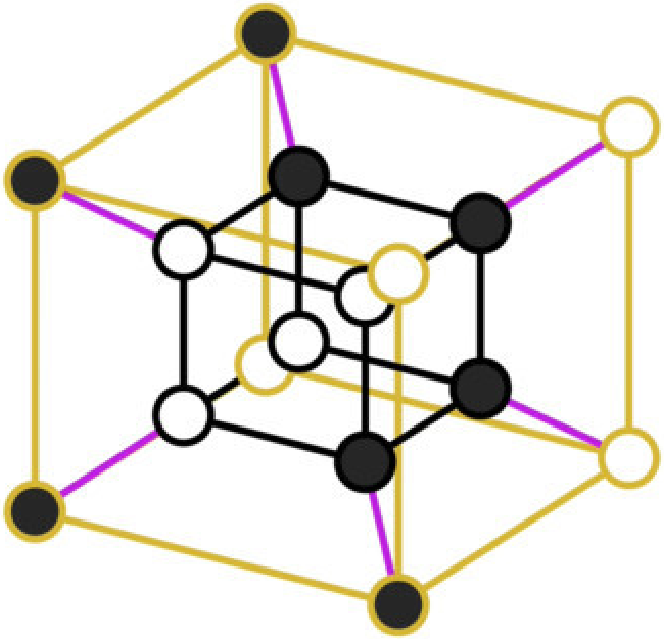
Concentration on the Boolean hypercube via pathwise stochastic analysis



Ronen Eldan Renan Gross

Weizmann Institute of Science

Horowitz Seminar, 11/05/20

Warning

This presentation shows explicit images of graphs.

Viewer discretion is advised.

What we'll do today

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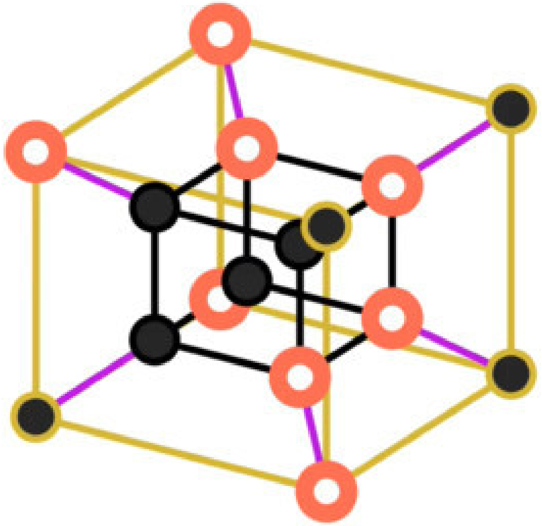

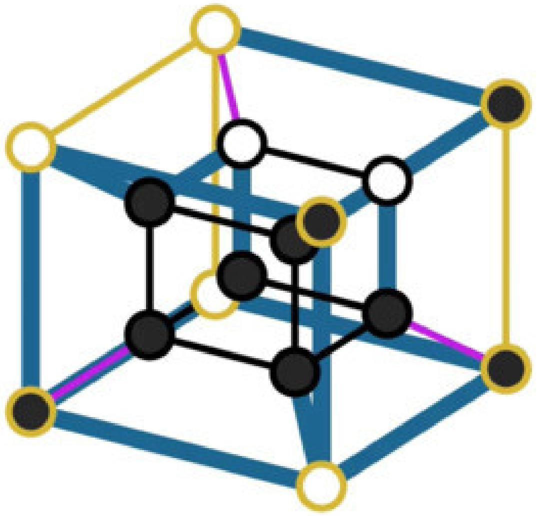

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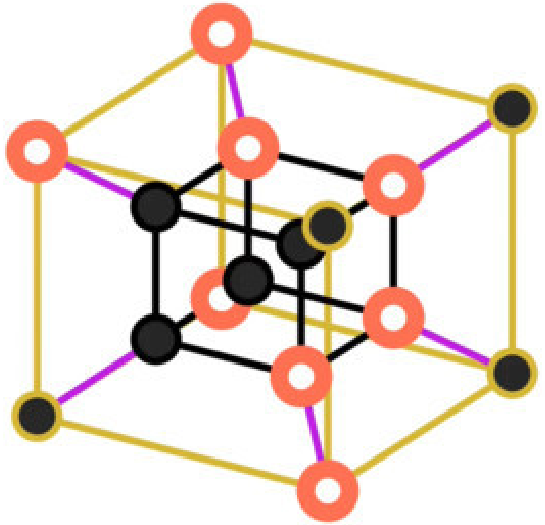

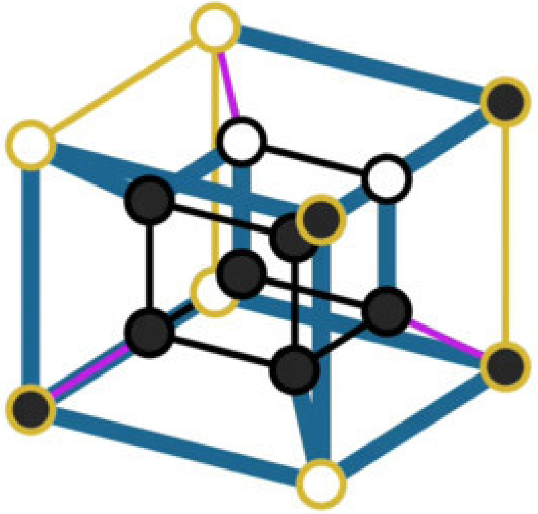

$$\mathrm{Var}[f] \le \sum_{i=1}^{n} \mathrm{Inf}_i(f)$$

$$\mathrm{Var}[f] \le C \frac{\sum_{i=1}^{n} \mathrm{Inf}_i(f)}{\log(1/\max_i \mathrm{Inf}_i(f))}$$

$$\mathrm{Var}[f] \le C \mathbb{E}\sqrt{h_f}$$

???

