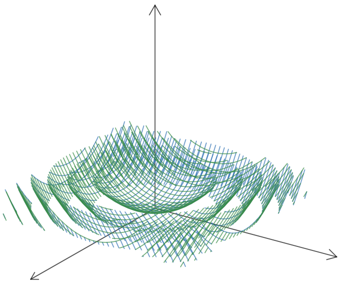
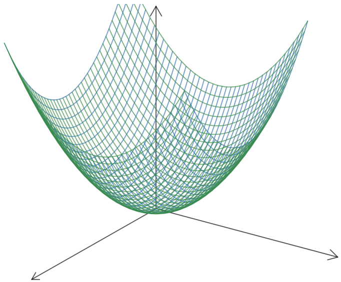
Statistical reconstruction of the Gaussian free field and Kosterlitz-Thouless transition

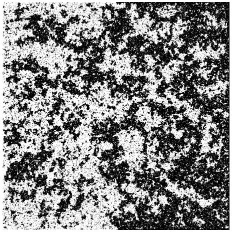
Christophe Garban (Univ Lyon 1)
joint with Avelio Sepúlveda (Univ Lyon 1)





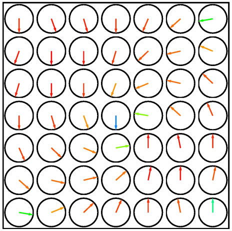
Spin systems on $\mathbb{Z}^2$

Ising model



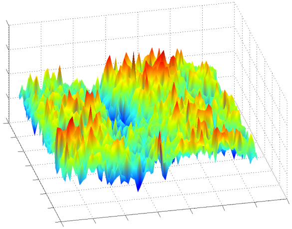
$\sigma \in \{-1, 1\}^{\mathbb{Z}^2}$

XY model



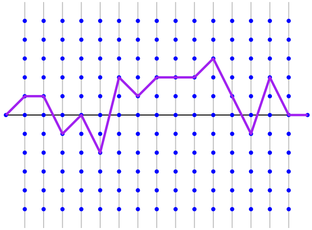
$\sigma : \mathbb{Z}^2 \to \mathbb{S}^1$

GFF



$\sigma (= \phi) : \mathbb{Z}^2 \to \mathbb{R}$

Int-valued GFF



$\phi : \mathbb{Z}^2 \to \mathbb{Z}$

Gibbs measure

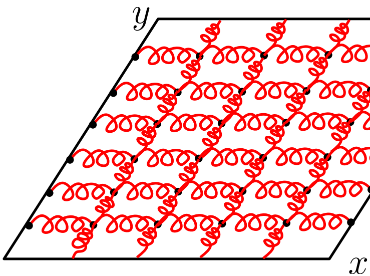
$$\mathbb{P}_\beta(\sigma) \propto \exp\Big(-\frac{\beta}{2} \sum_{i \sim j} \|\sigma_i - \sigma_j\|^2\Big) \left(= \frac{1}{Z_\beta} \exp\Big(-\frac{\beta}{2} \sum_{i \sim j} \|\sigma_i - \sigma_j\|^2\Big) \right)$$

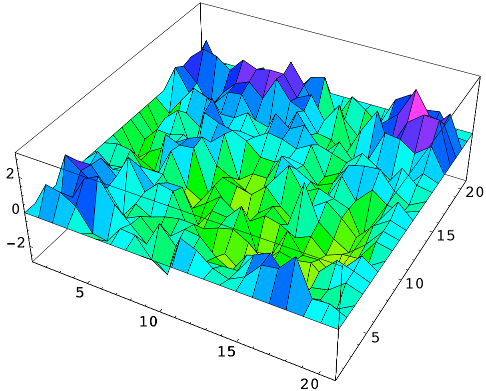
Gaussian Free Field (GFF)

Definition

On $\Lambda_n := \frac{1}{n}\mathbb{Z}^2 \cap [-1,1]^2$

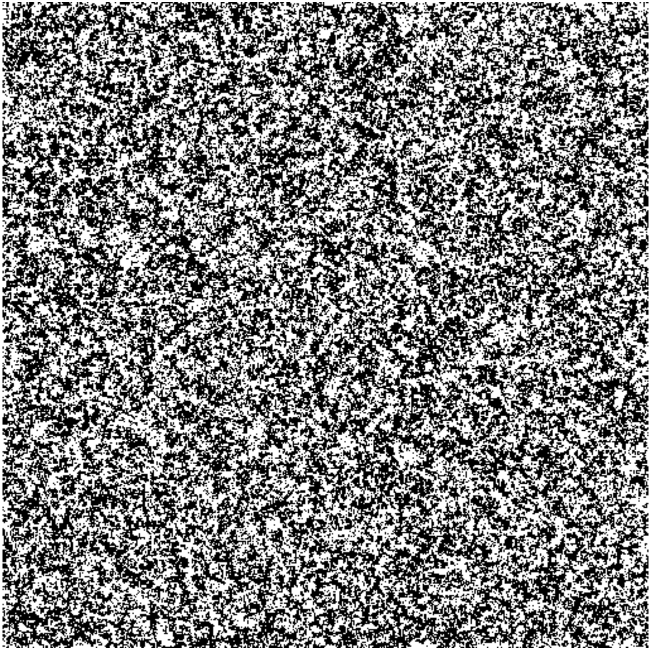
$$d\mathbb{P}_\beta^{\mathrm{GFF}}[\phi] \propto \exp\Big(-\frac{\beta}{2}\sum_{x\sim y}(\phi(x)-\phi(y))^2\Big)\prod_{x\in\Lambda_n} d\phi(x)$$



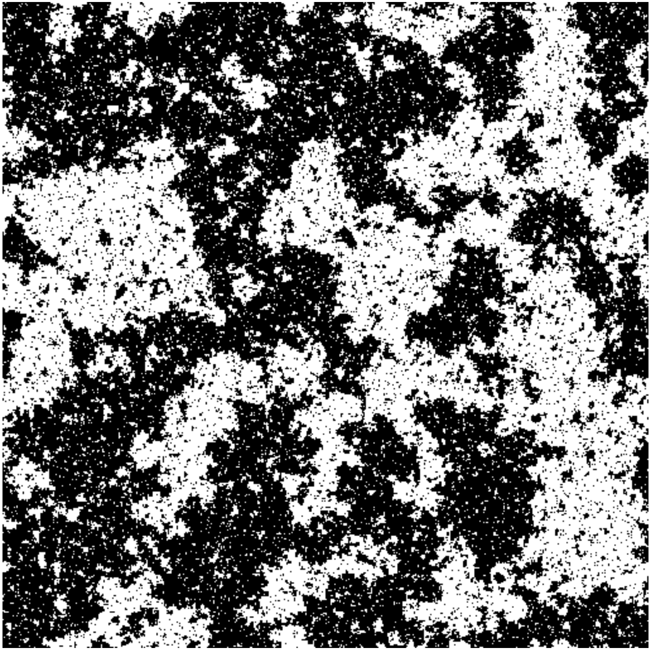


$$\mathrm{Var}\big[\phi((0,0))\big] \sim \frac{1}{\beta 2\pi}\log n$$

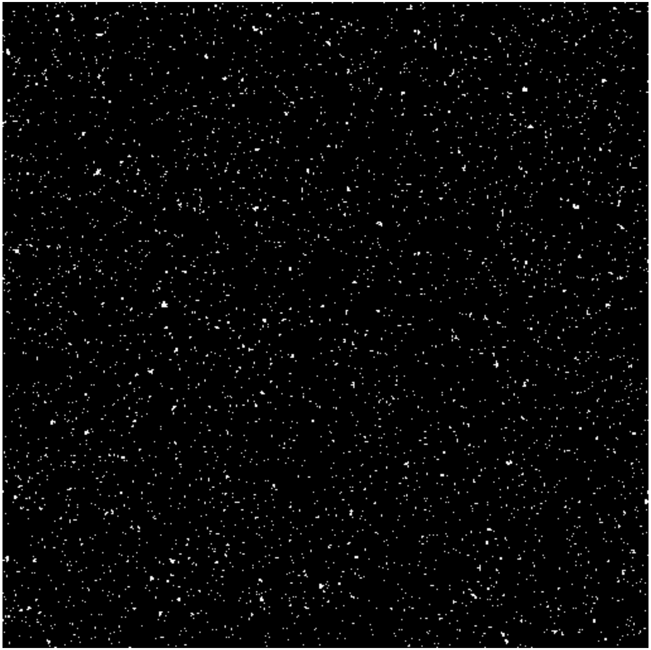
Ising model, $\sigma \in \{-1,1\}^\Lambda$, $T \gg 1$



