frac{\rho(x,\omega)\,r(x,\omega)\,\mathrm{d}\Omega}{|\nabla\theta(x,\omega)|^n} = D_{r,\rho}(x),$$

which implies (10). For odd n, we again obtain $d_n = D_{r,\rho}$, and (10) follows.

Fixing a point $x \in X$ and setting $f_\varepsilon(y) = e_\varepsilon(y-x)f(y)$ for a C^n-function f in X gives

$$Q_\rho M_r(f_\varepsilon)(x) = c_n \int_\Sigma \frac{\rho M_r(f_\varepsilon)\mathrm{d}\lambda\mathrm{d}\Omega}{(\theta-\lambda)^n} = c_n \operatorname{Re}\int_X \frac{\rho(x,\omega)r(y,\omega)e_\varepsilon(y-x)f(y)\mathrm{d}V(y)}{(\theta(y,\omega)-\theta(x,\omega)-i0)^n},$$

where $c_n = \pi_n(n-1)!$. By Lemma 5.3, the right hand side tends to $D_{r,\rho}(x)f(x)$ as $\varepsilon \to 0$. The operator $f \mapsto D_{r,\rho}f$ acting in $L_2(X)_{\mathrm{comp}}$ is obviously bounded; and by Lemma 5.2, the residue $A_{r,\rho} = Q_\rho M_r - D_{r,\rho}\mathrm{Id}$ is the off-diagonal kernel of $Q_\rho M_r$ and is a bounded operator $L_2(X)_{\mathrm{comp}}(X) \to L_2(X)_{\mathrm{loc}}$. Take an arbitrary function

$f \in L_2(X)_{\mathrm{comp}}$ that vanishes in a neighborhood of x and calculate

$$\begin{aligned} Q_\rho M_r f(x) &= c_n \int_\Sigma \frac{M_r f(\lambda,\omega)\mathrm{d}\lambda\,\rho(x,\omega)\mathrm{d}\Omega}{(\theta(x,\omega)-\lambda)^n} \\ &= c_n \int_\Omega \rho(x,\omega)\mathrm{d}\Omega \int_{\mathbb{R}} \frac{\mathrm{d}\lambda}{(\theta(x,\omega)-\lambda)^n} \int_{Z(\lambda,\omega)} r(y,\omega) f(y) q \\ &= c_n \int_X \left(\int_\Omega \frac{\rho(x,\omega) r(y,\omega)\mathrm{d}\Omega}{(\theta(x,\omega)-\theta(y,\omega))^n} \right) f(y)\mathrm{d}\theta \wedge q \\ &= c_n \int_X \operatorname{Re}\Theta(x,y) f(y)\mathrm{d}V, \end{aligned}$$

where we have applied (4) and Θ is as in (11). The relation $\mathrm{d}\lambda = \mathrm{d}\theta$ holds in Z, and the equation $\mathrm{d}\theta \wedge q = \mathrm{d}V$ is satisfied in X, by definition. Thus the function $-c_n \operatorname{Re}\Theta$ is the off-diagonal kernel of the operator $Q_\rho M_r$. A similar equation holds for odd n with the kernel $\operatorname{Im}\Theta$. □

6 Off-diagonal kernel

Proposition 6.1. *Under the assumptions of Theorem 5.1, the operator* $A_{r,\rho} = Q_\rho M_r - D_{r,\rho}\mathrm{Id}$ *is a singular integral operator of Sobolev order* 0 *with leading term*

$$a_0(x,s) = \begin{cases} \operatorname{Re} q_0(x,s) & \text{for even } n, \\ -\operatorname{Im} q_0(x,s)/\pi & \text{for odd } n, \end{cases}$$

where

$$q_0(x,s) = -(n-1)! \int_\Omega \frac{\rho(x,\omega) r(x,\omega)\mathrm{d}\Omega}{(\langle \nabla\theta(x,\omega), s\rangle - i0)^n}, \quad s = y - x. \tag{13}$$

Proof. We show that the kernel Θ defines an operator of Sobolev order 0. Applying the Lagrange formula to $\theta(y,\omega)$ at $y = x$ and estimating the remainder, we get

$$\theta(y,\omega) - \theta(x,\omega) = \langle \nabla\theta(x,\omega), s\rangle + \xi(x,s,\omega),$$

where

$$\max_{|i|+|j|\le\kappa-2} \max_{x\in K} \max_{\omega} |s|^{|i|} |D_s^i D_x^j \xi(x,s,\omega)| \le C_K |s|^2. \tag{14}$$