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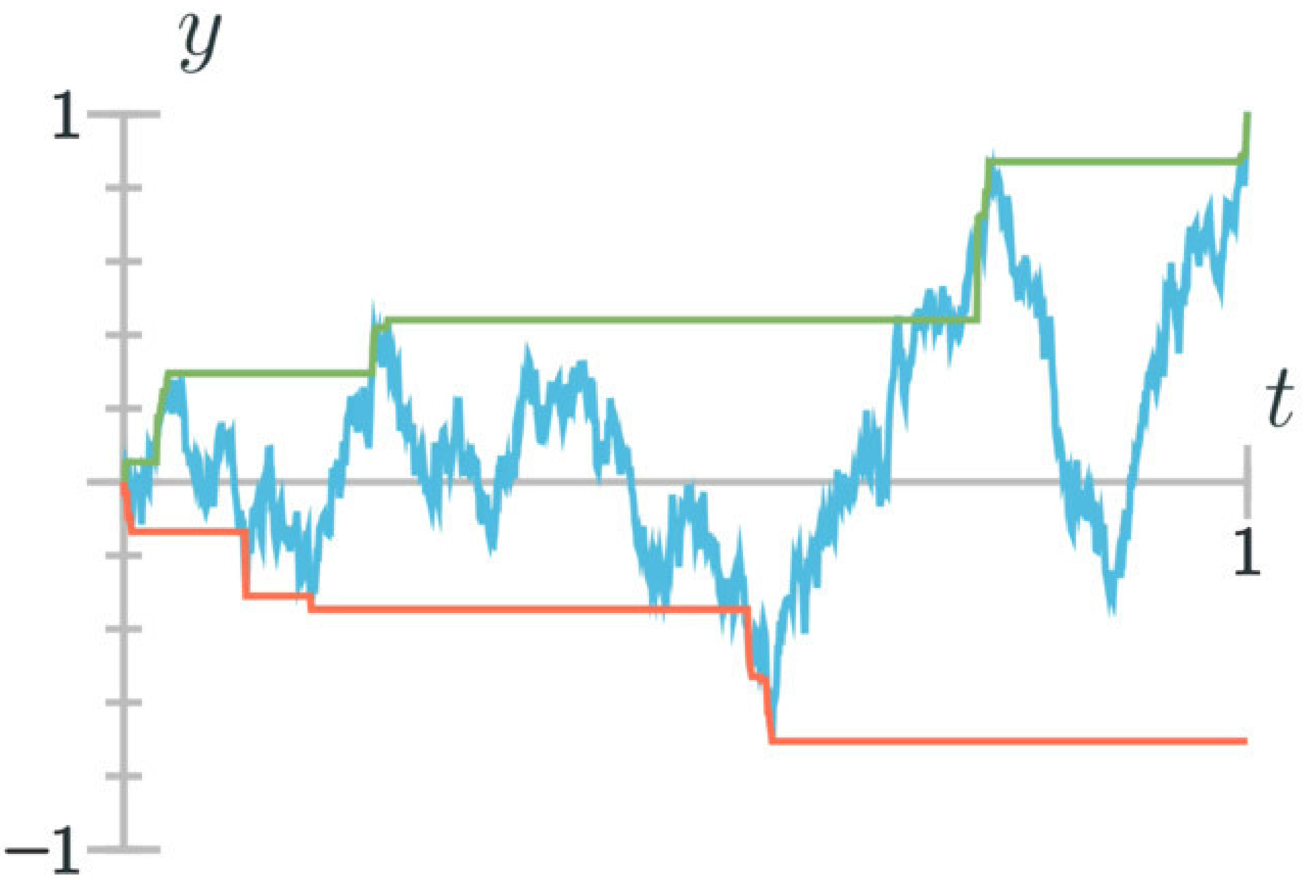

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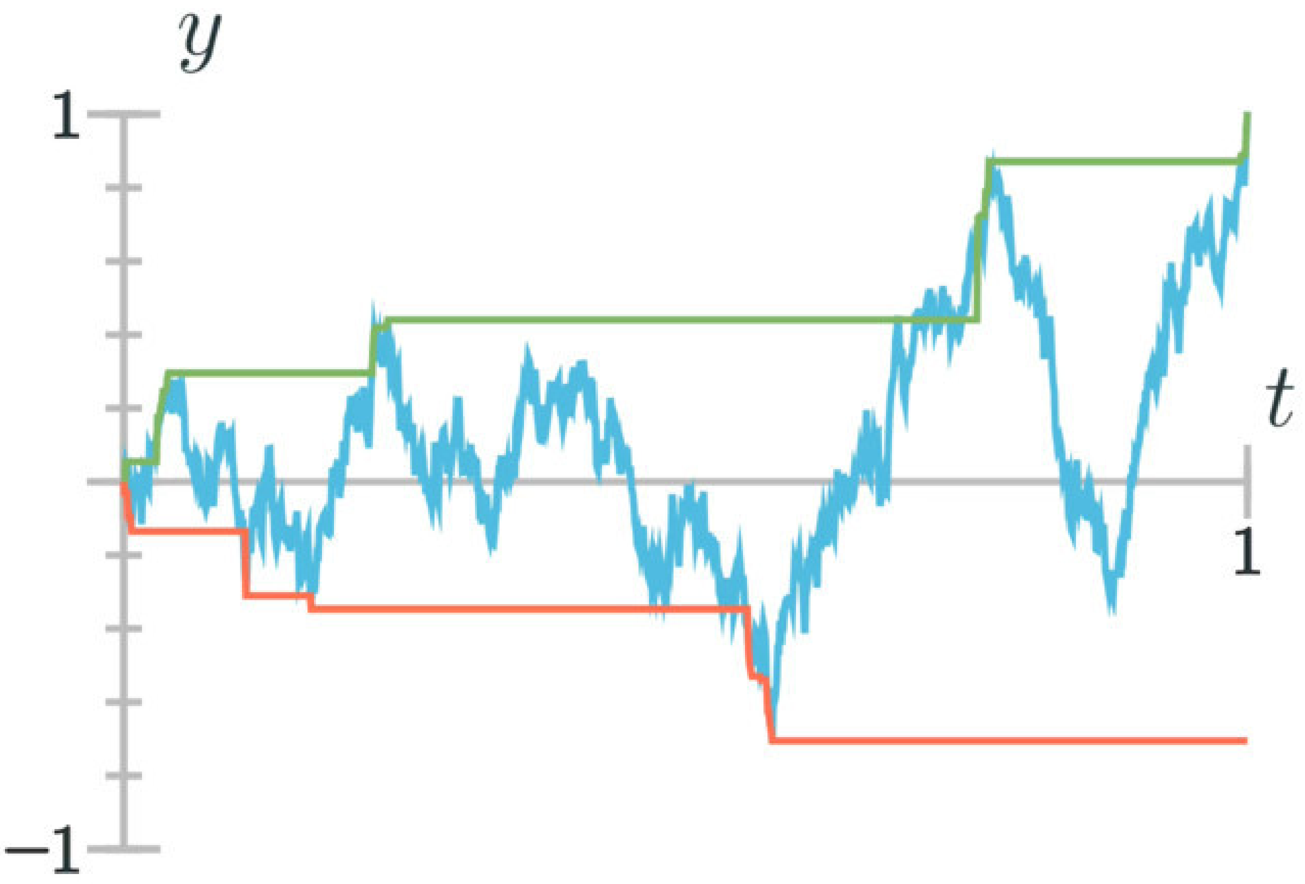