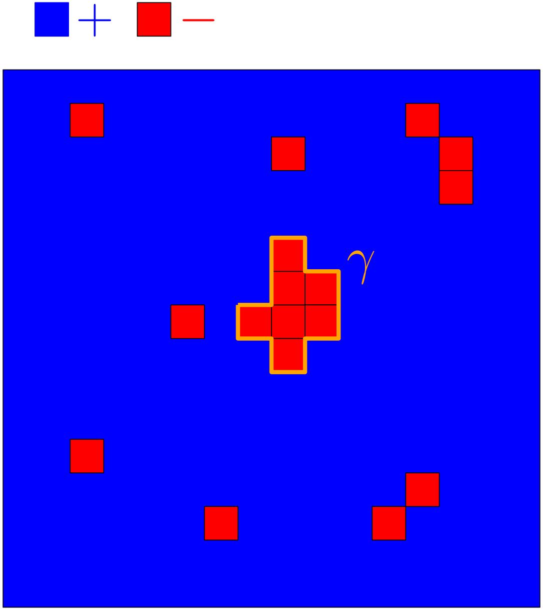
Ising model, $\sigma \in \{-1, 1\}^\Lambda$, $T \approx T_c$



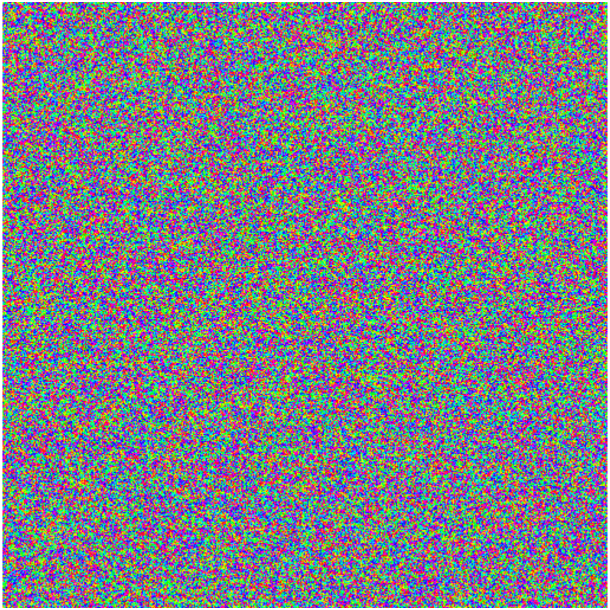
Ising model, $\sigma \in \{-1, 1\}^\Lambda$, $T \ll 1$



Long-Range-Order ≡ Peierls argument



XY model, $\sigma \in (\mathbb{S}^1)^\Lambda$, $T \gg 1$



$x \in \Lambda, \ \sigma_x \in \mathbb{S}^1$

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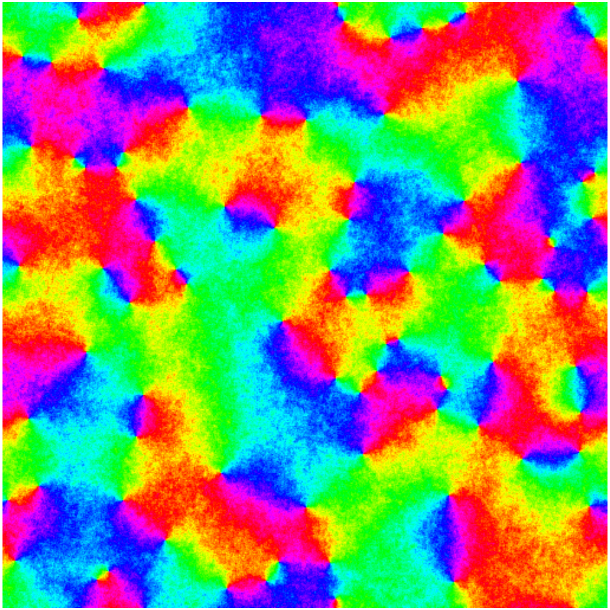
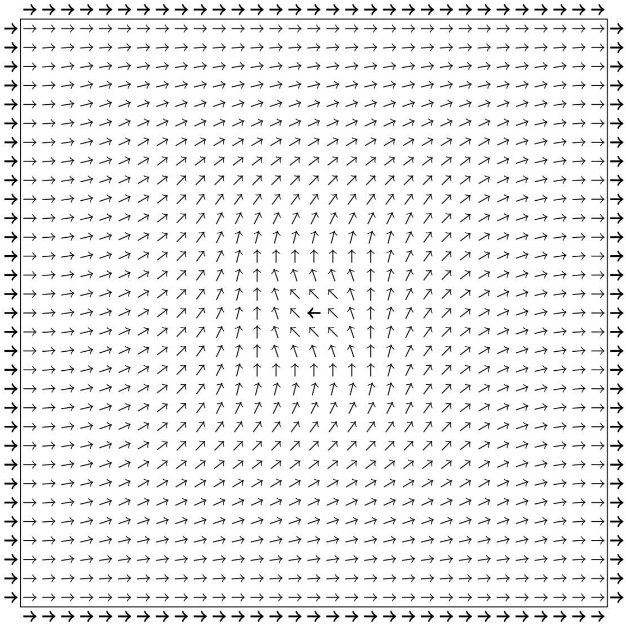
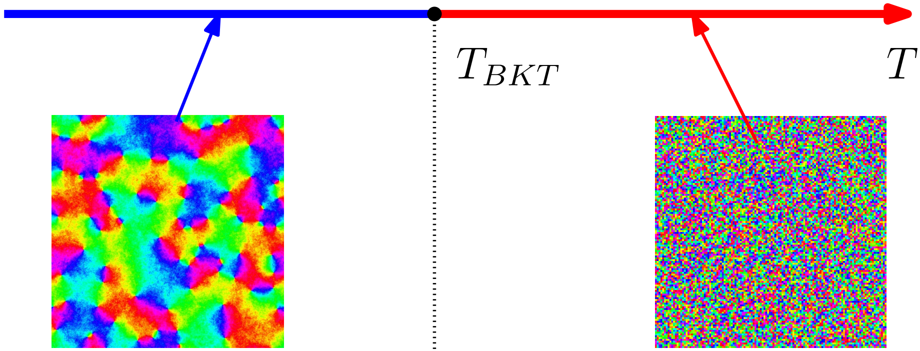


$x \in \Lambda, \ \sigma_x \in \mathbb{S}^1$

No Long-Range-Order ? Spin-waves!