For $F(p) = (1-p)^{-n}$, the Lagrange formula yields

$$(1+p)^{-n} = 1 - np \int_0^1 (1+tp)^{-n-1}\mathrm{d}t = 1 - np + \frac{n(n+1)}{2} p^2 \int_0^1 (1+tp)^{-n-2}\mathrm{d}t. \tag{15}$$

Taking $p = \xi(x, s, \omega)(\langle\nabla\theta(x, \omega), s\rangle - i0)^{-1}$ and multiplying by

$$(\langle\nabla\theta(x, \omega), s\rangle - i0)^{-n}$$

yields

$$\frac{1}{(\theta(y, \omega) - \theta(x, \omega) - i0)^n} = \frac{1}{(\langle\nabla\theta(x, \omega), s\rangle - i0)^n} - n\int_0^1 \frac{\xi \mathrm{d}t}{(\langle\nabla\theta(x, \omega), s\rangle + t\xi - i0)^{n+1}}.$$

Integrating against the density $\rho(x, \omega)r(y, \omega)\mathrm{d}\Omega$ yields

$$\Theta(x, y) = \int \frac{\rho(x, \omega)r(y, \omega)\mathrm{d}\Omega}{(\theta(y, \omega) - \theta(x, \omega) - i0)^n} = q_0(x, s) + r_1(x, s),$$

where the first term on the right hand side is as in (13) and

$$\begin{aligned} r_1(x, s) = -n\int_0^1 \mathrm{d}t \int_\Omega &\frac{\xi(x, s, \omega)\rho(x, \omega)r(x, \omega)\mathrm{d}\Omega}{(\langle\nabla\theta(x, \omega), s\rangle + t\xi - i0)^{n+1}} \\ &+ \int \frac{\rho(x, \omega)r(y, \omega) - r(x, \omega)\mathrm{d}\Omega}{(\theta(y, \omega) - \theta(x, \omega) - i0)^n}. \end{aligned} \tag{16}$$

For each $i = 1, 2, \ldots, n$, we can find a smooth function t_{0i} and a smooth tangent field t_{1i} in Ω such that $t_{0i}\nabla_x\theta + t_{1i}(\nabla_x\theta) = e_i$, where the gradient $\nabla_x\theta$ is viewed as a column vector and e_i is the i-th column of the unit $n \times n$-matrix. For a local construction, we use columns of the inverse to the matrix

$$\begin{pmatrix} \theta_1 & \partial_{\omega_1}\theta_1 & \cdots & \partial_{\omega_{n-1}}\theta_1 \\ \theta_2 & \partial_{\omega_1}\theta_2 & \cdots & \partial_{\omega_{n-1}}\theta_2 \\ \vdots & \vdots & \vdots & \vdots \\ \theta_n & \partial_{\omega_1}\theta_n & \cdots & \partial_{\omega_{n-1}}\theta_n \end{pmatrix},$$

where $\theta_k = \partial\theta/\partial x_k$, $k = 1, \ldots, n$. By (2), this matrix is invertible for any local coordinate system $\omega_1, \ldots, \omega_{n-1}$ in Ω. We extend the functions t_{0i} and fields t_{1i} to the whole sphere by means of a partition of unity. Consider the differential operator

$$t_0 + t_1,\ t_0 = |s|^{-2}\sum_1^n s_i t_{0i},\ t_1 = |s|^{-2}\sum s_i t_{1i},$$

where t_0 is a function and t_1 is a tangent field on the sphere with coefficient depending on s. Consider the function $\nu = \upsilon(x, s) \doteq \langle\nabla\theta(x, \omega), s\rangle - i0$. We have

$$t_0\upsilon + t_1(\upsilon) = |s|^{-2}\sum s_i^2 = 1.$$

Therefore $\tau_{-k}(\nu^{-k}) = (t_0 v + k^{-1} t_1(v)) v^{-k-1} = \nu^{-k-1}$ for an arbitrary integer $k \neq 0$, where $\tau_{-k} = t_0 + k^{-1} t_1$ and $t_0 + t_1(\log \nu) = \nu^{-1}$. Integration by parts yields

$$q_0 = \int_\Omega \rho v^{-n} \mathrm{d}\Omega = \int_\Omega \tau_{1-n}(\nu^{1-n}) \rho \mathrm{d}\Omega = \frac{1}{1-n} \int_\Omega \nu^{1-n} \tau^*_{1-n}(\rho) \mathrm{d}\Omega,$$

where τ^*_k is the adjoint differential operator to τ_k which is a homogeneous function of s of degree -1. Integrating by parts n times, we obtain

$$\begin{aligned} q_0(s) = \cdots &= b_n \int_\Omega \nu^{-1} \tau^*_{-1}(\cdots \tau^*_{1-n}(\rho)) \mathrm{d}\Omega \\ &= b_n \int_\Omega [\log \nu \; t^*_1 + t_0] \tau^*_{-1}(\cdots \tau^*_{1-n}(\rho)) \mathrm{d}\Omega, \end{aligned} \tag{17}$$

