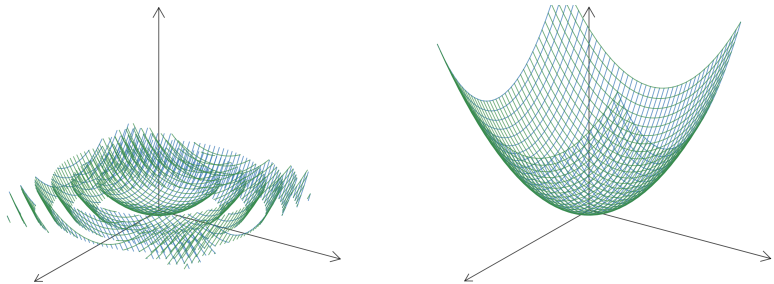
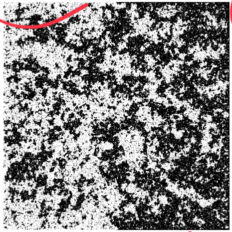
Statistical reconstruction of the Gaussian free field and Kosterlitz-Thouless transition

Christophe Garban (Univ Lyon 1)
joint with Avelio Sepúlveda (Univ Lyon 1)



Spin systems on $\mathbb{Z}^2$

Ising model

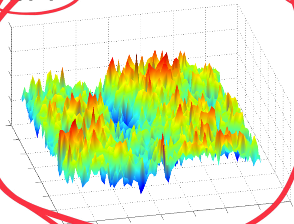


$\sigma \in \{-1, 1\}^{\mathbb{Z}^2}$

XY model

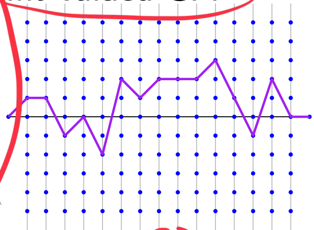
$\sigma : \mathbb{Z}^2 \to \mathbb{S}^1$

GFF



$\sigma (= \phi) : \mathbb{Z}^2 \to \mathbb{R}$

Int-valued GFF



$\phi : \mathbb{Z}^2 \to \mathbb{Z}$

Gibbs measure

$$\mathbb{P}_\beta(\sigma) \propto \exp\Big(-\frac{\beta}{2} \sum_{i \sim j} \|\sigma_i - \sigma_j\|^2\Big) \left(= \frac{1}{Z_\beta} \exp\Big(-\frac{\beta}{2} \sum_{i \sim j} \|\sigma_i - \sigma_j\|^2\Big) \right)$$

$\beta \gg 1$ $\quad \beta = \frac{1}{T}$ $\quad T \ll 1$

$\beta \ll 1$ $\quad \beta$ $\quad T \gg 1$

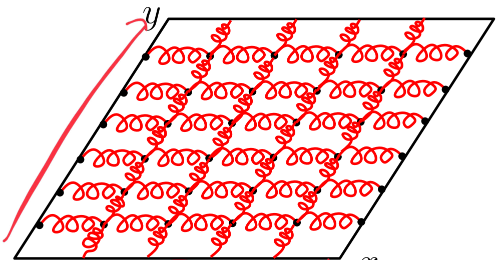
Gaussian Free Field (GFF)

Quadratic Form ⇒ Gaussian Vector $\in \mathbb{R}^n$

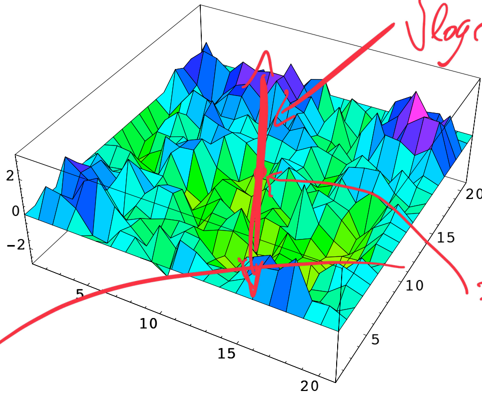
Definition

On $\Lambda_n := \frac{1}{n}\mathbb{Z}^2 \cap [-1,1]^2$

$$d\mathbb{P}^{\mathrm{GFF}}_{\beta}[\phi] \propto \exp\left(-\frac{\beta}{2}\sum_{x\sim y}(\phi(x)-\phi(y))^2\right)\prod_{x\in\Lambda_n} d\phi(x)$$



$n \times n$ $\Lambda_n \subset \mathbb{Z}^2$



$\sqrt{\log n}$

x_0

$$\mathrm{Var}[\phi((0,0))] \sim \frac{1}{\beta 2\pi}\log n$$

$\frac{1}{\beta} = T$

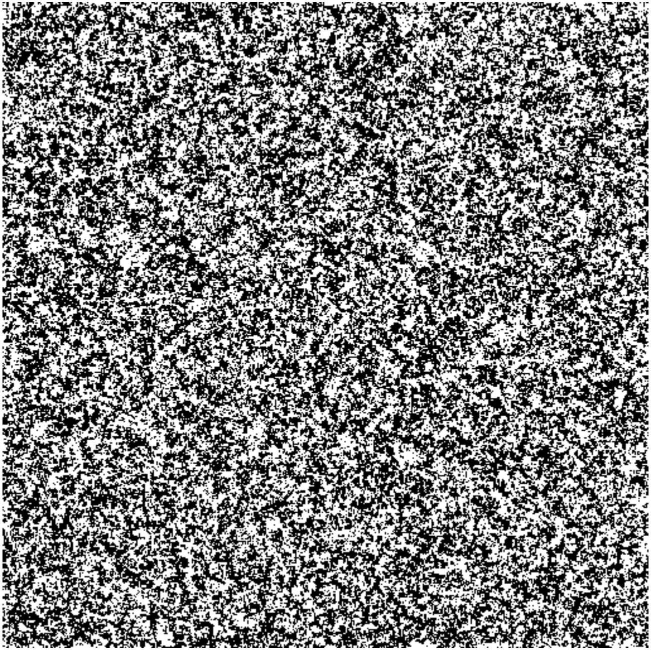
Ising model, $\sigma \in \{-1,1\}^{\Lambda}$, $T \gg 1$