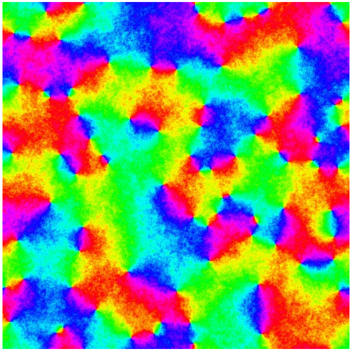
BKT transition



$$\langle \sigma_x \sigma_y \rangle_\beta \approx \frac{1}{|x-y|^{\alpha(T)}}$$

$$\langle \sigma_x \sigma_y \rangle_\beta \approx e^{-c(T)\,|x-y|}$$

The physics way ($\to$ "topological phase transitions")



$$P_\beta(\sigma) \propto \exp(-\frac{\beta}{2}\sum_{i\sim j}\|\sigma_i - \sigma_j\|^2)$$

$$\propto \exp(\beta \sum_{i\sim j}\cos(\theta_i - \theta_j))$$

$$\propto\approx \exp(-\frac{\beta}{2}\sum_{i\sim j}(\theta_i - \theta_j)^2)$$

GFF !!

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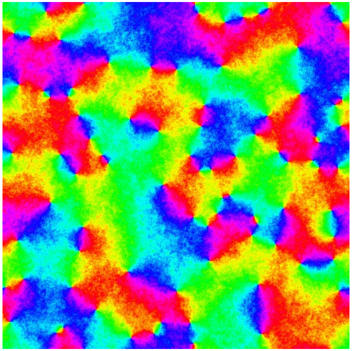

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$$\exp(-\tfrac{\beta}{2}\int(d\theta)^2)$$

$d\theta$ $\equiv$ 1-form on “$\mathbb{R}^2$”

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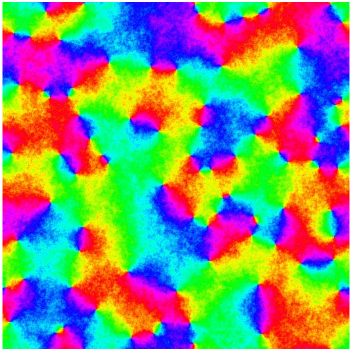

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d

0-forms $\phi : \mathbb{R}^2 \to \mathbb{R}$

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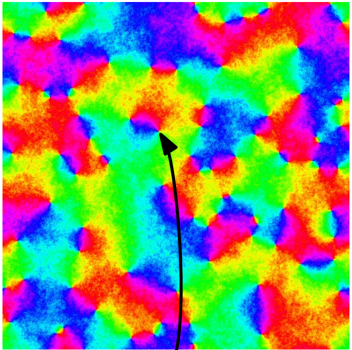

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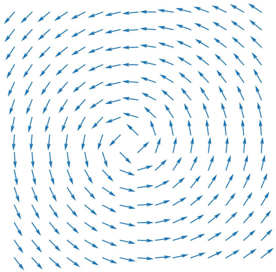


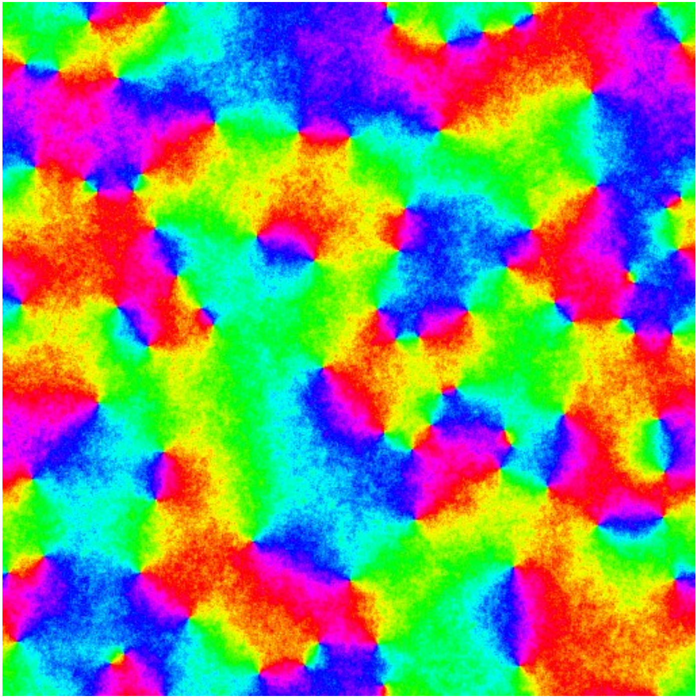
d

$d^\star$

0-forms $\phi : \mathbb{R}^2 \to \mathbb{R}$

2-forms $q : \mathbb{R}^2 \setminus \{z_i\}$

The physics way (→ "**topological phase transitions**")



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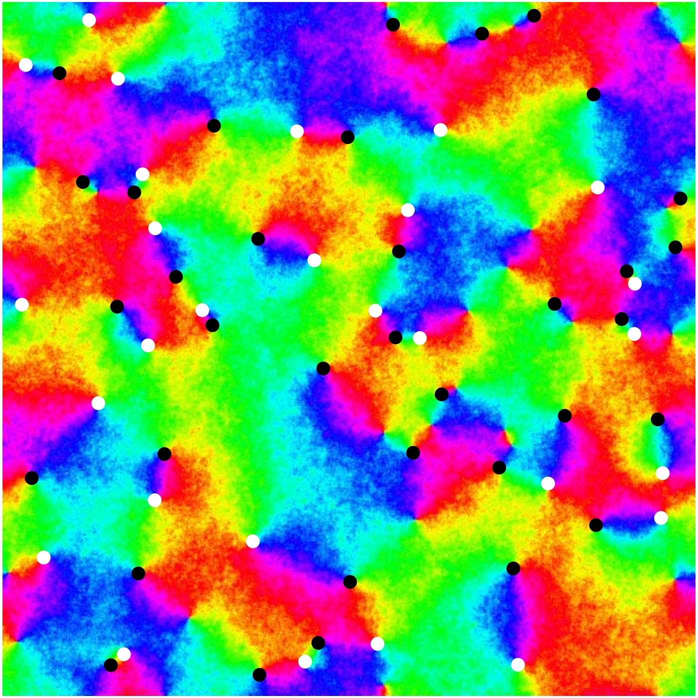

Mathwise : **duality** with integer-valued fields

XY model

$$Z_\beta^{XY} = \int_{[0,2\pi)^\Lambda} \prod_{i\sim j} e^{\beta \cos(\theta_i - \theta_j)} \prod d\theta_i$$

$$= \int_{[0,2\pi)^\Lambda} \prod_{i\sim j} \left(\sum_k \hat{f}_\beta(k) e^{ik(d\theta)_{ij}} \right) \prod d\theta_i$$

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$f(\theta_i - \theta_j) = e^{\beta\cos(\theta_i-\theta_j)}$ $\rightsquigarrow$ $f(\theta_i - \theta_j) := \sum_{m\in\mathbb{Z}} e^{-\frac{\beta}{2}(2\pi m+\theta_i-\theta_j)^2}$

$$\boxed{\hat f(k) = e^{-\frac{k^2}{2\beta}}}$$

Mathwise : **duality** with integer-valued fields

Villain model

$$Z_\beta^{\text{Villain}} = \int_{[0,2\pi)^\Lambda} \prod_{i\sim j} \sum_{m\in\mathbb{Z}} e^{-\frac{\beta}{2}(2\pi m+\theta_i-\theta_j)^2} \prod d\theta_i$$

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$$= \int_{[0,2\pi)^\Lambda} \prod_{i\sim j} \left(\sum_k e^{-\frac{k^2}{2\beta}} e^{ik(d\theta)_{ij}} \right) \prod d\theta_i$$