where $b_n = (-1)^{n-1}/(n-1)!$. The function $t^*_1 \tau^*_{-1}(\cdots \tau^*_{1-n}(\rho))$ as well as the function $t_0 \tau^*_{-1}(\cdots \tau^*_{1-n}(\rho))$ has homogeneous coefficients of degree $-n$ with respect to s. We check that the integral (17) is a homogeneous function of degree $-n$. Indeed,

$$\log \nu = \log(\nu/|s|) + \log |s|,$$

where the first term in the right hand side is homogeneous of degree 0 and the second term vanishes since

$$\int t^*_1 \tau^*_{-1}(\ldots \tau^*_{1-n}(\rho)) \log |s| \mathrm{d}\Omega = \int \tau^*_{-1}(\ldots \tau^*_{1-n}(\rho)) t_1(\log |s|) \mathrm{d}\Omega = 0$$

and $\log |s|$ does not depend on ω. This proves that the kernel q_0 is homogeneous in s of degree $-n$. To get a bounded kernel in the right hand side of (17), we integrate by parts one more time and obtain an integral with bounded kernel $\nu \log \nu$. Integrating by parts again, we get, for an arbitrary natural number $\kappa \geq n+1$,

$$\max_{i+j \leq \kappa-n-1} \max_{x \in K} |s|^{n+i} |D^i_s D^j_x q_0(x,s)| \leq C_K, \;\; s \neq 0$$

for an arbitrary compact set $K \subset X$. Similar arguments applied to (16) show that for sufficiently small ε and $|s| < \varepsilon$,

$$\max_{i+j \leq \kappa-n-1} \max_{x \in K} |s|^{n+i} |D^i_s D^j_x r_1(x,s)| \leq C_K |s|, \;\; s \neq 0. \tag{18}$$

Therefore, the right hand side of (11) has the structure of equations (33)–(34) below with principal term (13).

To show that the kernel $\mathrm{Re}(i^n q_0)$ satisfies (32), we invoke the following fact, whose proof is given in [11, Lemma 4.3].

Proposition 6.2. *Let $v \in \mathbb{R}^n$ and $a \in \mathbb{R}$ be such that $|a| < |v|$. Then for even $n \geq 2$,*

$$\mathrm{Re} \int_\Omega \frac{i^n \mathrm{d}\Omega}{(\langle s, v \rangle - a - i0)^n} = 0,$$

where $\mathrm{d}\Omega$ is the euclidean volume form on the unit sphere Ω in $\mathbb{R}^n$.

This yields

$$\operatorname{Re}\int_{\Omega}\frac{i^{n}\mathrm{d}\Omega}{(\langle\nabla\theta(x,\omega),s\rangle-i0)^{n}}=0$$

for all $x\in X$. Integrating over Ω and changing the order of integrals yields

$$\operatorname{Re}\int_{\Omega}i^{n}q_{0}(x,s)\mathrm{d}\Omega=0,$$

which completes the proof of Proposition 6.1. □

7 Vanishing of the off-diagonal kernel and parametrix

Assume that Ω is oriented by the volume form $\mathrm{d}\Omega$. A key point of our construction is the following proposition.

Proposition 7.1. *The integral* (13) *vanishes if* $\eta=r\rho$ *satisfies*

$$\eta(x,\omega)\,\mathrm{d}\Omega=\frac{1}{(n-1)!}\nabla\theta\wedge(\mathrm{d}_{\omega}\nabla\theta)^{\wedge n-1}. \tag{19}$$

Proof. Fix $x\in X$ and consider the hypersurface

$$H=\operatorname{Im}\{\nabla\theta(x,\cdot):\Omega\to E^{n}\}.$$

Equation (19) can be written in the form $\eta(x,\omega)\mathrm{d}\Omega=\nabla\theta\wedge\mathrm{d}h$, where the differential form $\mathrm{d}h=1/(n-1)!(\mathrm{d}_{\omega}\nabla\theta)^{\wedge n-1}$ is the euclidean area form of H expressed in coordinates ω. Define $z:H\to\mathrm{S}^{n-1}$ by $z(h)=h/|h|$. Then

$$\eta(x,\omega)\mathrm{d}\Omega=\nabla\theta\wedge\mathrm{d}h=|\nabla\theta|^{n}\mathrm{d}z, \tag{20}$$

where $\mathrm{d}z$ is the area form of S^{n-1}. This yields

$$\int_{\Omega}\frac{\eta(x,\omega)\mathrm{d}\Omega}{(\langle\nabla\theta(x,\omega),s\rangle-i0)^{n}}=\int_{\Omega}\frac{\mathrm{d}z(\omega)}{(\langle z,s\rangle-i0)^{n}}, \tag{21}$$

where $x\in X$, $s\neq 0$, and

$$z=z(\omega)=\frac{\nabla\theta(x,\omega)}{|\nabla\theta(x,\omega)|}. \tag{22}$$