$\beta \ll 1$

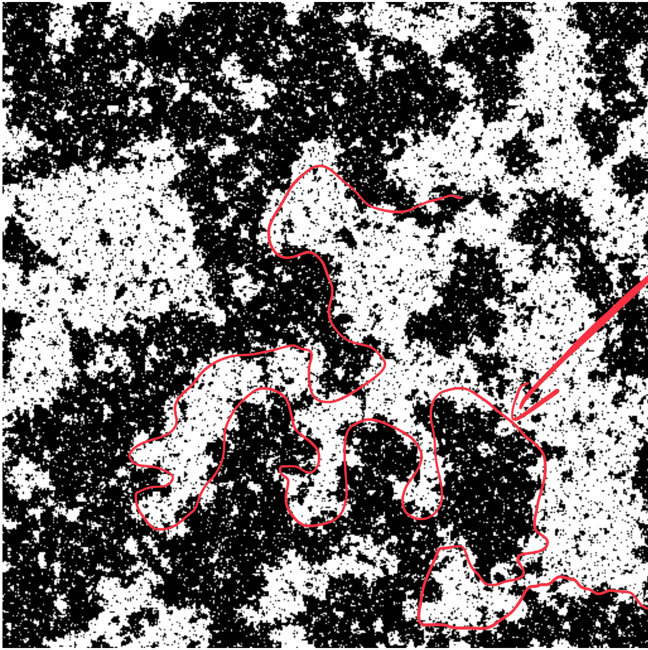
$\{-1,1\}^{\Lambda_n}$

$\{1,1\}^{\mathbb{Z}^2}$

$\exp\left(-\frac{\beta}{2}\sum|\sigma_i-\sigma_j|^2\right)$



Ising model, $\sigma \in \{-1, 1\}^{\Lambda}$, $T \approx T_c$

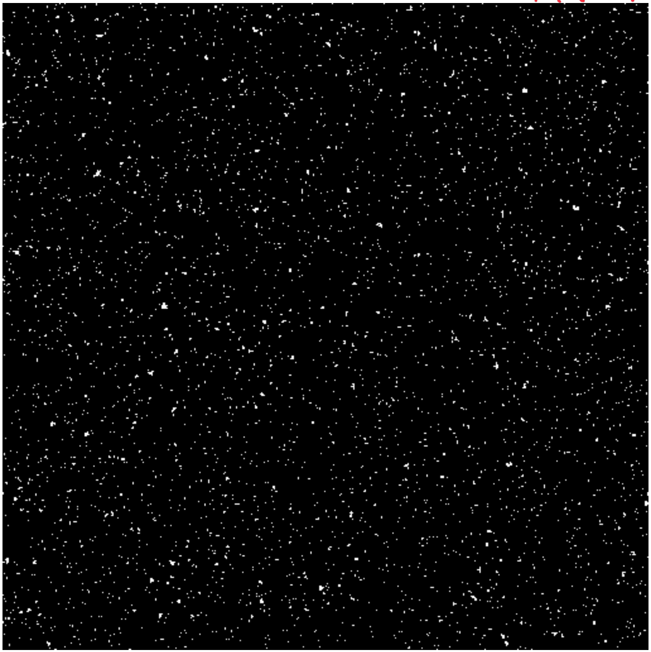


SLE_3

$\beta \gg 1$

LONG-Range ORDER

+ + + + + + +

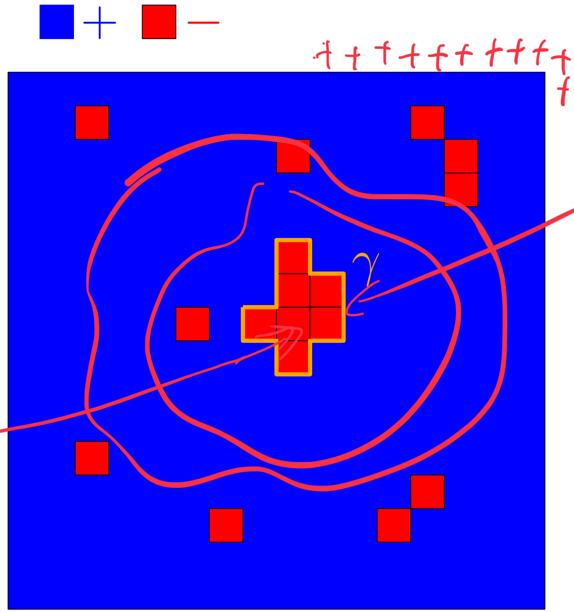


+ + + + +

Long-Range-Order ≡ Peierls argument

$\beta \gg 1$

$T \ll 1$



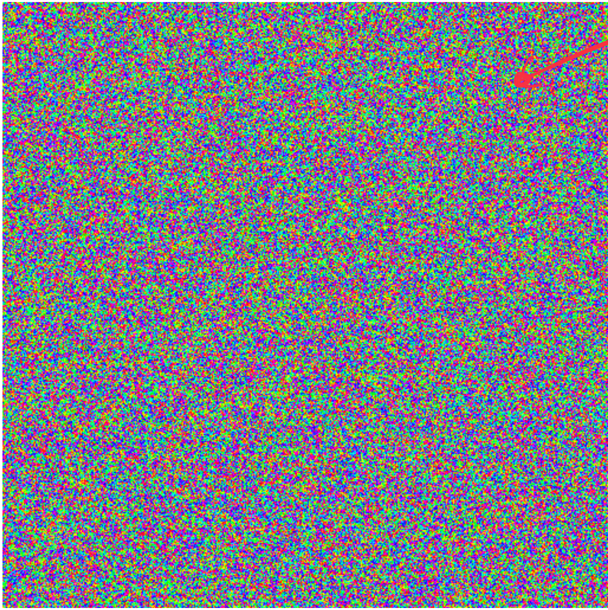
$e^{-\beta|\gamma|}$

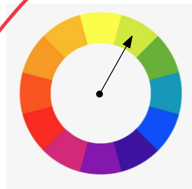
COSTLY!

⊖

XY model, $\sigma \in (\mathbb{S}^1)^\Lambda$, $T \gg 1$

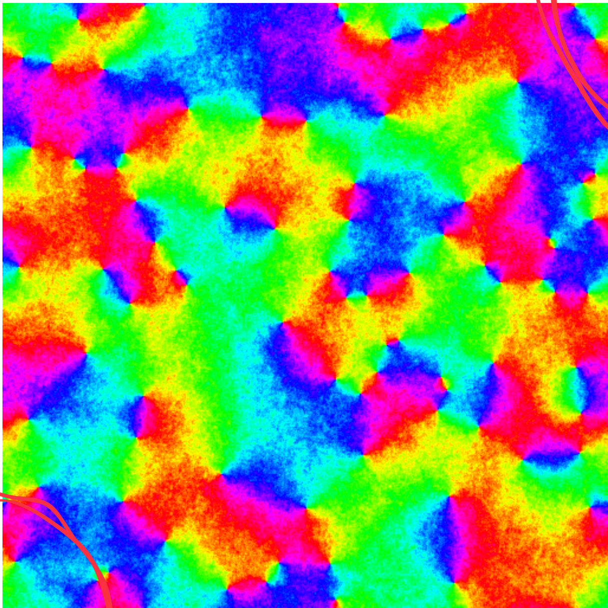
$x \in \mathbb{Z}^2$, $\sigma_x \in \mathbb{S}^1$





$x \in \Lambda,\ \sigma_x \in \mathbb{S}^1$

XY model, $\sigma \in (\mathbb{S}^1)^\Lambda$, $T \ll 1$



$x \in \Lambda, \ \sigma_x \in \mathbb{S}^1$

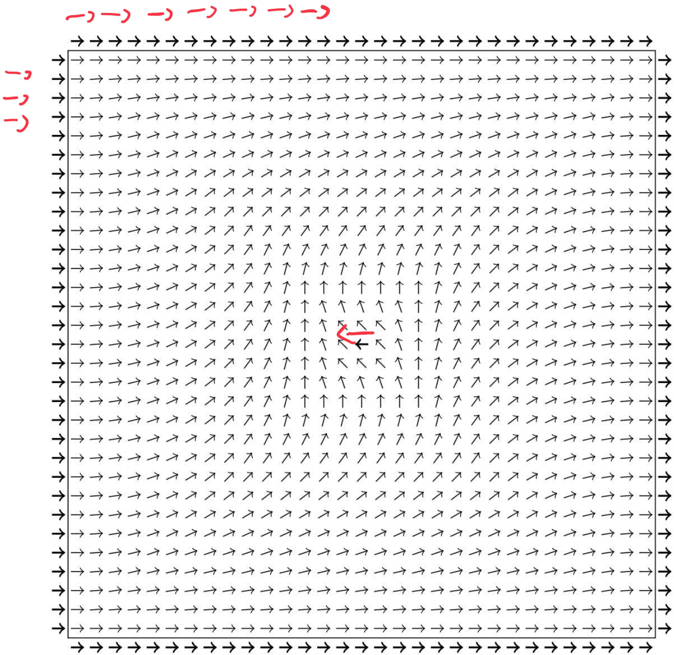
No Long-Range-Order ? Spin-waves!

© Velenik

th

Mermin
-Wagner

60's



BKT transition

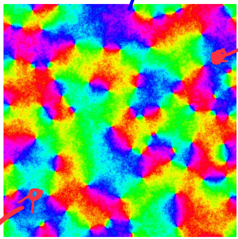
$T=0$

Quasi-Long-Range order

BKT transition

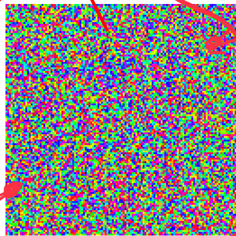
T_{BKT}

T



φ

x

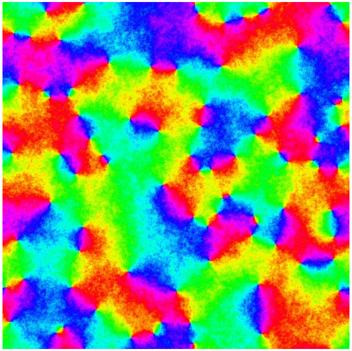


$$\langle \sigma_x \sigma_y \rangle_\beta \approx \frac{1}{|x-y|^{\alpha(T)}}$$

$$\boxed{\langle \sigma_x \sigma_y \rangle_\beta \approx e^{-c(T)\,|x-y|}}$$

$\alpha_{Ising} = \frac{1}{4}$ at T_c^{Ising}

The physics way ($\to$ “topological phase transitions”)



$\beta \gg 1$

$T \ll 1$

$$P_\beta(\sigma) \propto \exp\Big(-\frac{\beta}{2}\sum_{i\sim j}\|\sigma_i-\sigma_j\|^2\Big)$$

$$\propto \exp\Big(\beta\sum_{i\sim j}\cos(\theta_i-\theta_j)\Big)$$

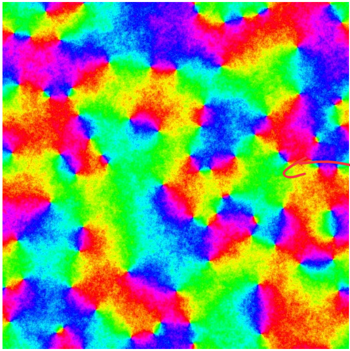
TAYLOR EXPAND

$$\propto\approx \exp\Big(-\frac{\beta}{2}\sum_{i\sim j}(\theta_i-\theta_j)^2\Big)$$

$\beta\big(\cancel{1} - \frac{1}{2}(\theta_i-\theta_j)^2 + \dots$

GFF !!