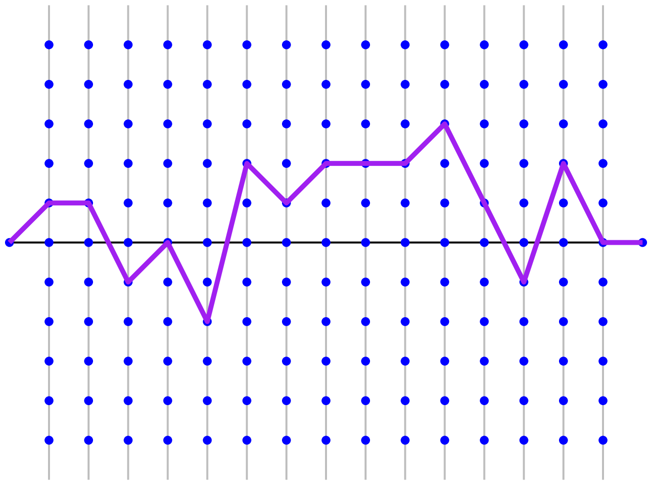
$$= \sum_{\mathbf{k}\in\mathbb{Z}^{E_\Lambda}} \prod_{i\sim j} e^{-\frac{\mathbf{k}_{ij}^2}{2\beta}} \prod_{x\in\Lambda} \int_{[0,2\pi)} e^{i(\nabla\cdot\mathbf{k})_x\theta} d\theta$$

Integer-Valued Gaussian Free Field (IV-GFF):

$\psi : \Lambda^\star \to \mathbb{Z}$

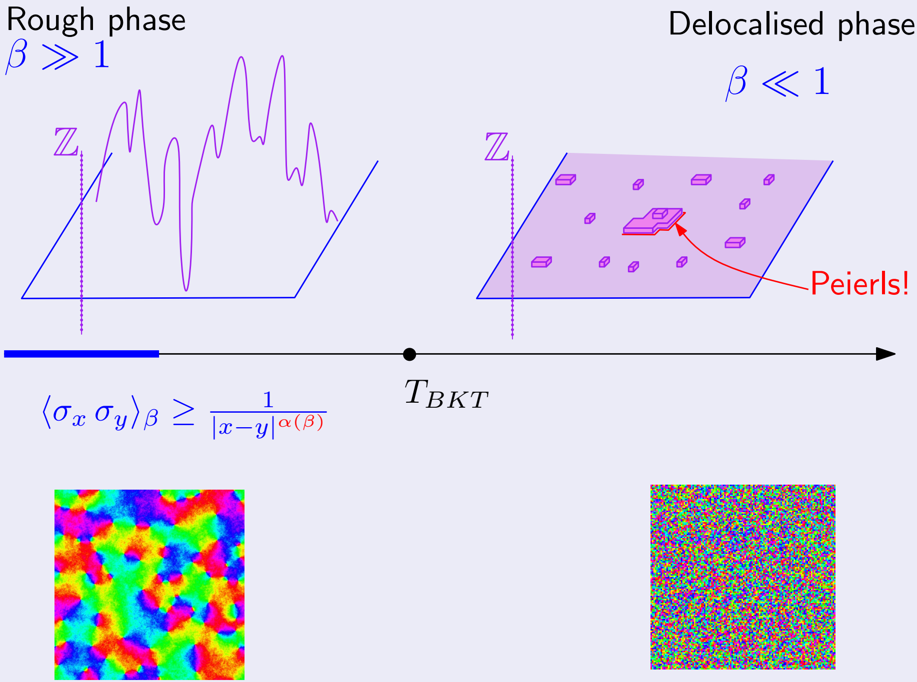
$$\mathbb{P}[\psi] \propto \exp(-\tfrac{1}{2\beta} \textstyle\sum_{f\sim g}(\psi_f - \psi_g)^2)$$

Integer-valued GFF in $d = 1$



$$Z^{\mathrm{IV}}_{1/\beta} = \sum_{\mathbf{m} \in \mathbb{Z}^{\Lambda}, \mathbf{m}_{|\partial\Lambda \equiv 0}} \exp\left(-\frac{1}{2\beta} \sum_{i \sim j} (\mathbf{m}_i - \mathbf{m}_j)^2 \right)$$

Theorem (**Fröhlich-Spencer**, 1981)



Theorem (**Fröhlich-Spencer**, 1981)

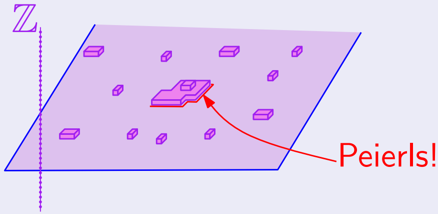
Rough phase

$\beta \gg 1$

$\mathrm{Var}[\psi(0)] \geq \frac{1-\epsilon}{2\pi\beta} \log n$

Delocalised phase

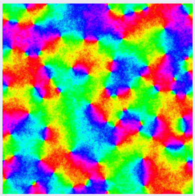
$\beta \ll 1$

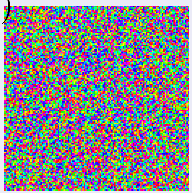


T_{BKT}

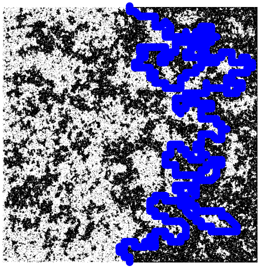
$\langle \sigma_x \, \sigma_y \rangle_\beta \geq \frac{1}{|x-y|^{\alpha(\beta)}}$

$\frac{1}{2\pi\beta} \leq \alpha(\beta) \leq \frac{1}{2\pi\beta}(1 + \epsilon(\beta))$

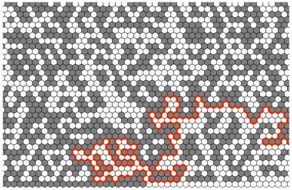




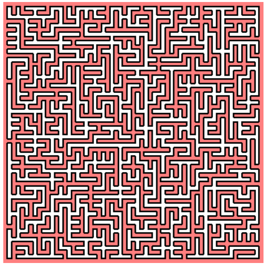
Large scale structures for $\mathbb{S}^1$ spin systems ??



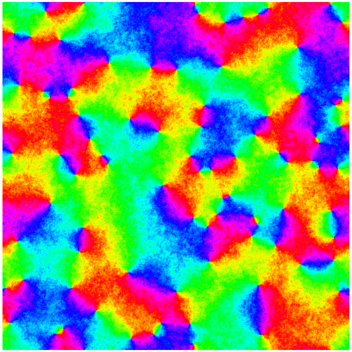
SLE_3



SLE_6



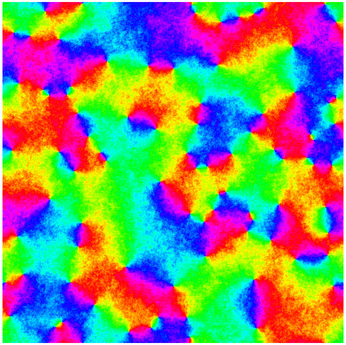
SLE_8



Macroscopic structures ?

Curves ?

Spin-waves and vortices



Conjecture (Fröhlich-Spencer 1983)

$$\{e^{i\theta_x}\}_x \sim \mathbb{P}_\beta^{\mathrm{Villain}}$$

$$\overset{d}{\approx} \{e^{i\frac{1}{\sqrt{\beta^\star}}\phi_x}\}_x, \;\; \phi \sim \mathbb{P}_{\mathbb{Z}^2}^{\mathrm{GFF}}$$

Theorem (G., Sepúlveda, 2020)

Vortices contribute to the log-fluctuations.