Consider the map $\zeta:\Omega\to S^{n-1}$, $\omega\mapsto z(\omega)$, and choose an orientation of Ω. The invariant $\deg\zeta$ is well-defined and does not vanish because of (2). Replacing the variables ω by $z\in\mathrm{S}^{n-1}$ on the right hand side of (21), we obtain

$$Z_{n}\doteq\deg\zeta\int_{\Omega}\frac{\mathrm{d}z}{(\langle z,s\rangle-i0)^{n}}.$$

According to [11, Proposition 4.3], $\operatorname{Re}i^{n}Z_{n}=0$ for all $n\geq 2$. □

Remark. The map z is proper and, by (2), is locally bijective. Choosing an orientation of Ω such that $\eta > 0$, we have deg $\zeta > 0$ and deg $\zeta = 1$ if $n > 2$.

Corollary 7.2. *If Φ is a generating function as in Theorem 5.1 and $r\rho = \eta$ is as in* (19)*, then the operator $P_1 = D_{r,\rho}^{-1} Q_\rho$, where Q_ρ is as in* (8)–(9) *and $D_{r,\rho}$ is as in* (10)*, can be written in the form*

$$P_1 g(x) = \frac{\pi_n}{\deg \zeta} \int_\Omega \int_{\mathbb{R}} \frac{g^{(n-1)}(\lambda, \omega) \mathrm{d}\lambda}{\theta(x, \omega) - \lambda} |\nabla \theta(x, \omega)|^n \mathrm{d}z(\omega) \tag{23}$$

for even n and in the form

$$P_1 g(x) = \frac{\pi_n}{\deg \zeta} \int_\Omega g^{(n-1)}(\theta(x, \omega), \omega) |\nabla \theta(x, \omega)|^n \mathrm{d}z(\omega) \tag{24}$$

for odd n, where z is defined by (22).

Proof. By (10) and (20), we have

$$D_{r,\rho}(x) = \frac{1}{|\mathrm{S}^{n-1}|} \int_\Omega \frac{\eta(x, \omega) \mathrm{d}\Omega}{|\nabla \theta(x, \omega)|^n} = \frac{1}{|\mathrm{S}^{n-1}|} \int_\Omega \mathrm{d}z = \deg \zeta; \tag{25}$$

so (23) and (24) follow from (8) and (9). □

In the next section, we show that P_1 is a 1-parametrix.

8 Calculation of a remainder

Theorem 8.1. *Let X be an open set in euclidean space E^n, $\Phi = \theta - \lambda$ a regular generating function in $X \times \mathbb{R} \times \Omega$ of class C^κ, where $\kappa > n + 4$, and $r\rho = \eta \in C^\kappa(X \times \Omega)$ as in* (19)*. Then the operator P_1 is a* 1*-parametrix for M_r, and the remainder $R_1 = P_1 M - \mathrm{Id}$ is a* 1*-smoothing integral operator with leading term*

$$b_1(x, s) = \frac{1}{D_{r,\rho}(x)} \begin{cases} n \operatorname{Re} q_1(x, s) & \text{for even } n \\ -n \operatorname{Im} q_1(x, s)/\pi & \text{for odd } n, \end{cases} \tag{26}$$

where

$$q_1(x, s) = \int \frac{\mu(x, s, \omega) \eta(x, \omega) \mathrm{d}\Omega}{(\langle \nabla \theta(x, \omega), s \rangle - i0)^{n+1}}, \quad \mu(x, s, \omega) = \frac{1}{2} \langle \nabla_x^2 \theta(x, \omega), s^2 \rangle. \tag{27}$$

Proof. By Propositions 6.1 and 7.1, for even n,

$$P_1 M_r f(x) - f(x) = D_{r,\rho}^{-1}(x) \int \operatorname{Re} \Theta(x, y) f(y) \mathrm{d}V = D_{r,\rho}^{-1}(x) \int \operatorname{Re} r_1(x, y) f(y) \mathrm{d}V,$$

and the kernel r_1 defined in (16) satisfies (18); the same conclusion holds for odd n. Therefore, the function $D_{r,\rho}^{-1} \operatorname{Re} r_1$ is the kernel of R_1 and q_0 vanishes.