The physics way ($\to$ "topological phase transitions")



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$[0, 2\pi)$

$$\propto\approx \exp\Big(-\frac{\beta}{2}\sum_{i\sim j}(\theta_i-\theta_j)^2\Big)$$

1 form $\mathbb{R}^2$

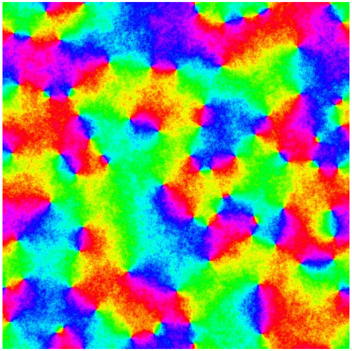
$d\theta$

GFF !!

$$\exp\Big(-\frac{\beta}{2}\int (d\theta)^2\Big)$$

$d\theta \equiv$ 1-form on "$\mathbb{R}^2$"

The physics way ($\to$ "topological phase transitions")



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$$\propto\approx \exp(-\frac{\beta}{2}\sum_{i\sim j}(\theta_i - \theta_j)^2)$$

GFF !!

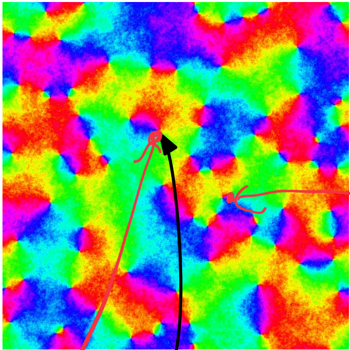
$$\exp(-\tfrac{\beta}{2}\int (d\theta)^2)$$

$d\theta \equiv$ 1-form on "$\mathbb{R}^2$"

$\xrightarrow{d}$

0-forms $\phi : \mathbb{R}^2 \to \mathbb{R}$

The physics way ($\to$ "topological phase transitions")



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$$\propto \exp\Big(\beta\sum_{i\sim j}\cos(\theta_i-\theta_j)\Big)$$

$$\propto\approx \exp\Big(-\frac{\beta}{2}\sum_{i\sim j}(\theta_i-\theta_j)^2\Big)$$

GFF !!

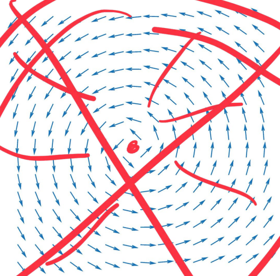
$\exp(-\frac{\beta}{2}\int (d\theta)^2)$

$d\theta \equiv$ 1-form on "$\mathbb{R}^2$"

d

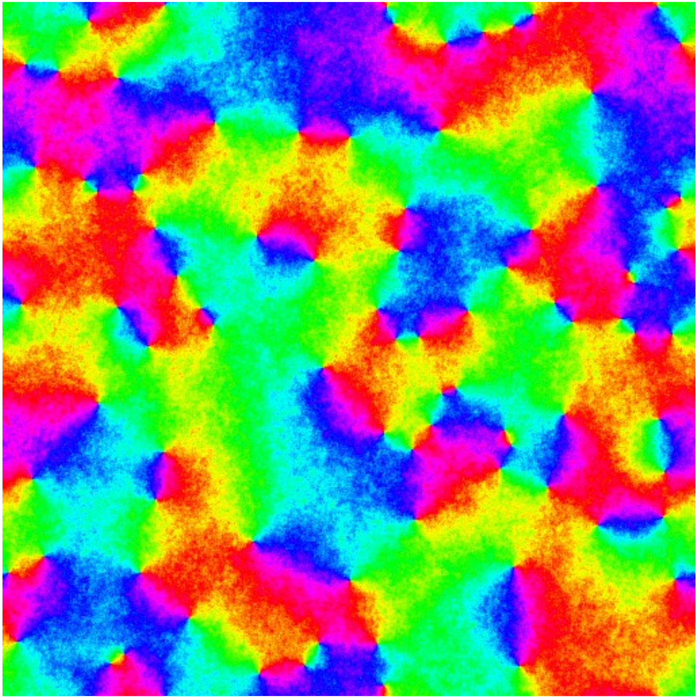
d^*

0-forms $\phi : \mathbb{R}^2 \to \mathbb{R}$

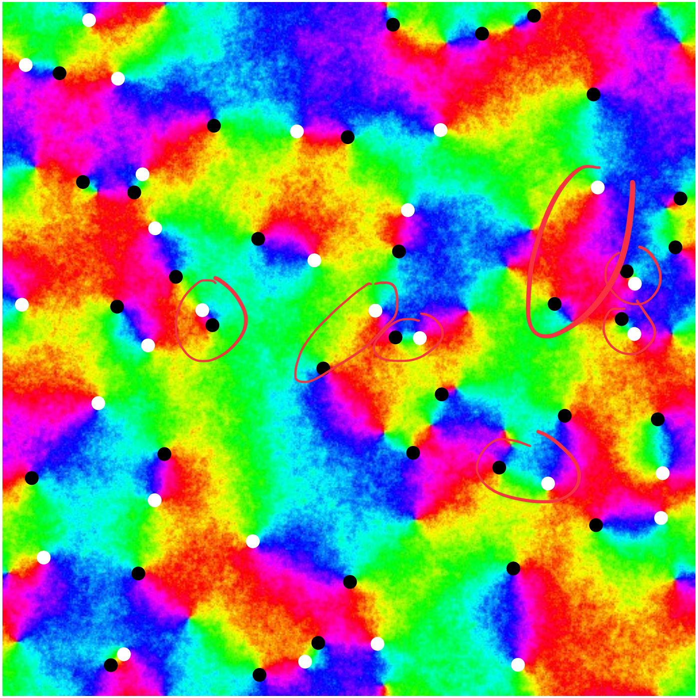


~~2-forms $q : \mathbb{R}^2 \setminus \{z_i\}$~~

The physics way (→ “**topological phase transitions**”)



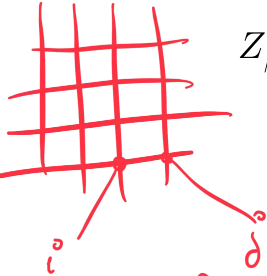
The physics way (→ **“topological phase transitions”**)



Mathwise : **duality** with integer-valued fields

Fröhlich - Spencer 80's

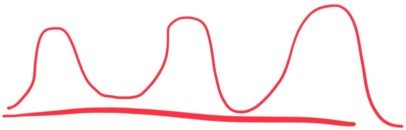
XY model



$$Z_\beta^{XY} = \int_{[0,2\pi)^\Lambda} \prod_{i\sim j} e^{\beta\cos(\theta_i-\theta_j)} \prod d\theta_i$$

$$= \int_{[0,2\pi)^\Lambda} \prod_{i\sim j} \left(\sum_k \hat{f}_\beta(k) e^{ik(d\theta)_{ij}} \right) \prod d\theta_i$$

$\theta \mapsto e^{\beta \cos\theta}$



XY model

$$Z_\beta^{XY} = \int_{[0,2\pi)^\Lambda} \prod_{i\sim j} e^{\beta\cos(\theta_i-\theta_j)} \prod d\theta_i$$

Fourier

$$= \int_{[0,2\pi)^\Lambda} \prod_{i\sim j} \left(\sum_k \hat{f}_\beta(k) e^{ik(d\theta)_{ij}} \right) \prod d\theta_i$$