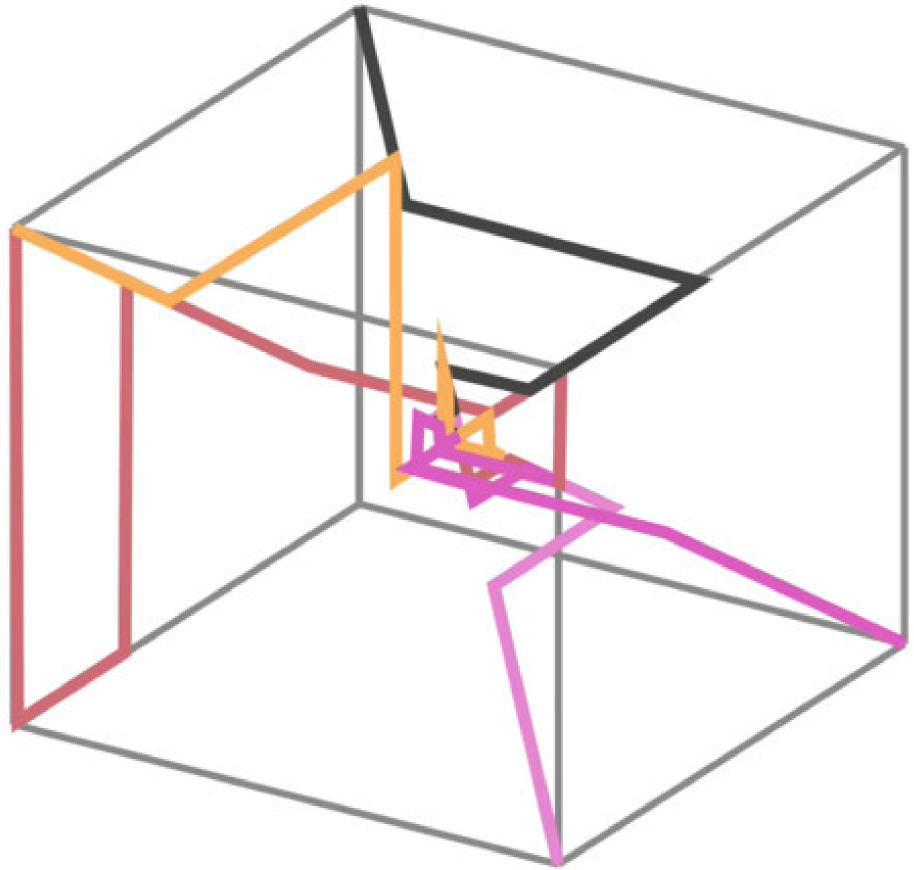

Meet the players

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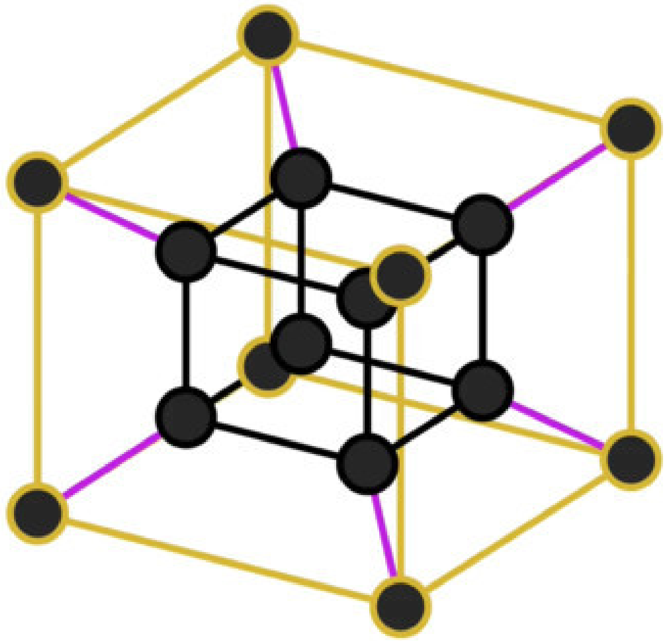