We can specify the structure of r_1. First, applying the Lagrange formula for ξ, we have $\xi(x, s, \omega) = \mu(x, s, \omega) + \sigma(x, s, \omega)$, where the remainder σ satisfies an inequality like (14), with the power $|s|^3$ instead of $|s|^2$. Set

$$p = \frac{\mu + \sigma}{\langle \nabla\theta(x, \omega), s\rangle - i0}$$

in the right hand side of (15) to obtain $r_1(x, y) = -nq_1(x, s) + r_2(x, s)$, where q_1 is as in (27) and the remainder r_2 admits an estimation like (18) with the factor $|s|^2$ instead of $|s|$ on the right hand side. The kernel q_1 is homogeneous of degree $1 - n$, which implies (26). By Proposition 10.4, r_1 is a 1-smoothing operator. □

Remarks. 1. Beylkin [1] has constructed a parametrix for M_r in terms of Fourier integral operators. His construction depends on the assumption that $\theta(x, -\omega) = -\theta(x, \omega)$, which is not satisfied in the case of photo-acoustic acquisition geometry.

2. Higher parametrices P_k and remainders R_k, $k = 2, 3, 4, \ldots$, can be calculated as in Section 2 by means of the Lagrange formula like (15) with more terms.

Proposition 8.2. *Suppose that $\tilde{\theta}(x, \xi) \doteq |\xi|\theta(x, \xi/|\xi|)$ is a linear function of $\xi \in \mathbb{R}^n$ and that the function $\theta(y, \omega) - \theta(x, \omega)$ has at least one root $\omega \in \Omega$ for each $x \neq y \in X$ and $r = 1$. Then the parametrix P_1 coincides with the left inverse operator constructed in* [11]*, namely,*

$$P_1 g(x) = \frac{\pi_n}{D_1(x)} \int_\Omega \int_{\mathbb{R}} g^{(n-1)}(\lambda, \omega) \frac{\mathrm{d}\lambda \mathrm{d}\omega}{\theta(x, \omega) - \lambda}$$

for even n, and

$$P_1 g(x) = \frac{\pi_n}{D_1(x)} \int_\Omega .g^{(n-1)}(\lambda, \omega)|_{\lambda = \theta(x,\omega)} \mathrm{d}\omega$$

for odd n.

Proof. We have

$$(n-1)!\eta \mathrm{d}\Omega \doteq \nabla\theta \wedge (\mathrm{d}_\omega \nabla\theta)^{\wedge n-1} = |\xi|^{-n} (\mathrm{d}_\xi \nabla\tilde{\theta})^{\wedge n} / \mathrm{d}|\xi|,$$

where $\omega = \xi/|\xi|$. The quotient $\rho \doteq (\mathrm{d}_\xi \nabla\tilde{\theta})^{\wedge n} / \mathrm{d}\xi_1 \wedge \cdots \wedge \mathrm{d}\xi_n$ does not depend on ξ since $\tilde{\theta}$ is a linear function of ξ. Therefore, the right hand side equals

$$\rho(\mathrm{d}\xi_1 \wedge \ldots \wedge \mathrm{d}\xi_n)/\mathrm{d}|\xi| = \rho \mathrm{d}\Omega,$$

that is, $\eta = \rho$; hence η does not depend on ω. Thus, the factor ρ in (8) cancels the same factor in the (10) for $D_{1,\rho}$. By [11, Theorem 3.1], it follows that (8) and (9) with $\rho = 1$ and the factor $D_{1,\rho}^{-1}$ define a left inverse operator L to M_1. Therefore, the parametrix P_1 coincides with L. □

9 Photo-acoustic acquisition geometries

Let Γ be a hypersurface in euclidean space E^n, and let

$$\mathrm{R}_\Gamma f(r, \xi) = \int_{|x-\xi|=r} f\, dS, \quad \xi \in \Gamma,\ r > 0,$$

be the corresponding spherical integral transform of a function f on E^n. The problem of inverting this transform has been studied for at least a decade in view of applications to photo-acoustic tomography. A special construction of a parametrix was proposed by Popov and Sushko [13] based on a reduction to the Radon transform. For an arbitrary spherical central surface Γ, Finch et. al. [4], [3] found exact reconstruction formulas. Other reconstruction formulas were given by Kunyansky [8] and Xu and Wang [15]. In [9], explicit reconstruction formulas were found for an arbitrary ellipsoid $\Gamma \subset E^3$, and in [11], a reconstruction is given for ellipsoids Γ in a space of arbitrary dimension. Natterer [9] showed that the same formula holds for an arbitrary convex surface Γ in E^3 up to an explicitly calculated remainder, and Haltmeier [5] did the same for E^2. It can be checked that the remainder does not vanish in the general case (in contrast with [7]). We prove below that the formulas in (29)-(30) below provide a 1-parametrix in arbitrary dimension. In three dimensional space, this coincides with the main term of Natterer's formula up to a 1-smoothing operator, and the remainders in Natterer's and Haltmeier's formulas are also 1-smoothing operators. An exact reconstruction formula for the operator R_Γ is still known only for a narrow class of algebraic curves and surfaces [11].