$e^{-\frac{\beta}{2}\theta^2}$

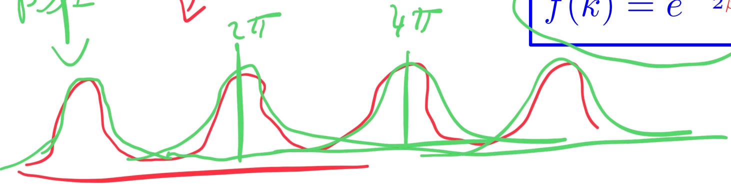
k^{th} modified Bessel functions

$$f(\theta_i-\theta_j) = e^{\beta\cos(\theta_i-\theta_j)}$$

$$f(\theta_i-\theta_j) := \sum_{m\in\mathbb{Z}} e^{-\frac{\beta}{2}(2\pi m+\theta_i-\theta_j)^2}$$

$$\boxed{\hat{f}(k) = e^{-\frac{k^2}{2\beta}}}$$

Poisson

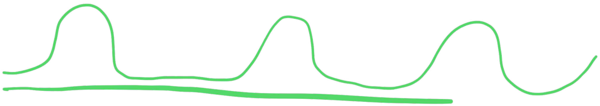


Mathwise : **duality** with integer-valued fields

Villain model

$$Z_\beta^{\text{Villain}} = \int_{[0,2\pi)^\Lambda} \prod_{i\sim j} \sum_{m\in\mathbb{Z}} e^{-\frac{\beta}{2}(2\pi m+\theta_i-\theta_j)^2} \prod d\theta_i$$

$$= \int_{[0,2\pi)^\Lambda} \prod_{i\sim j} \left(\sum_k e^{-\frac{k^2}{2\beta}} e^{ik(d\theta)_{ij}} \right) \prod d\theta_i$$



Mathwise : **duality** with integer-valued fields

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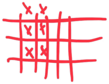
$$= \int_{[0,2\pi)^\Lambda} \prod_{i\sim j} \Big(\sum_k e^{-\frac{k^2}{2\beta}} e^{ik(d\theta)_{ij}} \Big) \prod d\theta_i$$

$$= \sum_{\mathbf{k}\in\mathbb{Z}^{E_\Lambda}} \prod_{i\sim j} e^{-\frac{\mathbf{k}_{ij}^2}{2\beta}} \prod_{x\in\Lambda} \int_{[0,2\pi)} e^{i(\nabla\cdot\mathbf{k})_x\theta} d\theta$$

Duality

① $\Lambda \to \Lambda^*$



② ~~tem~~ Inversion of length

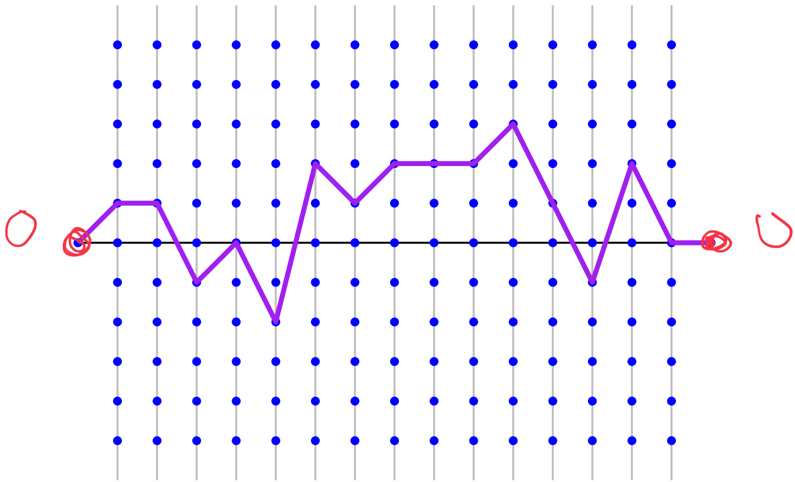
$\beta_{\text{Villain}} \mapsto \frac{1}{\beta}$

Integer-Valued Gaussian Free Field (IV-GFF):

$\psi : \Lambda^\star \to \mathbb{Z}$

$$\mathbb{P}[\psi] \propto \exp\Big(-\frac{1}{2\beta} \sum_{f\sim g} (\psi_f - \psi_g)^2\Big)$$

Integer-valued GFF in $d = 1$

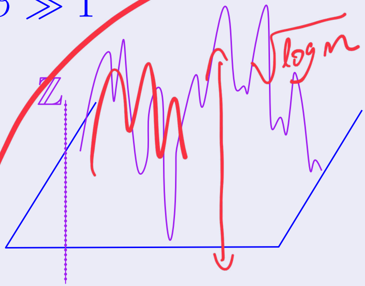


$$Z^{\mathrm{IV}}_{1/\beta} = \sum_{\mathbf{m}\in\mathbb{Z}^{\Lambda},\mathbf{m}_{|\partial\Lambda}\equiv 0} \exp\left(-\frac{1}{2\beta}\sum_{i\sim j}(\mathbf{m}_i-\mathbf{m}_j)^2\right)$$

Theorem (**Fröhlich-Spencer**, 1981)

Rough phase

$\beta \gg 1$

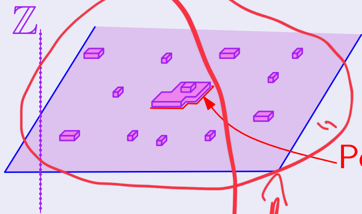


Delocalised phase

$\beta \ll 1$

Villain

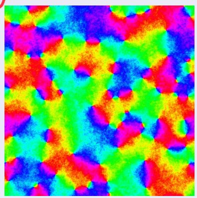
$\beta_{\text{ir-eff}} \gg 1$



T_{BKT}

$$\langle \sigma_x \, \sigma_y \rangle_\beta \geq \frac{1}{|x-y|^{\alpha(\beta)}}$$

FOURIER



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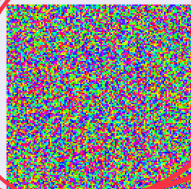


Theorem (**Fröhlich-Spencer**, 1981)

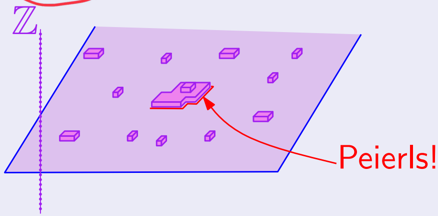
Rough phase

$\beta \gg 1$

$$\mathrm{Var}[\psi(0)] \geq \frac{1-\epsilon}{2\pi\beta} \log n$$

Delocalised phase

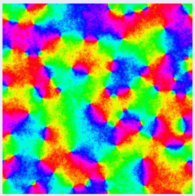
$\beta \ll 1$

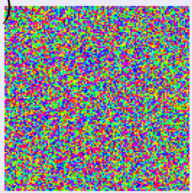


T_{BKT}

$$\langle \sigma_x \, \sigma_y \rangle_\beta \geq \frac{1}{|x-y|^{\alpha(\beta)}}$$

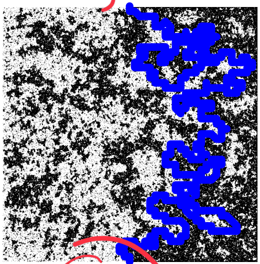
$$\frac{1}{2\pi\beta} \leq \alpha(\beta) \leq \frac{1}{2\pi\beta}(1+\epsilon(\beta))$$





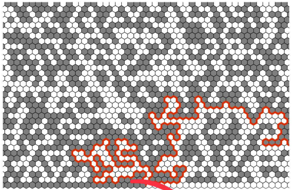
Large scale structures for $\mathbb{S}^1$ spin systems ??

Ising



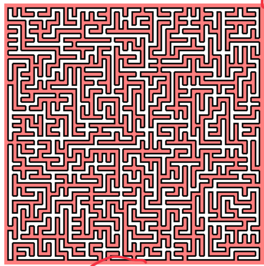
SLE_3

Percolat°



SLE_6

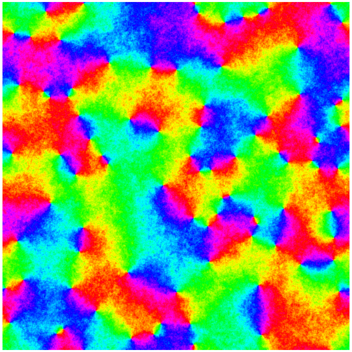
UST



SLE_8

continuous symmetry

$O(2)$

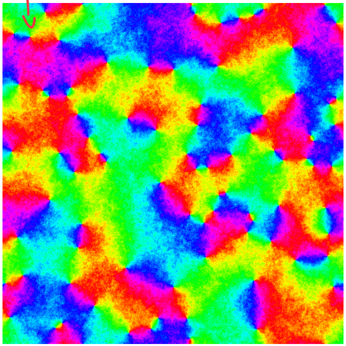


Macroscopic structures ?

Curves ?