- $\beta^* \leq \beta - e^{-4\beta}$
- $\langle \sigma_x \, \sigma_y \rangle_\beta^{\mathrm{Villain}} \leq \left(\frac{1}{|x-y|}\right)^{\frac{1}{2\pi\beta} + e^{-4\beta}}$ *Equivalently* $\boxed{\varepsilon(\beta) \geq e^{-4\beta}}$

Maximum of the integer-valued GFF

Theorem (Wirth, 2019)

$$\mathbb{P}^{\mathrm{IV}}_{\beta,\Lambda,0}\Big[\max_{x\in\Lambda_n}\psi(x) \geq \frac{c_1}{\sqrt{\beta}}\log n\Big] \geq 1-\varepsilon$$

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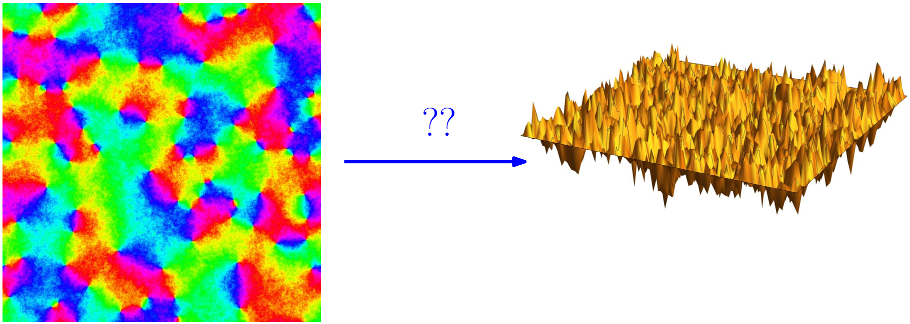
$$\mathbb{P}^{\mathrm{IV}}_{\beta,\Lambda,0}\Big[\max_{x\in\Lambda_n}\psi(x)\geq\frac{c_1}{\sqrt{\beta}}\log n\Big]\geq 1-\varepsilon$$

Theorem (G., Sepúlveda 2020)

$$\mathbb{P}^{\mathrm{IV}}_{\beta,\Lambda,0}\Big[\max_{x\in\Lambda_n}\psi(x)\leq\frac{1-e^{-4\beta}}{\sqrt{2\pi\beta}}2\log n\Big]\to 1$$

Statist. reconstr. of the macroscopic field ϕ given $e^{\frac{i}{\sqrt{\beta^*}}\phi}$?

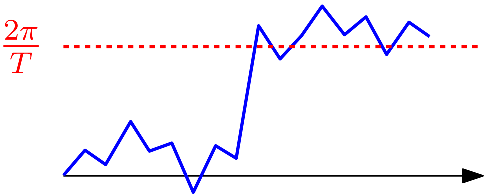


Question we address:

- Sample $\{\phi_x\}_{x\in\Lambda} \sim \mathbb{P}^{\mathrm{GFF}}_{\Lambda}$ i.e. $\propto \exp(-\frac{1}{2}\langle\nabla\phi, \nabla\phi\rangle)$
- Let $T > 0 \quad (T \equiv \frac{1}{\sqrt{\beta^*}})$

?? $\mathrm{Law}\left(\{\phi_x\}_x \mid \{e^{iT\phi_x}\}_x\right) = \mathrm{Law}\left(\{\phi_x\}_x \mid \phi \pmod{\frac{2\pi}{T}}\right)$??

$\phi \pmod{\frac{2\pi}{T}} \rightsquigarrow \phi$ in $d = 1$?



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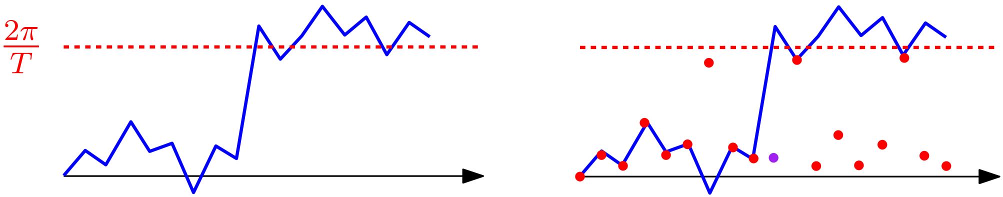

$$\mathbf{a} := \phi \pmod{\tfrac{2\pi}{T}} \rightsquigarrow \mathbf{a} \in [0, \tfrac{2\pi}{T})^N$$

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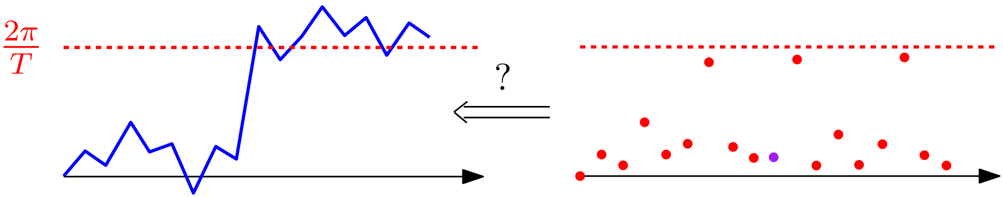

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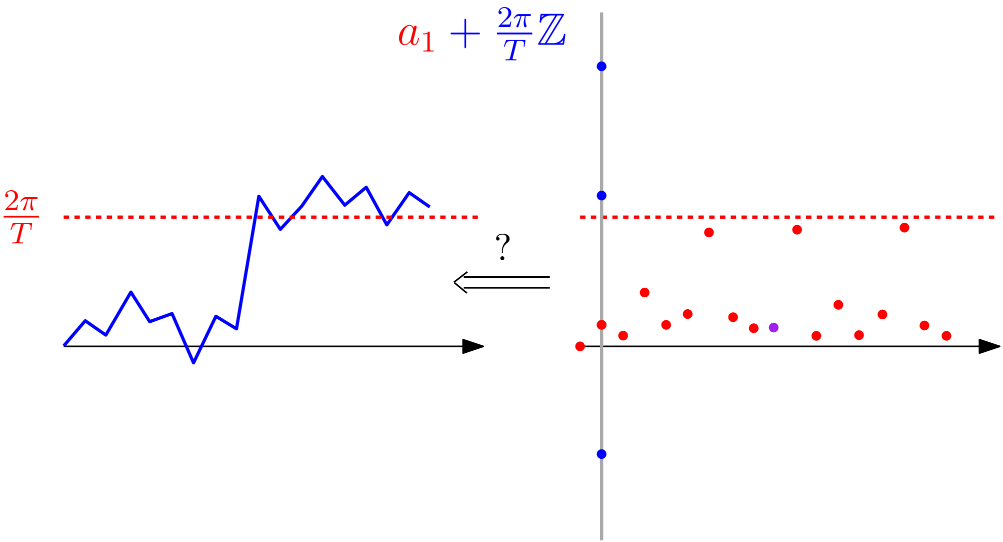