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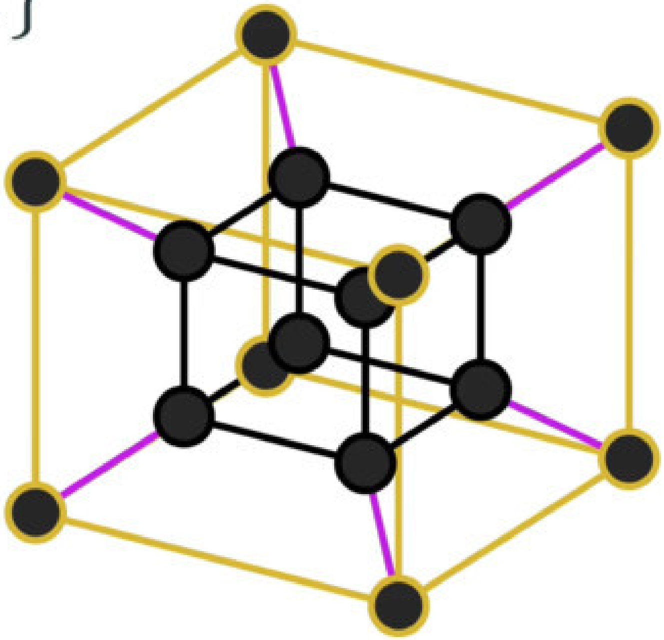

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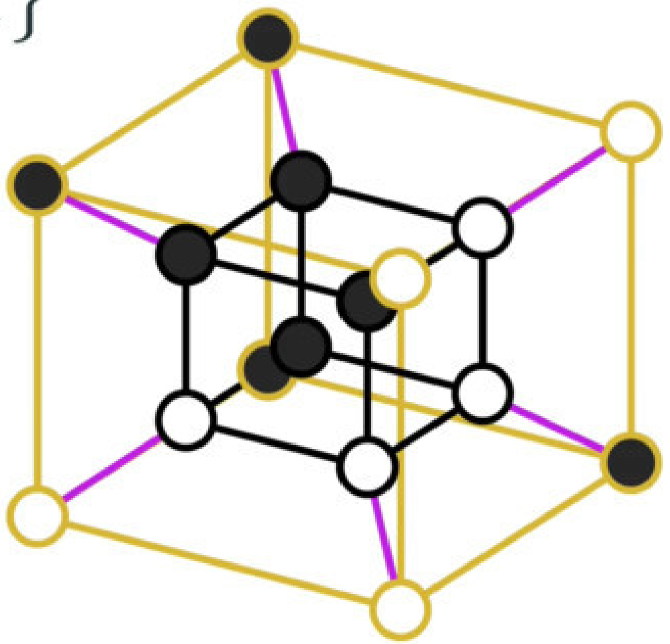

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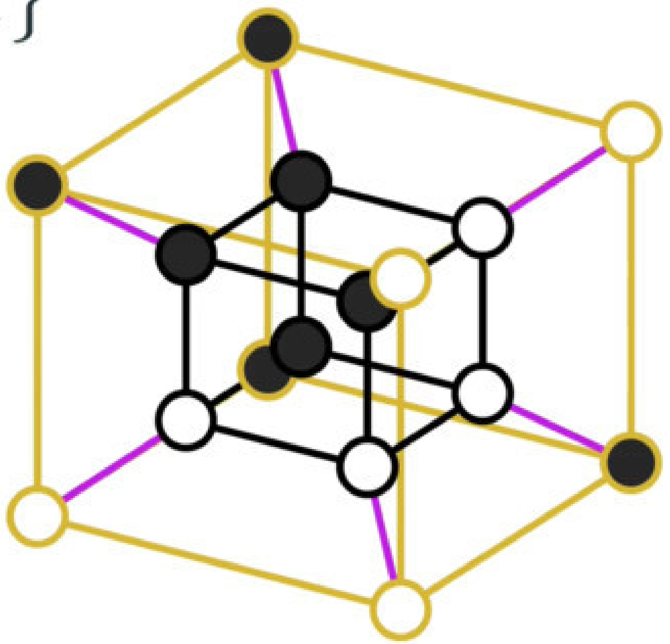

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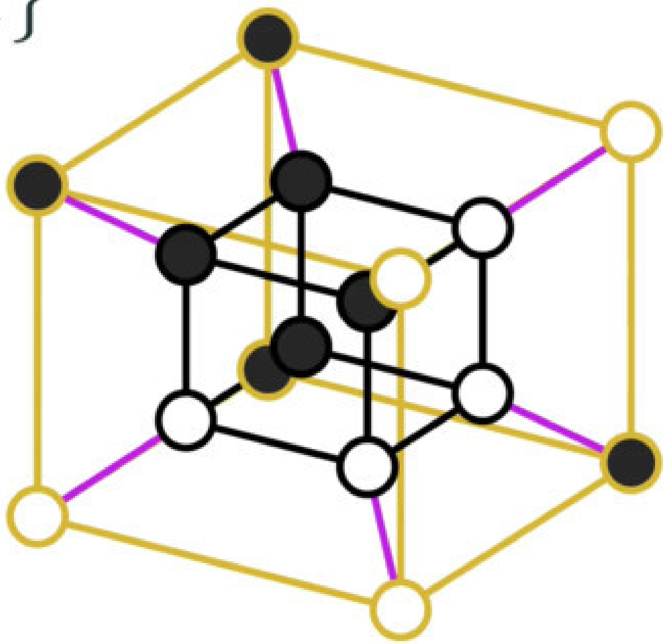