Spin-waves and vortices

$x \in \mathbb{Z}^2 \to \sigma_x \in \mathbb{S}^1$ $\nearrow$ $e^{i\theta_x}$



Conjecture (Fröhlich-Spencer 1983)

$\beta^\circ = \beta_{\text{effective}}$

$$\{e^{i\theta_x}\}_x \sim \mathbb{P}_\beta^{\mathrm{Villain}}$$

$$\overset{d}{\approx} \{e^{i\frac{1}{\sqrt{\beta^\star}}\phi_x}\}_x, \ \phi \sim \mathbb{P}_{\mathbb{Z}^2}^{\mathrm{GFF}}$$

Complex Gaussian Multiplicative Chaos

Theorem (G., Sepúlveda, 2020)

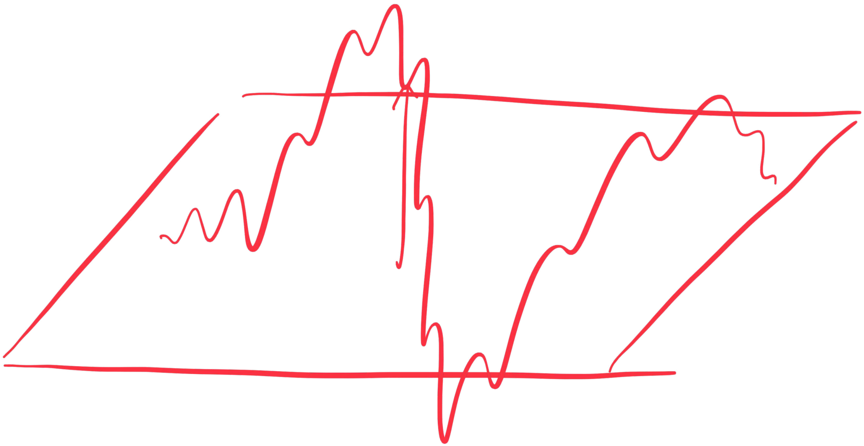
Vortices contribute to the log-fluctuations.

- $\beta^* \leq \beta - e^{-4\beta}$
- $\langle \sigma_x \sigma_y \rangle_\beta^{\mathrm{Villain}} \leq \left(\frac{1}{|x-y|}\right)^{\frac{1}{2\pi\beta} + e^{-4\beta}}$ *Equivalently* $\boxed{\varepsilon(\beta) \geq e^{-4\beta}}$

Maximum of the integer-valued GFF

Theorem (Wirth, 2019)

$$\mathbb{P}^{\mathrm{IV}}_{\beta,\Lambda,0}\left[\max_{x\in\Lambda_n}\psi(x) \geq \frac{c_1}{\sqrt{\beta}}\log n\right] \geq 1-\varepsilon$$



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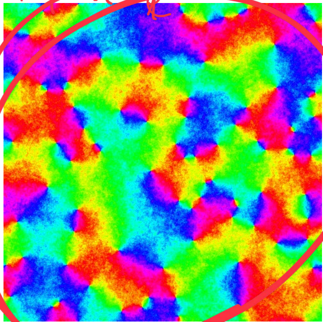
Theorem (G., Sepúlveda 2020)

$$\mathbb{P}^{\mathrm{IV}}_{\beta,\Lambda,0}\Big[\max_{x\in\Lambda_n}\psi(x) \leq \frac{1+e^{-4\beta}}{\sqrt{2\pi\beta}}2\log n\Big] \to 1$$

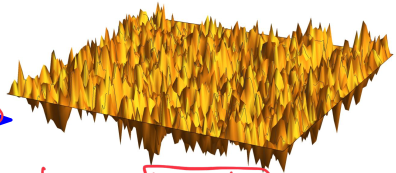
$$\frac{1}{\sqrt{2\pi\beta}}\sqrt{\log n \,(\log\log n)}$$

Statist. reconstr. of the macroscopic field ϕ given $e^{\frac{i}{\sqrt{\beta^*}}\phi}$?

$\{\sigma_x\} \propto \ldots \approx e^{i\frac{1}{\sqrt{\beta^*}}\phi_x}$



??



$\phi_x \to \boxed{e^{iT\phi_x}} \xrightarrow{?} \phi_x$

$T = \frac{1}{\sqrt{\beta^*}}$

Question we address:

- Sample $\{\phi_x\}_{x\in\Lambda} \sim \mathbb{P}^{\mathrm{GFF}}_{\Lambda}$ i.e. $\propto \exp(-\frac{1}{2}\langle \nabla\phi, \nabla\phi\rangle)$
- Let $T > 0$ $\left(T \equiv \frac{1}{\sqrt{\beta^*}}\right)$

?? $\mathrm{Law}\left(\{\phi_x\}_x \mid \{e^{iT\phi_x}\}_x\right) = \mathrm{Law}\left(\{\phi_x\}_x \mid \phi \pmod{\frac{2\pi}{T}}\right)$??

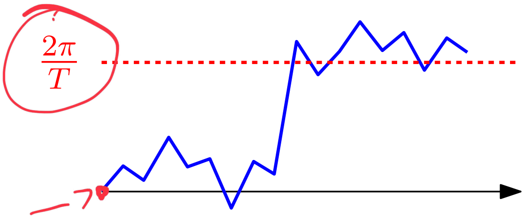
$\mathcal{L}\left(\phi \mid \phi\left(\frac{2\pi}{T}\right)\right)$

$\phi \pmod{\frac{2\pi}{T}} \rightsquigarrow \phi$ in $d = 1$?

GFF in $d=1$

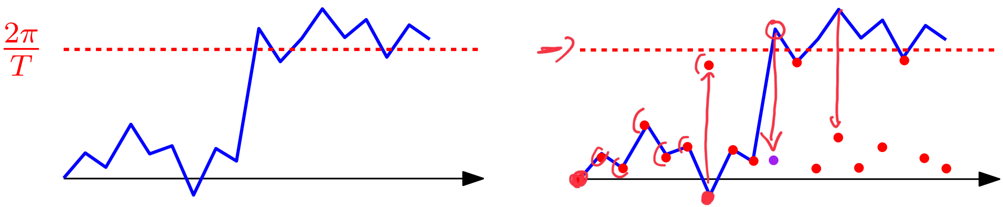
$\rightsquigarrow$ SRW on $\mathbb{Z}$

$\begin{cases} \text{starts at } 0 \\ \text{iid } \mathcal{N}(0,1) \end{cases}$



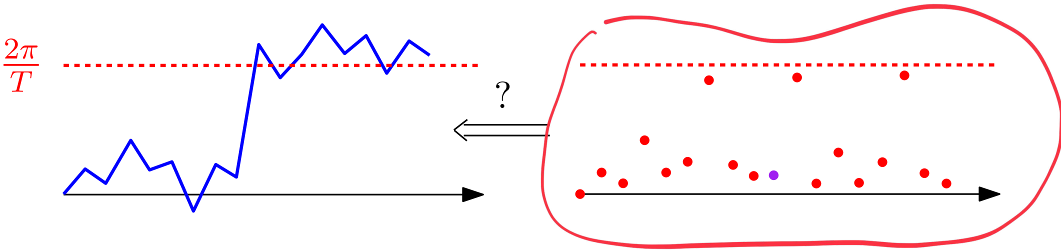
O in 0

$\phi \pmod{\frac{2\pi}{T}} \rightsquigarrow \phi$ in $d = 1$?



$$\mathbf{a} := \phi \pmod{\tfrac{2\pi}{T}} \rightsquigarrow \mathbf{a} \in [0, \tfrac{2\pi}{T})^N$$

$\phi \ (\mathrm{mod}\ \frac{2\pi}{T}) \rightsquigarrow \phi$ in $d = 1$?



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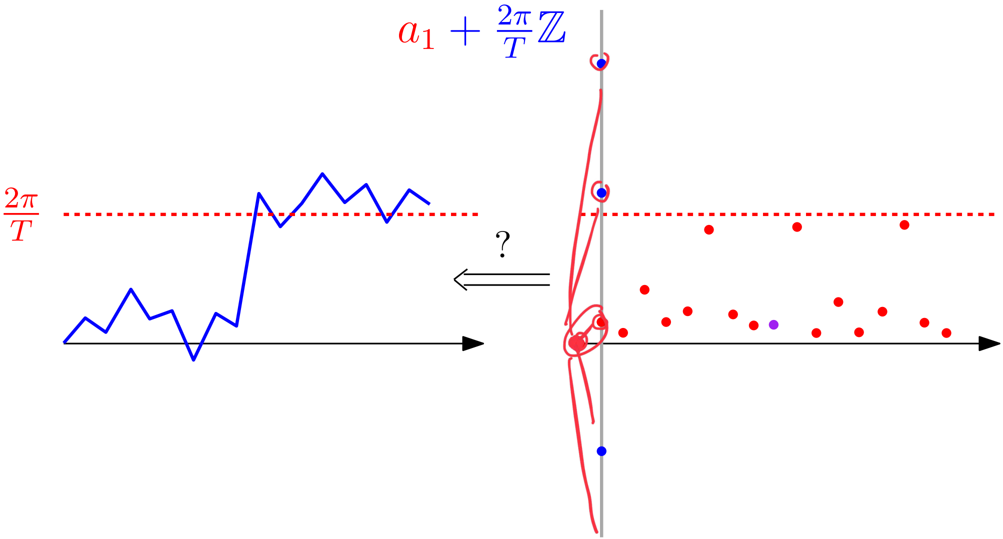