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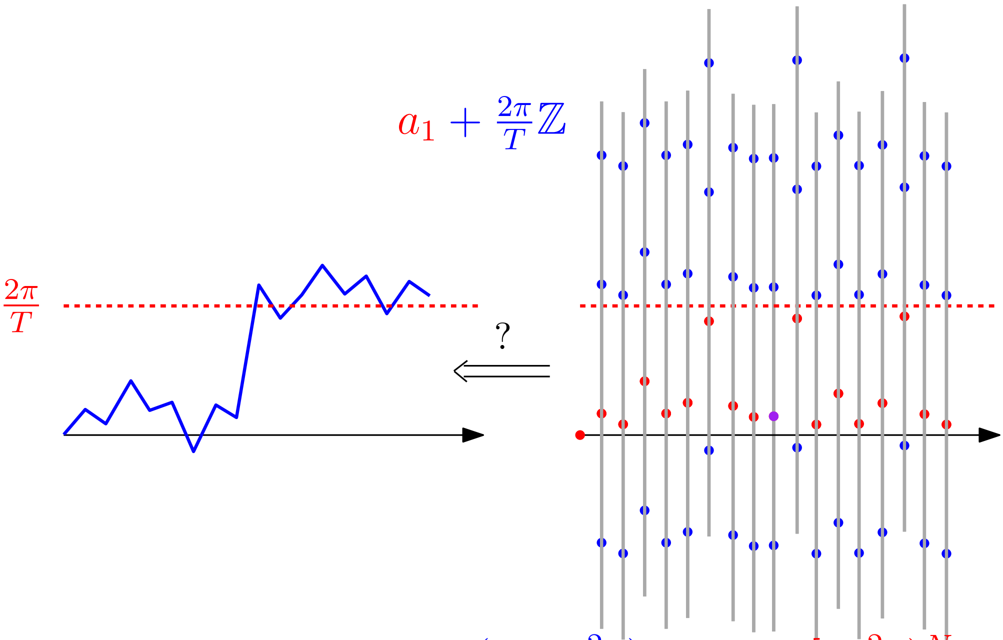

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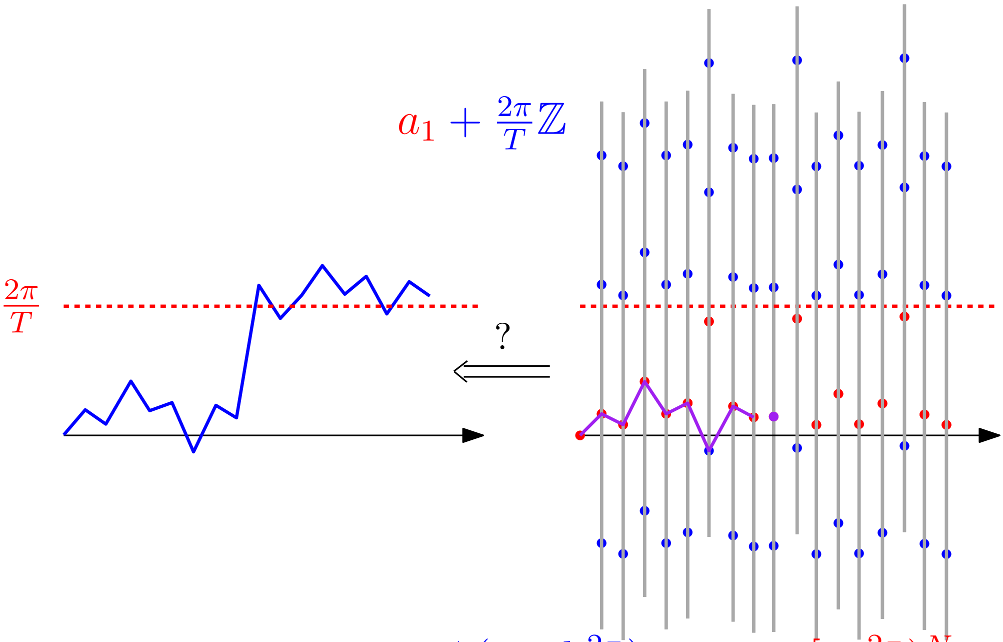

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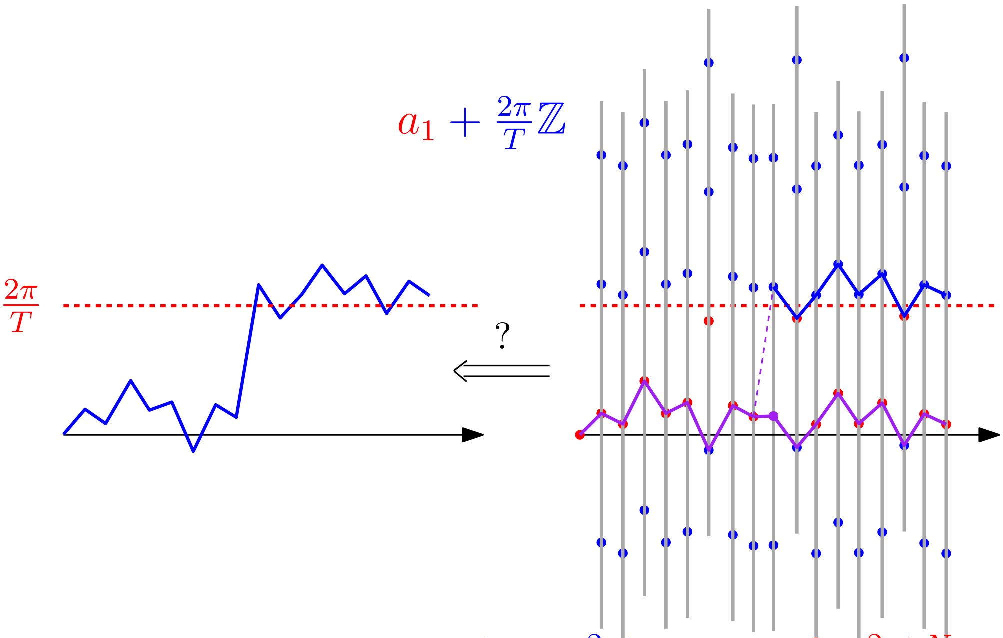

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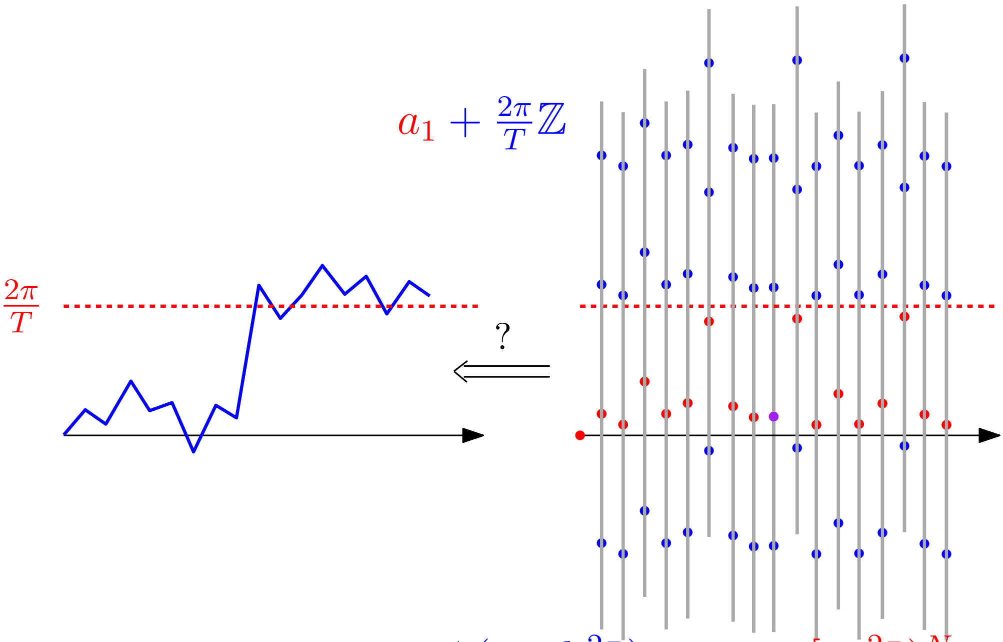

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The conditional law if a **shifted** Integer-valued GFF