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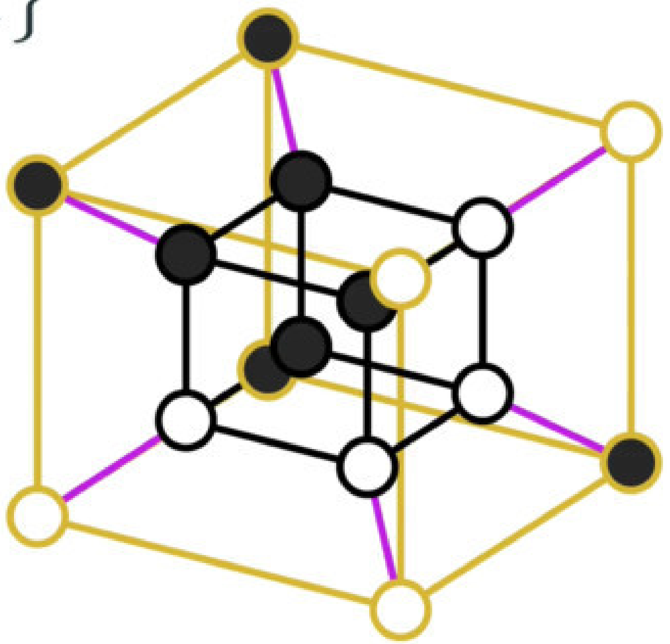

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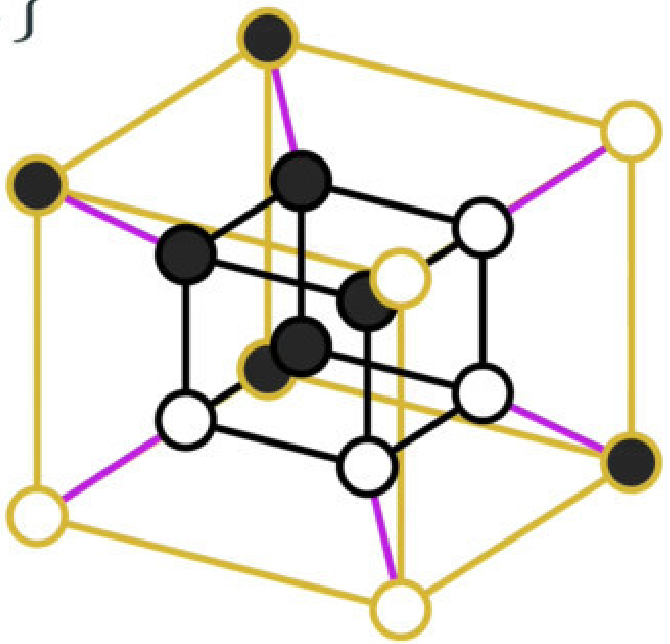

$\mathrm{Var}(f) = 1 - (\mathbb{E}f)^2 = (1 - \mathbb{E}f)(1 + \mathbb{E}f) = \mathcal{O}(1 - |\mathbb{E}f|)$

For the analyst

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$$\partial_i f(x) = \frac{f\left(x^{i\to 1}\right) - f\left(x^{i\to -1}\right)}{2}$$

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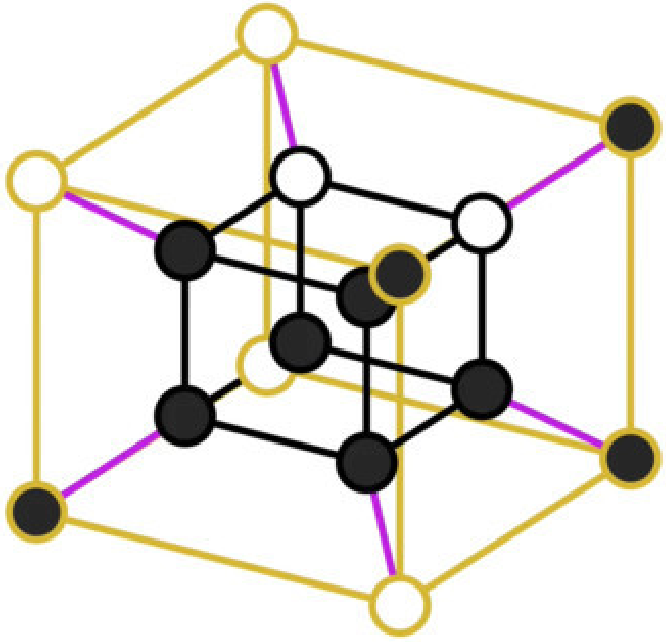

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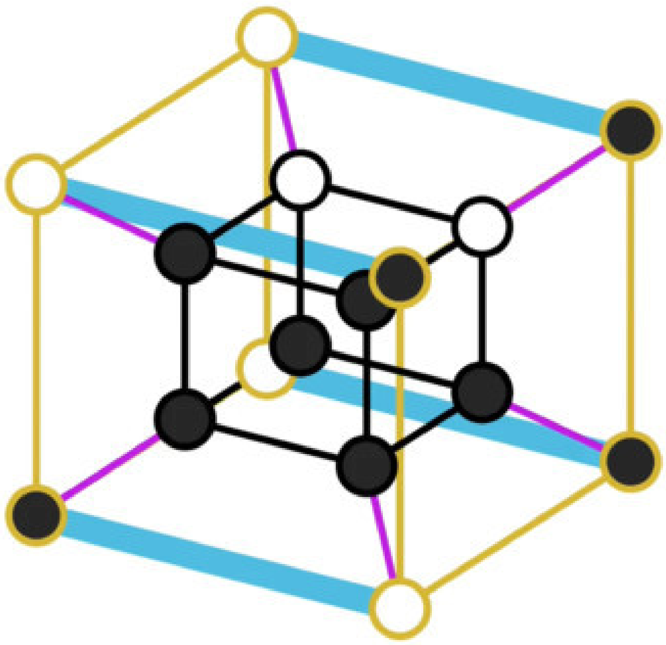