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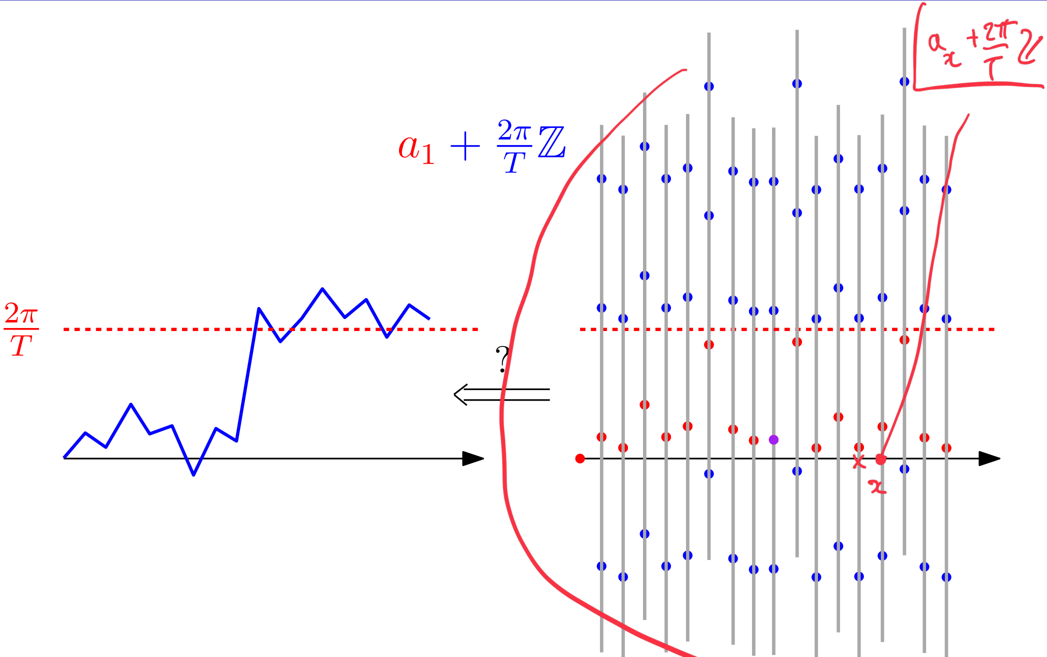

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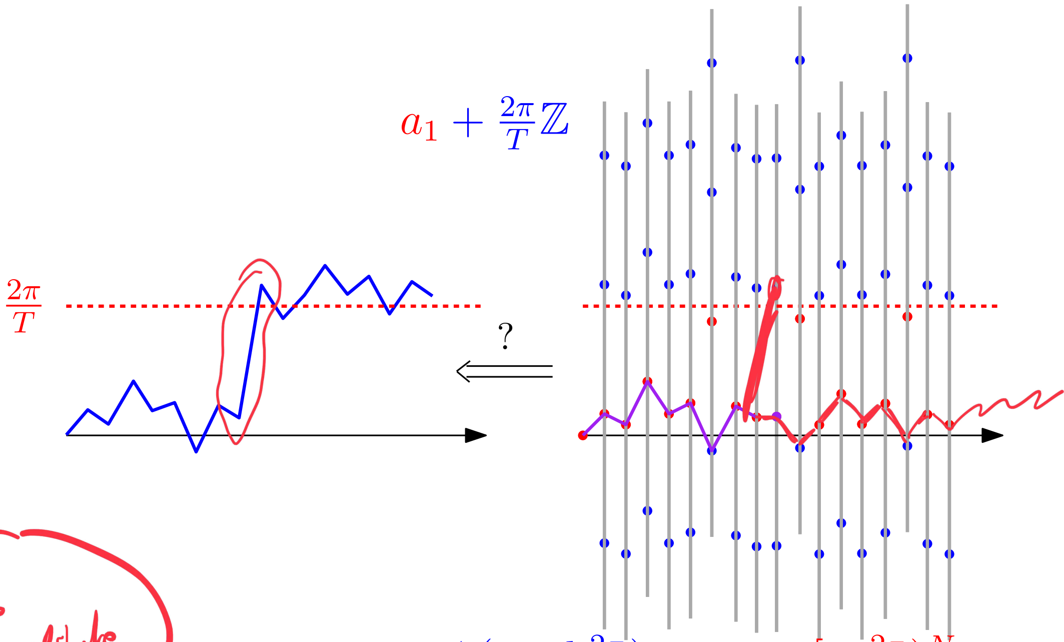

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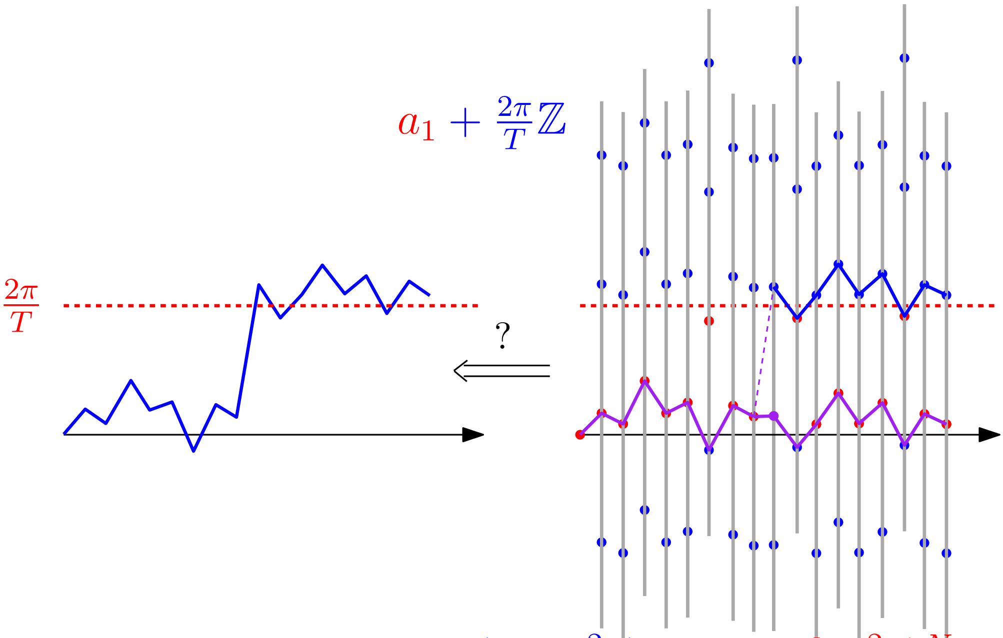
∃! groundstate

$\phi \pmod{\frac{2\pi}{T}} \rightsquigarrow \phi$ in $d = 1$?



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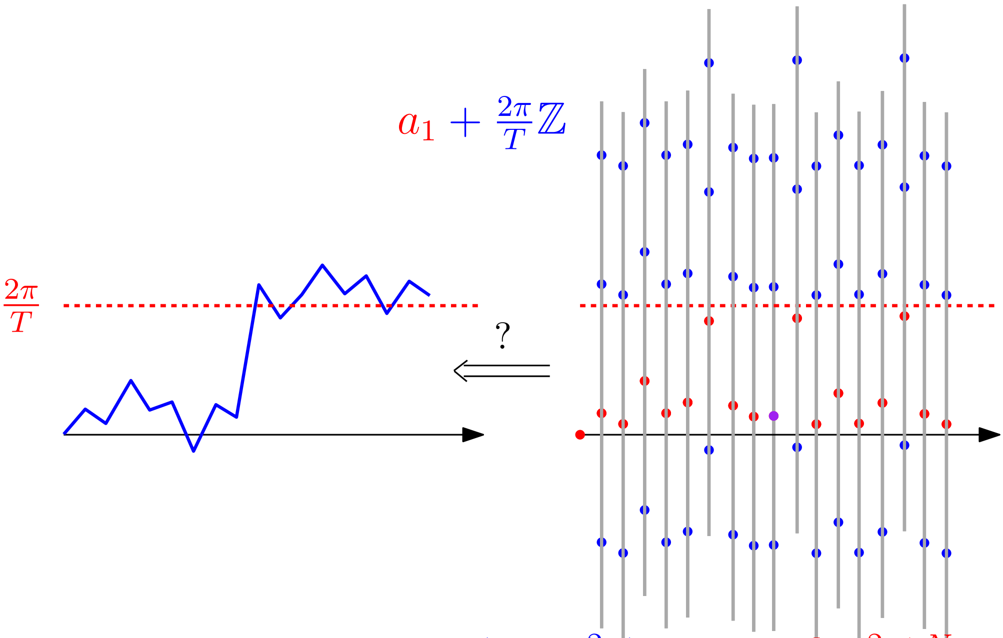