Proposition 9.1. *Let X be a compact convex domain in E^n with a boundary Γ parametrized by a smooth map of rank $n-1$ $x = \xi(\omega)$, $\omega \in \Omega$, where $\Omega = \mathrm{S}^{n-1}$ is the unit sphere in euclidean space E^n. Then the generating function $\Phi(x; \lambda, \omega) = |x - \xi(\omega)| - \lambda$ defined in $X \times \mathbb{R}_+ \times \Omega$ is regular.*

Proof. We have

$$\eta \mathrm{d}\Omega = \frac{1}{(n-1)!}\left(\frac{-1}{|x-\xi|}\right)^n (x-\xi) \wedge (\mathrm{d}\xi)^{\wedge(n-1)} \neq 0,$$

since the map ξ has rank $n-1$. This proves (2).

To prove (5), we consider the function

$$\theta(y, \omega) - \theta(x, \omega) = |y - \xi(\omega)| - |x - \xi(\omega)|$$

and show that for $x, y \in X$, $x \neq y$, each zero ω of this function is simple. Indeed, were ω a nonsimple zero, then we would have

$$0 = \theta(y,\omega) - \theta(x,\omega) = \mathrm{d}_\omega(\theta(y,\omega) - \theta(x,\omega)) = \frac{\langle x-y, \mathrm{d}\xi\rangle}{|x-\xi(\omega)|^2} \tag{28}$$

The second equation implies that the vector $x - y \neq 0$ is orthogonal to the tangent hyperplane T of Γ at $\xi(\omega)$. By the first equation, x and y are the same distance to $\xi(\omega)$ and are therefore symmetric with respect to T. This is impossible, since X is convex and hence lies on one side of T. □

Corollary 9.2. *For even n,*

$$f(x) - S_1 f(x) = \pi_n \int_\Omega \int_{\mathbb{R}} \frac{\mathrm{R}_\Gamma f^{(n-1)}(\lambda,\omega)\mathrm{d}\lambda}{|x-\xi|-\lambda}\mathrm{d}z, ; \tag{29}$$

and for odd n,

$$f(x) - S_1 f(x) = \pi_n \int_\Omega \mathrm{R}_\Gamma f^{(n-1)}(|x-\xi|,\xi)\mathrm{d}z, \tag{30}$$

where

$$z = \frac{\xi - x}{|\xi - x|},\ \mathrm{d}z = \frac{\cos\psi}{|x-\xi|^{n-1}}\mathrm{d}\xi$$

and S_1 is a 1-smoothing operator in X.

Proof. We have $|\nabla\theta| = 1$, $\nabla\theta(x,\omega) = z$; hence $Rf = M_1 f$, and we can apply Corollary 7.2. By (25), we have $D_{1,\rho} = \deg z$ and $\deg z = 1$ for all $x \in X$ since the map $z : \Omega \to \mathrm{S}^{n-1}$ is bijective. This yields $D_{1,\rho}(x) = 1$. □

For the case $n = 3$, we obtain

$$f(x) = \pi_3 \int_\Gamma \mathrm{R}_\Gamma f^{(2)}(|x-\xi|,\xi)\frac{\cos\psi}{|x-\xi|^2}\mathrm{d}\xi + S_1 f(x),$$

where $\mathrm{R}_\Gamma f^{(2)} = (\partial/\partial r)^2 \mathrm{R}_\Gamma f$. The first term coincides with that of the reconstruction [9], which is exact up to a 1-smoothing operator S_1.

10 Singular integral operators

Let E^n be euclidean space of dimension $n \geq 1$, and let $a(x,s)$ be a locally bounded function on $E^n \times (E^n \setminus \{0\})$. Consider the integral transform A defined by

$$Af(x) = \lim_{\varepsilon\to 0} \int_{|s|>\varepsilon} a(x,s) f(x+s)\mathrm{d}s \tag{31}$$

for functions $f \in L_2(E^n)_{\mathrm{comp}}$. Let Ω be the unit sphere in E^n.

Theorem 10.1. *Let $a : E^n \times (E^n \setminus \{0\}) \to \mathbb{R}$ be a positively homogeneous function of degree $-n$ in the variable s for each x, locally bounded on $E^n \times \Omega$ satisfying*

$$\int_\Omega a(x,s)\mathrm{d}\Omega(s) = 0,\ x \in E^n, \tag{32}$$

where $\mathrm{d}\Omega$ *is the euclidean volume form on* Ω*. Then* (31) *defines a continuous operator* $A : L_{2\mathrm{comp}} \to L_{2\mathrm{loc}}$.