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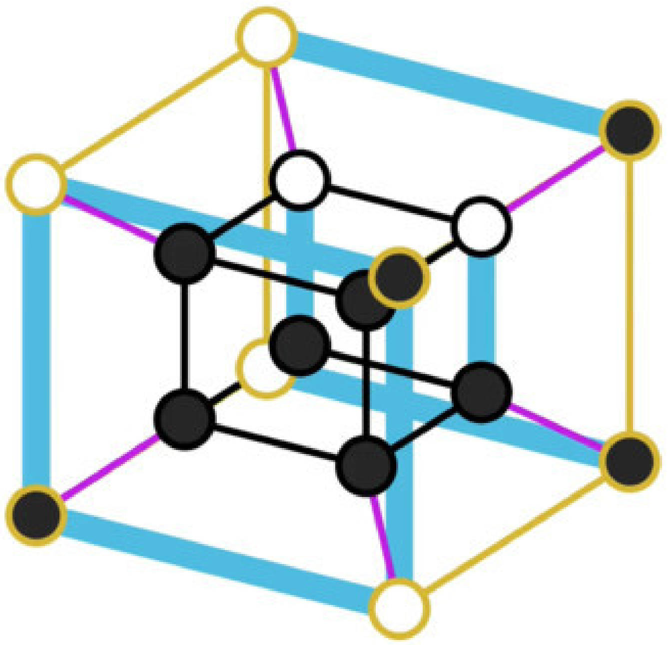

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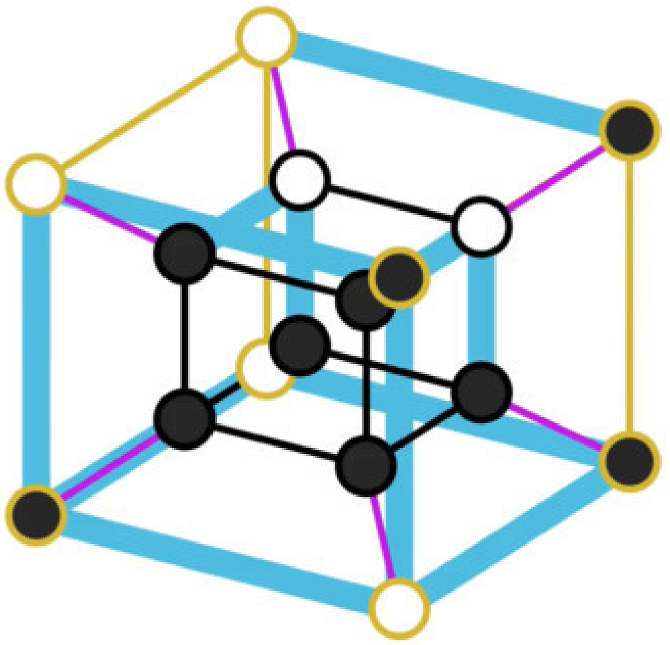

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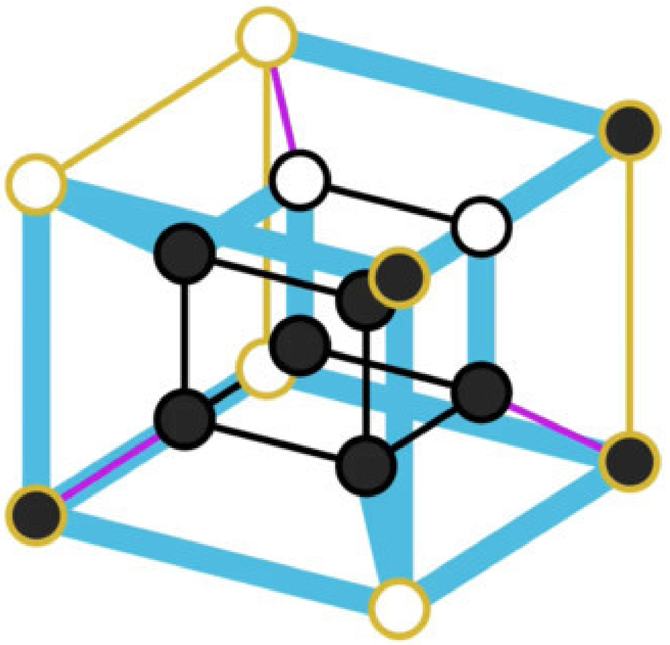

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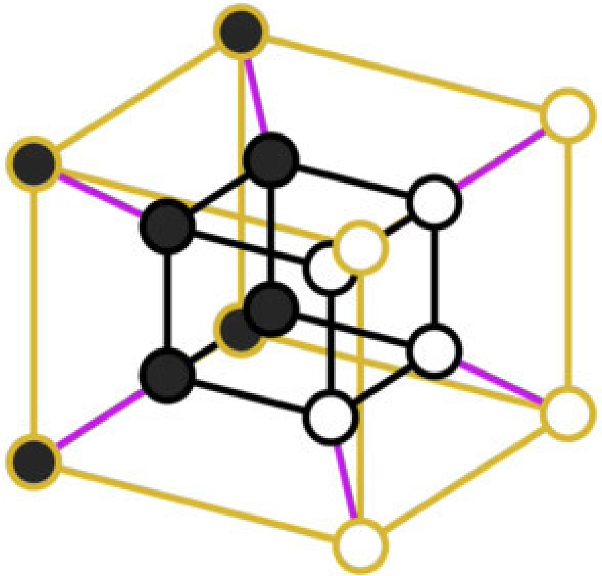
Dictator



