Convergence in Distribution (D), Convergence in Probability (P)

and Almost-Sure convergence (AS) of Martingales

Old Stuff with a New Twist

ABSTRACT: It is well known to every student of probability that $AS \Rightarrow P \Rightarrow D$ and that in general no two of these three basic modes of convergence are equivalent. There is however one celebrated instance, due to P. Levy (1937), for which both these implications are reversible, namely for sums of independent random variables. Thus a martingale with independent increments converges **AS** if it converges **D**. How about general martingales? Well, the first example of a **P**-convergent martingale which fails to converge **AS** was given by Baez-Duarte (1971), his martingale however had unbounded increments. Subsequently, Gilat (1972) exhibited a martingale with bounded increments converging **P** but not **AS**, and another one (this time merely with unbounded increments) converging **D** but not **P**.

These issues were recently (2014) revisited by Jim Pitman from a somewhat wider perspective. Pitman addresses the more general question: to what extent do the marginal distributions of a martingale determine its convergence behavior. This question makes sense in view of Doob's basic martingale convergence theorem (an $L_1 -$bounded martingale converges **AS**). Pitman exhibits a **D**-convergent sequence of martingale marginals, such that some martingales with these marginals converge **AS**, while others diverge **AS**. So by mixing, the probability of convergence of martingales with these marginals can be any number in [0,1]. The same phenomenon occurs for **P**-convergence.

In view of these examples, it is natural to ask: (1) What is a necessary and sufficient condition on martingale marginals for every martingale with these marginals to converge **AS**? Is $L_1 - \mathit{boundedness}$, known to be sufficient, also necessary? (2) Same as (1) for **P** convergence.

In my talk I shall review the various examples and draw attention to the questions (1) & (2) which so far (to the best of my knowledge) remain open.