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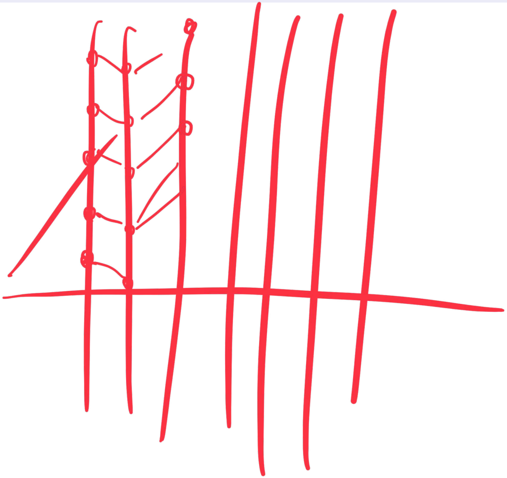
The conditional law if a **shifted** Integer-valued GFF

Definition

For any $\mathbf{a} \in [0, \frac{2\pi}{T})^{\mathbb{Z}^2}$, consider the **shifted integer-valued GFF:**

$$\mathbb{P}^{\mathbf{a},\mathrm{IV}}_{T,\Lambda}\left[d\phi\right] := \frac{1}{Z} \sum_{\mathbf{m} \in \mathbb{Z}^\Lambda, \mathbf{m}_{|\partial\Lambda} \equiv 0} \delta_{\frac{2\pi}{T}\mathbf{m}+\mathbf{a}}(d\phi) \exp(-\frac{1}{2}\langle \nabla\phi, \nabla\phi \rangle)$$



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And in $d = 2$?

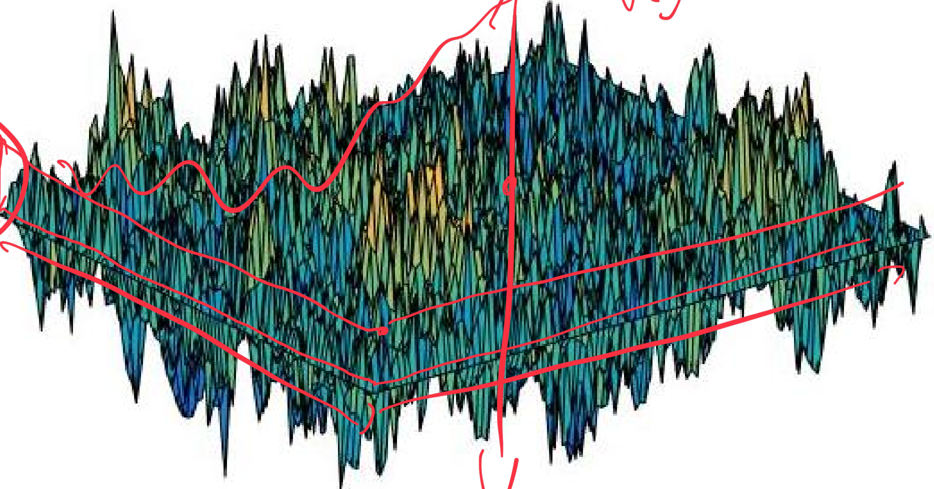
$\mathbf{a} = \phi \pmod{\frac{2\pi}{T}}$

$T \gg 1$ $\phi\left(\frac{2\pi}{T}\right)$ Powell

$T \ll 1$

$\sqrt{\log n}$

$\frac{2\pi}{T}$



Λ_n

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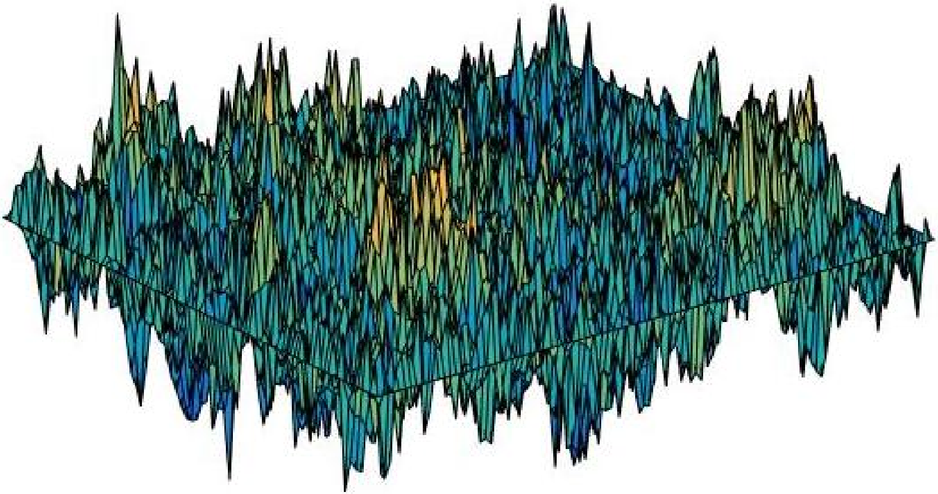
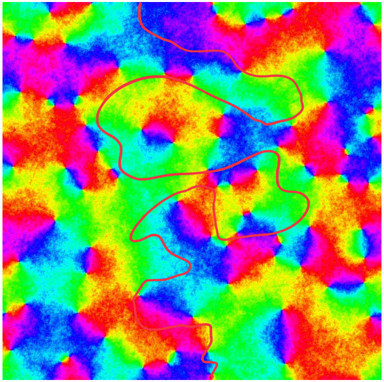


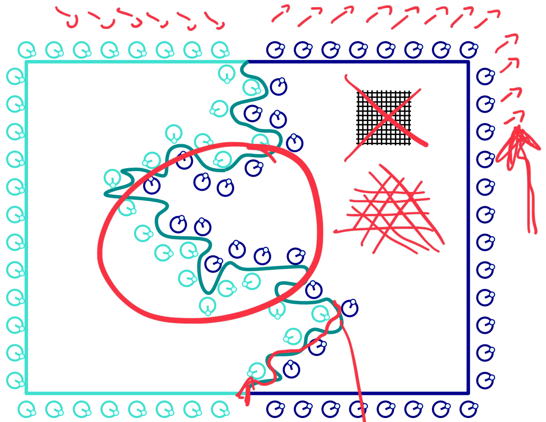
Motivations/context

$\sin(\theta_x) > 0$

Dobrushin B.C

1. Extract $GFF = GFF(\{\theta_x\}_{x\in\mathbb{Z}^2})$ when $\{\theta_x\} \sim \text{Villain}_\beta$



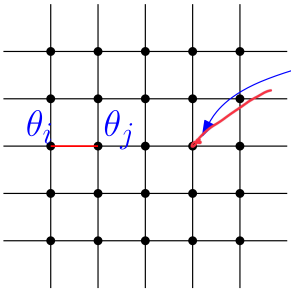


Conjecture: level lines of the **Villain model** when $T < \hat{T}_c$ converge to $\mathrm{SLE}_4, \mathrm{SLE}(4,\rho), \mathrm{ALE}$ process $\mathbb{A}_{-\lambda,\lambda}$.

$\Rightarrow \mathrm{SLE}_4(\rho)$

2 Statistical reconstruction problems.

- E. Abbe, L. Massoulie, A. Montanari, A. Sly, and N. Srivas-Tava. *Group synchronization on grids.*



$x \in \mathbb{Z}^d$

$\theta_x \in \mathfrak{S}$, compact group

$$\mathrm{Law}\left(\{\theta_x\} \,\middle|\, \theta_i \theta_j^{-1} + noise\right)$$

$$\mathrm{Law}\left(\{\phi_x\} \,\middle|\, \phi_i - \phi_j \pmod{\tfrac{2\pi}{T}}\right)$$

- Peres, Sly. *Rigidity and tolerance for perturbed lattices.*
- Etc.