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$$\mathrm{Var}(f) = 1$$

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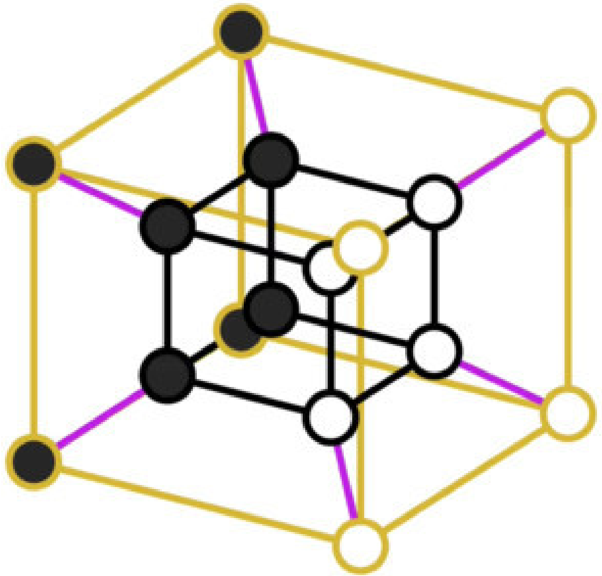

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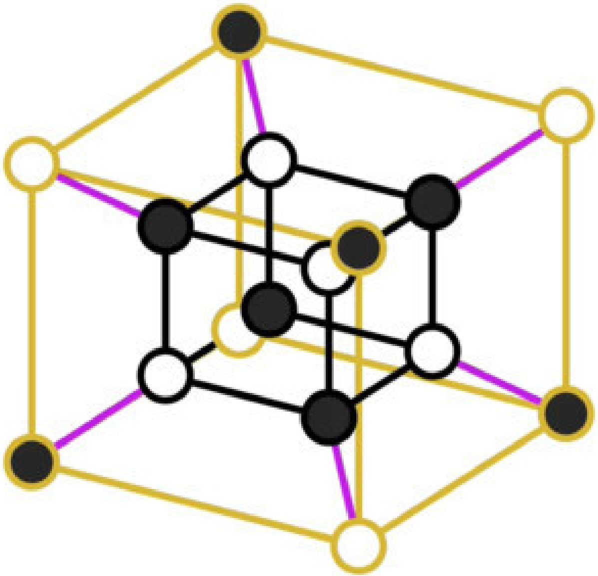

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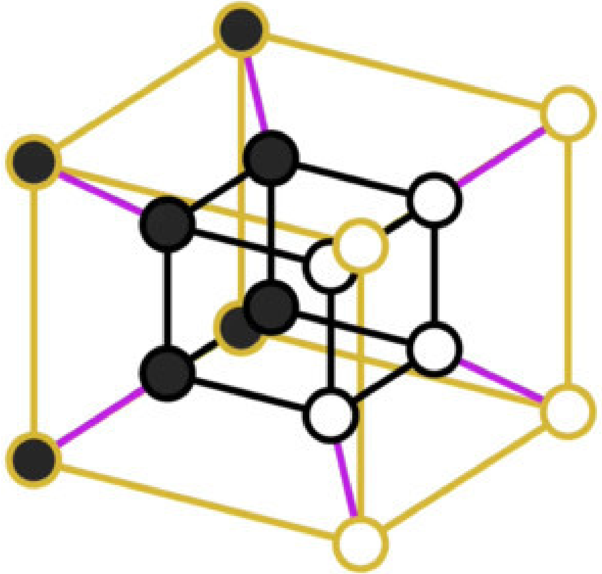

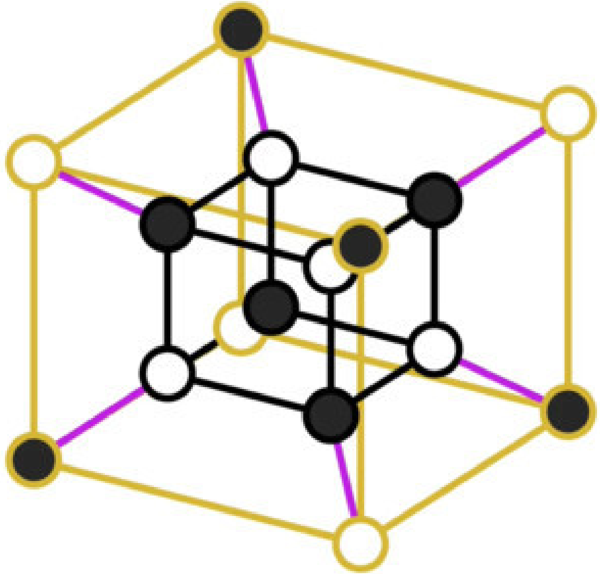

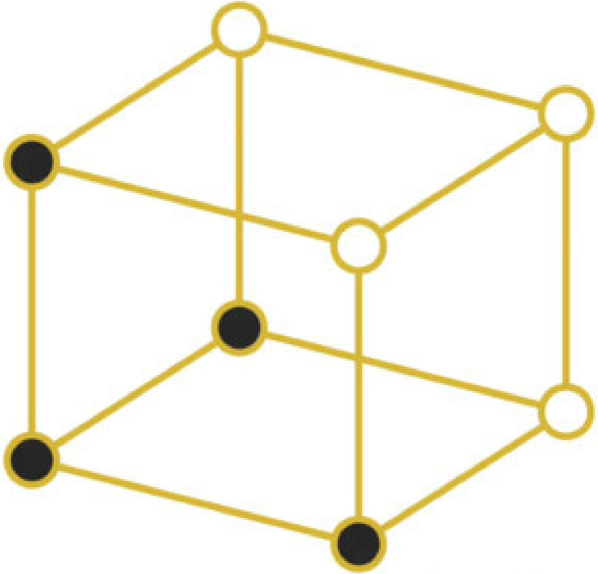

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Divide input intro tribes. Is there is a tribe that says yes?