Definition

For any $\mathbf{a} \in [0, \frac{2\pi}{T})^{\mathbb{Z}^2}$, consider the **shifted integer-valued GFF**:

$$\mathbb{P}^{\mathbf{a},\mathrm{IV}}_{T,\Lambda}\left[d\phi\right] := \frac{1}{Z} \sum_{\mathbf{m}\in\mathbb{Z}^\Lambda, \mathbf{m}_{|\partial\Lambda}\equiv 0} \delta_{\frac{2\pi}{T}\mathbf{m}+\mathbf{a}}(d\phi) \exp\left(-\frac{1}{2}\langle\nabla\phi, \nabla\phi\rangle\right)$$

The conditional law if a **shifted** Integer-valued GFF

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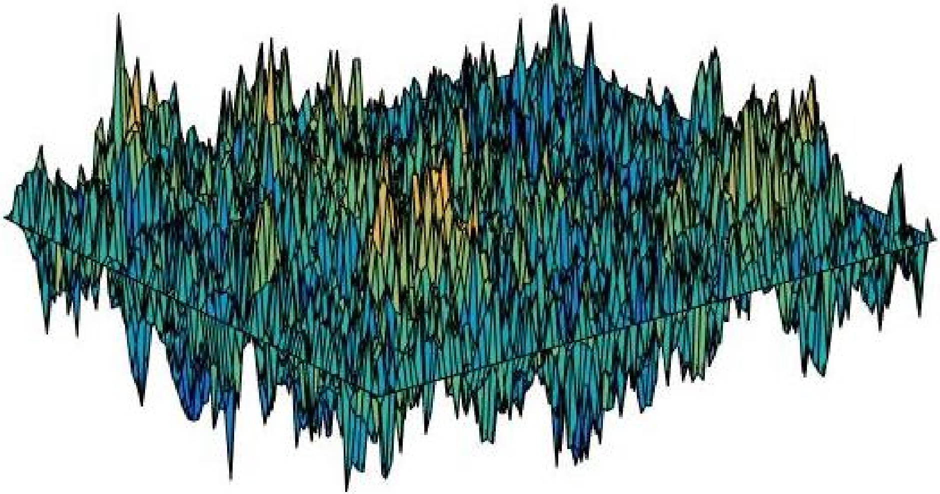
And in $d = 2$?

$\mathbf{a} = \phi \pmod{\frac{2\pi}{T}}$



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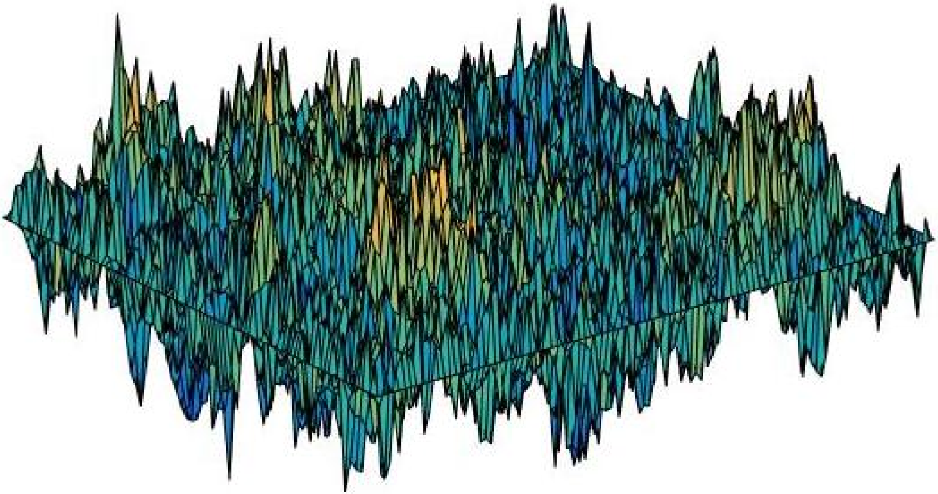
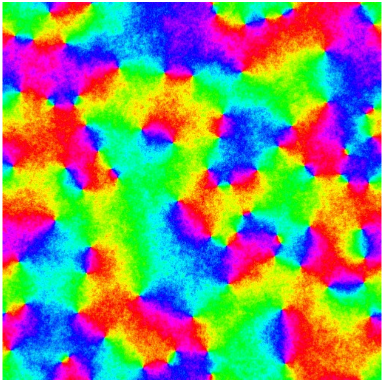
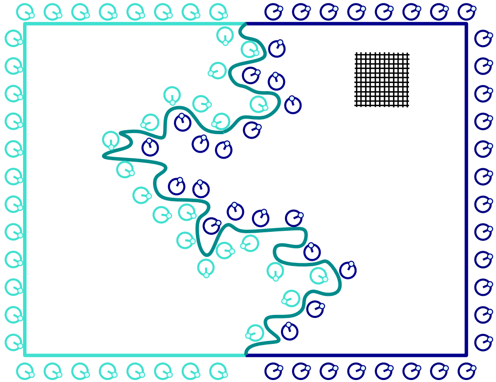


1. Extract $GFF = GFF(\{\theta_x\}_{x\in\mathbb{Z}^2})$ when $\{\theta_x\} \sim \text{Villain}_\beta$

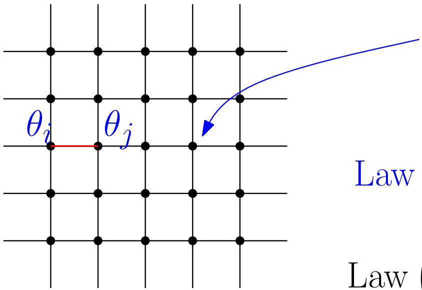




Conjecture: level lines of the **Villain model** when $T < \hat{T}_c$ converge to $\mathrm{SLE}_4, \mathrm{SLE}(4,\rho), \mathrm{ALE}$ process $\mathbb{A}_{-\lambda,\lambda}$.

2 Statistical reconstruction problems.

- E. Abbe, L. Massoulie, A. Montanari, A. Sly, and N. Srivas-Tava. *Group synchronization on grids.*



$x \in \mathbb{Z}^d$

$\theta_x \in \mathfrak{S}$, compact group

$$\text{Law}\left(\{\theta_x\} \,\middle|\, \theta_i \theta_j^{-1} + \textit{noise} \right)$$

$$\text{Law}\left(\{\phi_x\} \,\middle|\, \phi_i - \phi_j \, (\text{mod}\, \tfrac{2\pi}{T}) \right)$$

- Peres, Sly. *Rigidity and tolerance for perturbed lattices.*
- Etc.