Motivations/context

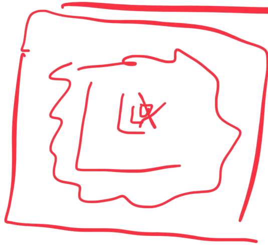
3. An "integrable model" for **IV-GFF** and a new interpretation of the KT transition.

$$\rightsquigarrow \text{ arguments in favor of } \varepsilon(\beta) \asymp \frac{1}{\beta} e^{-1/\beta}$$

4. Integer-valued random fields have seen an intense activity lately
   Square-ice model
   Uniform graph homomorphisms $\mathbb{Z}^2 \to \mathbb{Z}$

   - Duminil-Copin, Glazman, Peled, Spinka, 2017
   - Chandgotia, Peled, Sheffield, Tassy, 2018
   - Glazman, Manolescu, 2018.
   - Duminil-Copin, Harel, Laslier, Raoufi, Ray, 2019
   - XOR -trick

$\nabla\phi \in \{-1, 0, 1\}$



$$:e^{i\beta\phi}: \rightsquigarrow \phi$$

5. Complex Multiplicative Chaos. ($\to$ **Plasma phase** of Coulomb, i.e. $\beta^2 < 8\pi$).
   - N. Berestycki, S. Sheffield, X. Sun.
     $\boxed{:e^{\gamma\Phi}: \rightsquigarrow \Phi}$, $\forall \gamma < \gamma_c = 2$
   -

$\beta^2 = 4\pi$

$$\boxed{:e^{iT\Phi}: \rightsquigarrow \Phi \;\; ??}$$

6. Link with the **Random Phase Sine-Gordon** model

Random Phase Sine-Gordon model

Definition

Fix a **quenched disorder** $\mathbf{a} \sim$ i.i.d in $[0,1]^{\mathbb{Z}^2}$ and define the following quenched SG measure:

$$\mathbb{P}_{\beta}^{\mathbf{a},\mathrm{SG}}\left[d\phi\right] := \frac{1}{Z}\exp\left(-\frac{\beta}{2}\sum_{x\sim y}(\phi(x)-\phi(y))^2 + z\sum_{x}\cos(\phi(x)-\mathbf{a}(x))\right)$$

Random Phase Sine-Gordon model

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Sketch of proof

Low temperature case $T < T_{rec}^{-}$

Sketch of proof

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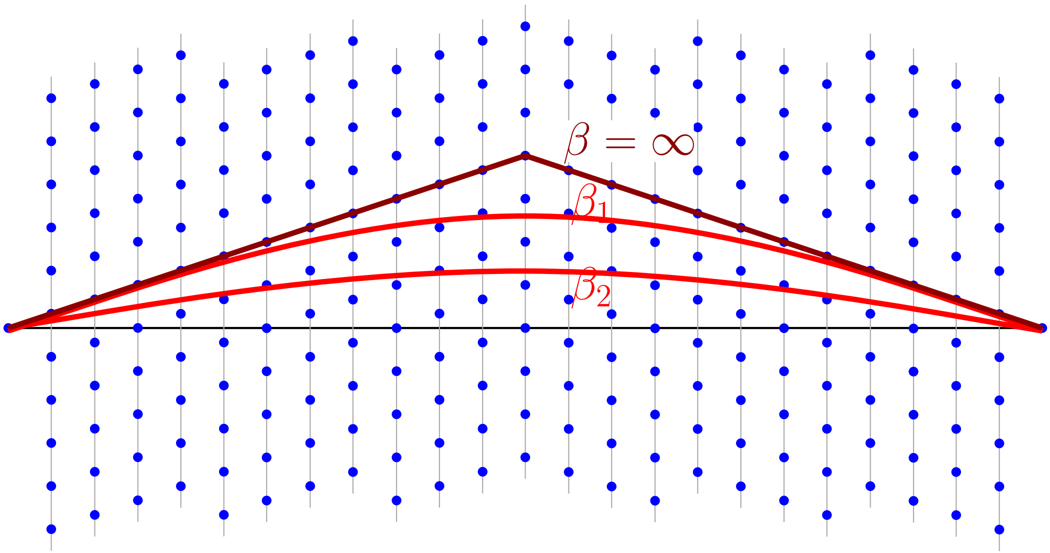


Sketch of proof

Low temperature case $T < T_{rec}^{-}$

Sketch of proof

Low temperature case $T < T_{rec}^{-}$

Sketch of proof

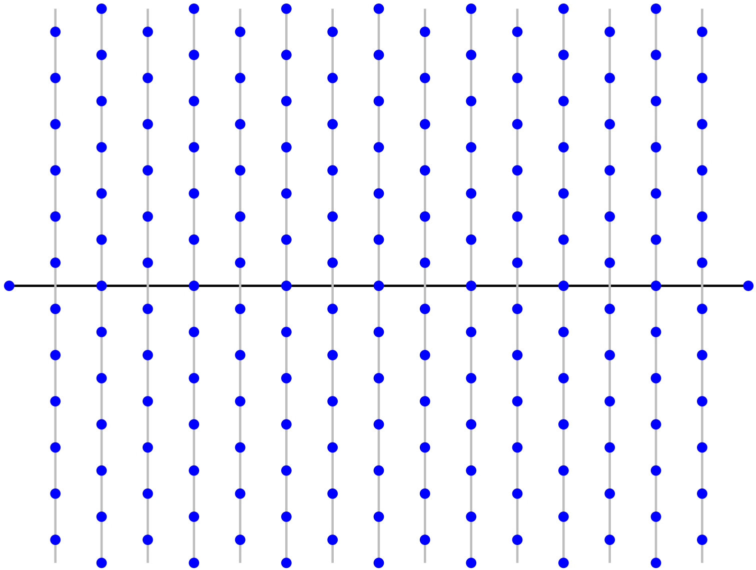
High temperature case $T > T_{rec}^{+}$

Sketch of proof

High temperature case $T > T^{+}_{rec}$

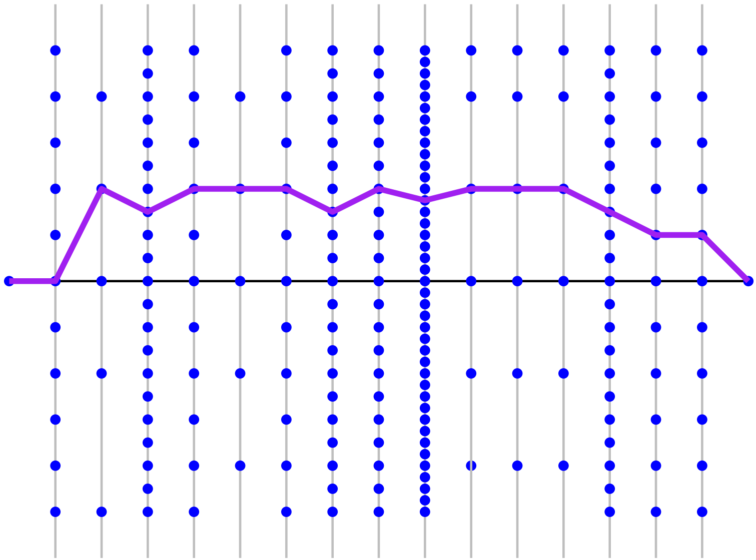
Sketch of proof

High temperature case $T > T^{+}_{rec}$



Sketch of proof

High temperature case $T > T_{rec}^{+}$



Sketch of proof

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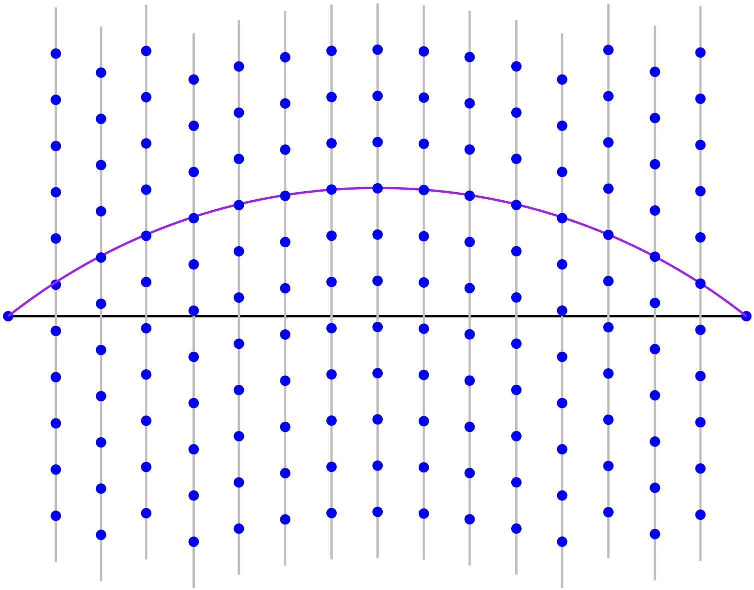

Sketch of proof

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Sketch of proof

High temperature case $T > T^{+}_{rec}$

Thank you!