This is a simplified version of the Calderón-Zygmund Theorem [2].

Lemma 10.2. *Let $k \in \mathbb{Z}$ and $A_k : H^k \to H^{k-d}$ and $A_{k+1} : H^{k+1} \to H^{k+1-d}$ be bounded linear operators such that A_{k+1} is the restriction of A_k. Then for $k < \alpha < k+1$, the restriction of A_k to H^α defines a bounded operator $A_\alpha : H^\alpha \to H^{\alpha-d}$ such that $\|A_\alpha\| \le C\|A_k\|^{\alpha-k}\|A_{k+1}\|^{k+1-\alpha}$, where C depends only on X.*

Proof. Since H^{k+1} is dense in H^k, the restrictions A_{k+1} and A_α are uniquely defined. The lemma follows from the fact that any Sobolev space H^α is a complex interpolation of spaces H^β and $H^{\beta+1}$, where $\beta < \alpha < \beta+1$ and $\varepsilon = \alpha - \beta$ is the exponent of interpolation; see [14]. □

Let $D_x^i = (\partial/\partial x_1)^{i_1} \cdots (\partial/\partial x_n)^{i_n}$ and $D_s^i = \cdots$, where $i = (i_1, \dots i_n)$ is a multiindex.

Theorem 10.3. *Let κ be a natural number and*

$$a(x,s) = a_0(x,s) + r_1(x,s) \tag{33}$$

be a kernel supported in $X \times (E^n \setminus \{0\})$ of class C^κ, where for each x, $a_0(x,s)$ is a homogeneous function of s of degree $-n$ satisfying (32) *and r_1 satisfies*

$$\max_{i+j\le\kappa} \max_{x\in X} |s|^{i+n}|D_s^i D_x^j r_1(x,s)| \le C|s|, \tag{34}$$

for some constant C. Then for each compact set $X \subset E^n$ with boundary of class C^κ, A defines a bounded operator $L_2(X) \to L_2(X)$ of Sobolev order 0.

This operator is called a **singular integral operator** of class C^κ with principal term a_0.

Proof. We abbreviate $L_2 = L_2(X)$ and $H^\alpha = H^\alpha(X)$ for $\alpha \in \mathbb{R}$. Assume that $f \in H^\alpha$ for some natural number $\alpha \le \kappa$ and apply a partial derivative D^j, $|j| \le \alpha$, to Af to obtain

$$D^j Af(x) = \int D_x^j a(x,s) f(x+s)\mathrm{d}s + \int a(x,s) D^j f(x+s)\mathrm{d}s. \tag{35}$$

The kernel $D_x^j a$ is a singular integral operator of class $C^{\kappa-\alpha}$, and the first term is contained in $L_2(X)$ by the Calderón-Zygmund Theorem. The same is true for the second term since $D^j f \in L_2$. This yields $D^j Af \in L_2$; hence $Af \in H^\alpha$.

If $\alpha \geq -\kappa$ is a negative integer, we use decreasing induction in α. An arbitrary function $f \in H^\alpha$ can be written in the form $f = D^j g_j$, where $g_j \in L_2$ and summation in j, $|j| = -\alpha$ is assumed. Reading (35) from right to left yields

$$Af(x) = \int a(x,s) D^j g_j(x+s) \mathrm{d}s = D^j A g_j(x) - \int D_x^j a(x,s) g_j(x+s) \mathrm{d}s.$$

The first term belongs to H^α since $Ag_j \in L_2$. The second term belongs to $L_2(X) \subset H^\alpha$ by the Calderón-Zygmund theorem, which implies $Af \in H^\alpha$.

If α is not an integer. we apply Lemma 10.2 to A. □

Proposition 10.4. *If a_l is a function in $X \times (E^n \setminus \{0\})$ of class C^κ which is homogeneous in s of degree $l - n$ for some natural number $l < \kappa$, and r_{l+1} satisfies* (34) *with right hand side $C|s|^{l+1}$, then the integral transform A with kernel $a = a_l + r_{l+1}$ is an l-smoothing operator of class $C^{\kappa-1}$.*