Motivations/context

3. An "integrable model" for **IV-GFF** and a new interpretation of the KT transition.

$$\rightsquigarrow \text{ arguments in favor of } \varepsilon(\beta) \asymp \frac{1}{\beta} e^{-1/\beta}$$

4. Integer-valued random fields have seen an intense activity lately
   Square-ice model
   Uniform graph homomorphisms $\mathbb{Z}^2 \to \mathbb{Z}$
   - Duminil-Copin, Glazman, Peled, Spinka, 2017
   - Chandgotia, Peled, Sheffield, Tassy, 2018
   - Glazman, Manolescu, 2018.
   - Duminil-Copin, Harel, Laslier, Raoufi, Ray, 2019

Motivations/context

5. Complex Multiplicative Chaos. ($\to$ **Plasma phase** of Coulomb, i.e. $\beta^2 < 8\pi$).
   - N. Berestycki, S. Sheffield, X. Sun.

$$\boxed{:e^{\gamma\Phi}: \rightsquigarrow \Phi}\,, \forall \gamma < \gamma_c = 2$$

   -

$$\boxed{:e^{iT\Phi}: \rightsquigarrow \Phi \;\; ??}$$

6. Link with the **Random Phase Sine-Gordon** model

Random Phase Sine-Gordon model

Definition

Fix a **quenched disorder** $\mathbf{a} \sim$ i.i.d in $[0,1]^{\mathbb{Z}^2}$ and define the following quenched SG measure:

$$\mathbb{P}_{\beta}^{\mathbf{a},\mathrm{SG}}[d\phi] := \frac{1}{Z} \exp\left(-\frac{\beta}{2} \sum_{x \sim y} (\phi(x) - \phi(y))^2 + z \sum_{x} \cos(\phi(x) - \mathbf{a}(x)) \right)$$

Random Phase Sine-Gordon model

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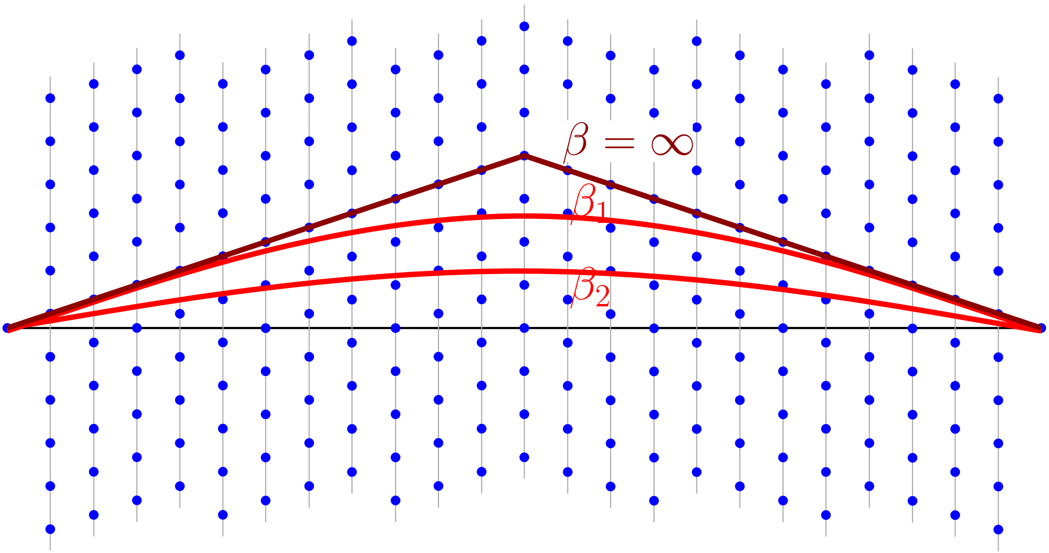
Sketch of proof

Low temperature case $T < T_{rec}^{-}$

Sketch of proof

Low temperature case $T < T_{rec}^{-}$



Sketch of proof

Low temperature case $T < T_{rec}^{-}$

Sketch of proof

Low temperature case $T < T_{rec}^{-}$

Sketch of proof

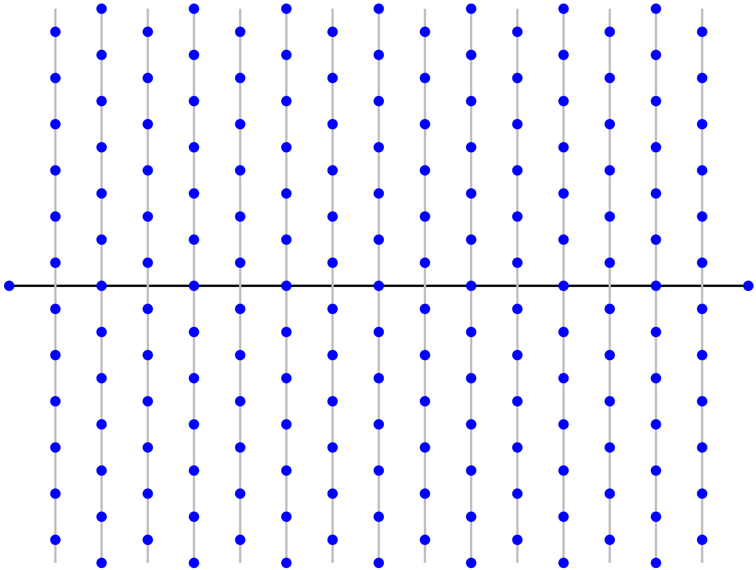
High temperature case $T > T_{rec}^{+}$

Sketch of proof

High temperature case $T > T_{rec}^{+}$

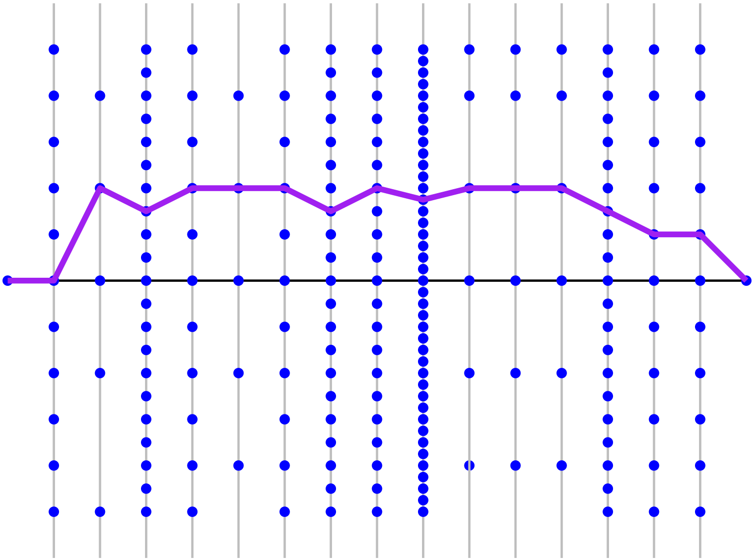
Sketch of proof

High temperature case $T > T_{rec}^{+}$



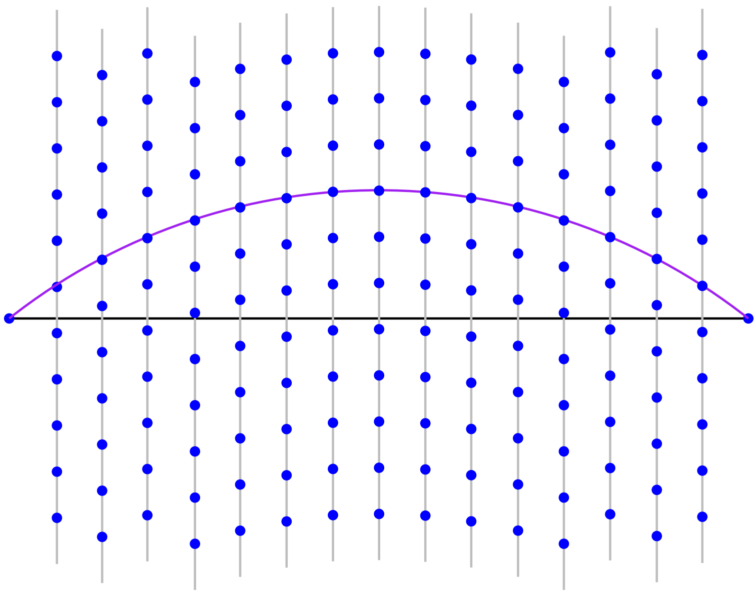
Sketch of proof

High temperature case $T > T_{rec}^{+}$



Sketch of proof

High temperature case $T > T_{rec}^{+}$



Sketch of proof

High temperature case $T > T_{rec}^{+}$

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High temperature case $T > T_{rec}^{+}$

Thank you!