Proof. First claim that the function $a_0 \doteq D_s^j a_1$ satisfies (32) for every multiindex j, $|j| = l$. Indeed, consider the differential form $\upsilon = a_0 \mathrm{d}\Omega = D_s^j a_l \mathrm{d}\Omega$ of degree $n-1$ in $E^n \setminus \{0\}$ (x is fixed). This form is closed since a_0 is homogeneous of degree $-n$. We can write $D_s^j = (\partial/\partial s_k) D_s^i$ for some k and i, and have $a_0 = \partial b/\partial s_k$, $b = D^i a_l$. Suppose that $k > 1$, and consider the hyperplanes $H_{\pm,0} = \{s_1 = \pm 1, 0\}$. The central projection $\pi(s) = s/|s|$ in $E^n \setminus \{0\}$ maps the set $H_+ \cup H_-$ to $\Omega \setminus H_0$. By Stokes' Theorem, $\int_{\Omega \setminus H_0} \upsilon = \int_{H_+ \cup H_-} \upsilon$, since the form υ is closed and decreases sufficiently fast at ∞. The left hand side equals $\int_\Omega \upsilon$, while the right hand side vanishes since $\upsilon = \pm \mathrm{d}(b \mathrm{d}s_2 \wedge \cdots \wedge \widehat{\mathrm{d}s_k} \wedge \cdots \wedge \mathrm{d}s_n)$ in $H_\pm$. This establishes the claim.

Now, for simplicity, suppose that $l = 1$. Substituting $D_x^j f(x+s) = D_s^j f(x+s)$ for $|j| = 1$ in the second term of (35) and integrating by parts, we arrive at

$$D^j Af(x) = \int D_x^j a(x,s) f(x+s)\, \mathrm{d}s - \int D_s^j a(x,s) f(x+s)\, \mathrm{d}s. \tag{36}$$

The kernel $D_x^j a = D_s^j a_1 + D_s^j r_2$ satisfies (34) and hence by the Calderón-Zygmund Theorem, defines an operator of order 0. The kernel $D_s^j a = D_s^j a_1 + D_s^j r_2$ is of the form of (33), where $a_0 \doteq D_s^j a_1$ is homogeneous of degree $-n$ in s, and belongs to class $C^{\kappa-1}$. The function $r_1 = D_s^j r_2$ satisfies (34) with $\kappa - 1$ instead of κ. From the claim and the Calderón-Zygmund Theorem, we conclude that the second term of (36) also defines an operator of order 0. The same is true for operators $D^j A$, $|j| = 1$, which implies that A is a 1-smoothing operator. □

REFERENCES

[1] G. Beylkin, *The inversion problem and applications of the generalized Radon transform*, Comm. Pure Appl. Math. **37** (1984), 579–599.

[2] A. P. Calderón and A. Zygmund, *On singular integrals*, Amer. J. Math. **78** (1956), 289–309.

[3] D. Finch, M. Haltmeier and Rakesh, *Inversion of spherical means and the wave equation in even dimensions*, SIAM J. Appl. Math. **68** (2007), 392–412.

[4] D. Finch, S. Patch, and Rakesh, *Determining a function from its mean values over a family of spheres*, SIAM J. Math. Anal. **35** (2004), 1213–1240.

[5] M. Haltmeier, *Inversion of circular means and the wave equation on convex planar domains*, Comput. Math. Appl. **65** (2013), 1025–1036.

[6] L. Hörmander, *The Analysis of Linear Partial Differential Operators IV, Fourier Integral Operators*, Springer, Berlin, 1985.

[7] M. Idemen and A. Alkumru, *On an inverse source problem connected with photo-acoustic and thermo-acoustic tomographies*, Wave Motion **49** (2012), 595–604.

[8] L. Kunyansky, *Reconstruction of a function from its spherical (circular) means with the centers lying on the surface of certain polygons and polyhedra*, Inverse Problems **27** (2011), 025012.

[9] F. Natterer, *Photo-acoustic inversion in convex domain*, Inverse Prob. Imaging **6** (2012), 315–320.

[10] V. Palamodov, *Remarks on the general Funk transform and thermoacoustic tomography*, Inverse Problems and Imaging **4** (2010), 693–702.

[11] V. Palamodov, *A uniform reconstruction formula in integral geometry*, Inverse Problems **28** (2012), 065014.

[12] L. Pestov and G. Uhlmann, *On characterization of the range and inversion formulas for the geodesic X-ray transform*, Int. Math. Res. Not. **2004**, 4331–4347.

[13] D. A. Popov and D. V. Sushko, *Image restoration in optical-acoustic tomography*, Problemy Peredachi Informacii **40** (2004), no.3, 81-107; translation in Probl. Inf. Transm. **40** (2004), 254–278.

[14] L. Tartar, *An Introduction to Sobolev Spaces and Interpolation Spaces*, Springer, Berlin, 2007.

[15] M. Xu and L. V. Wang, *Universal back-projection algorithm for photoacoustic computed tomography*, Phys. Rev. E **71** (2005), 016706.

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