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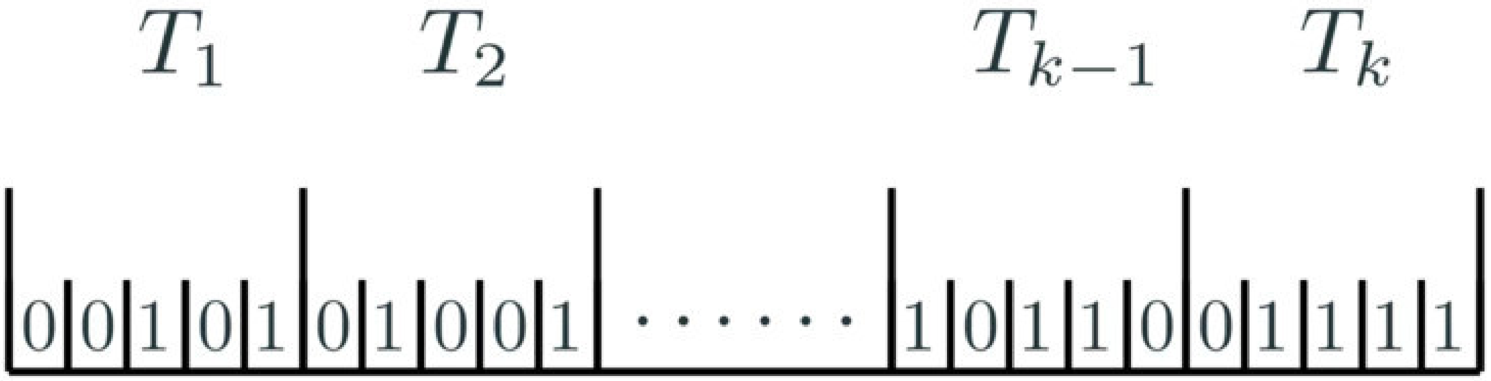

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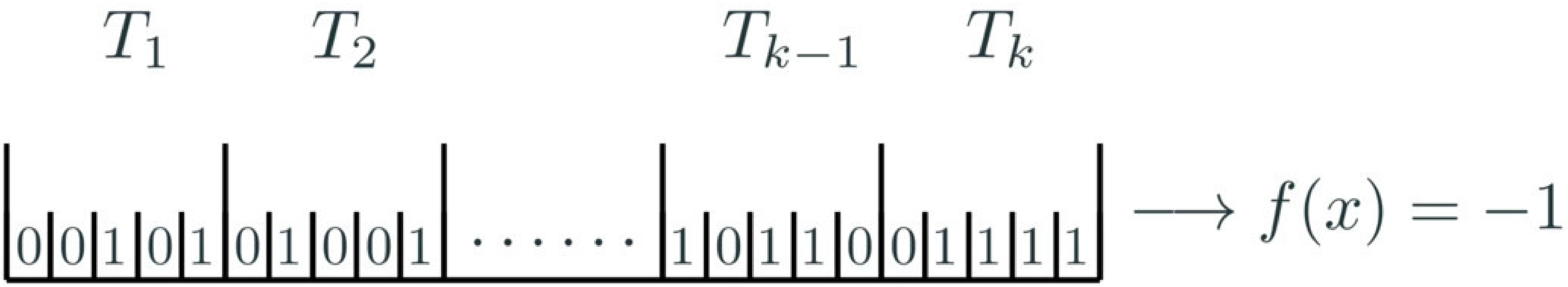

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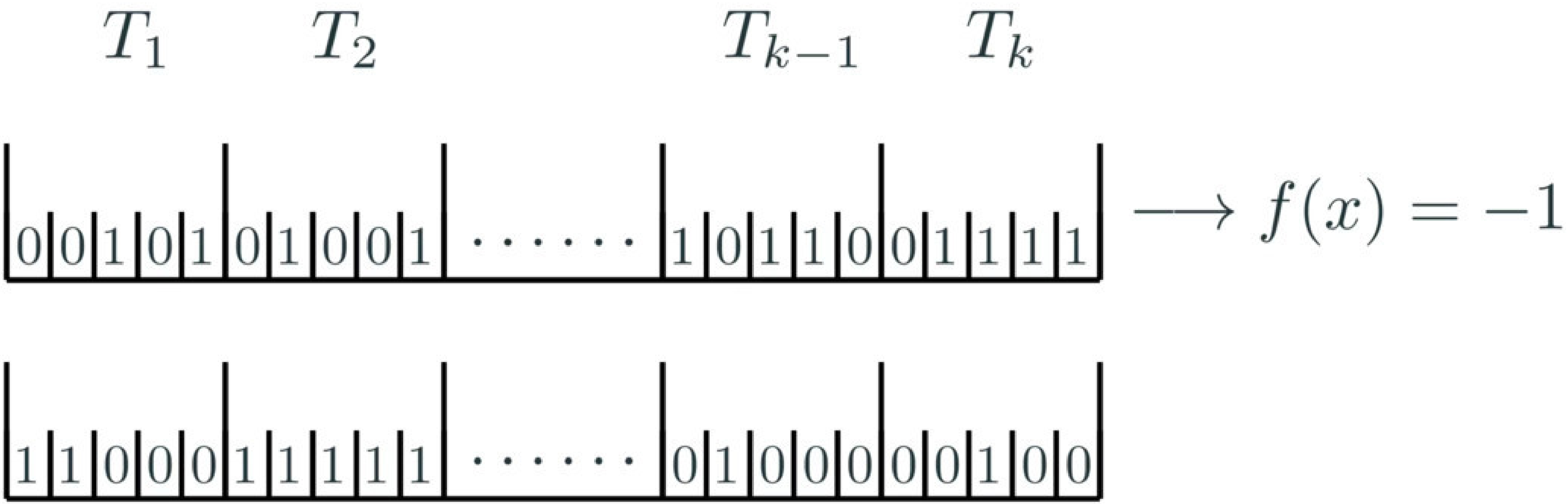

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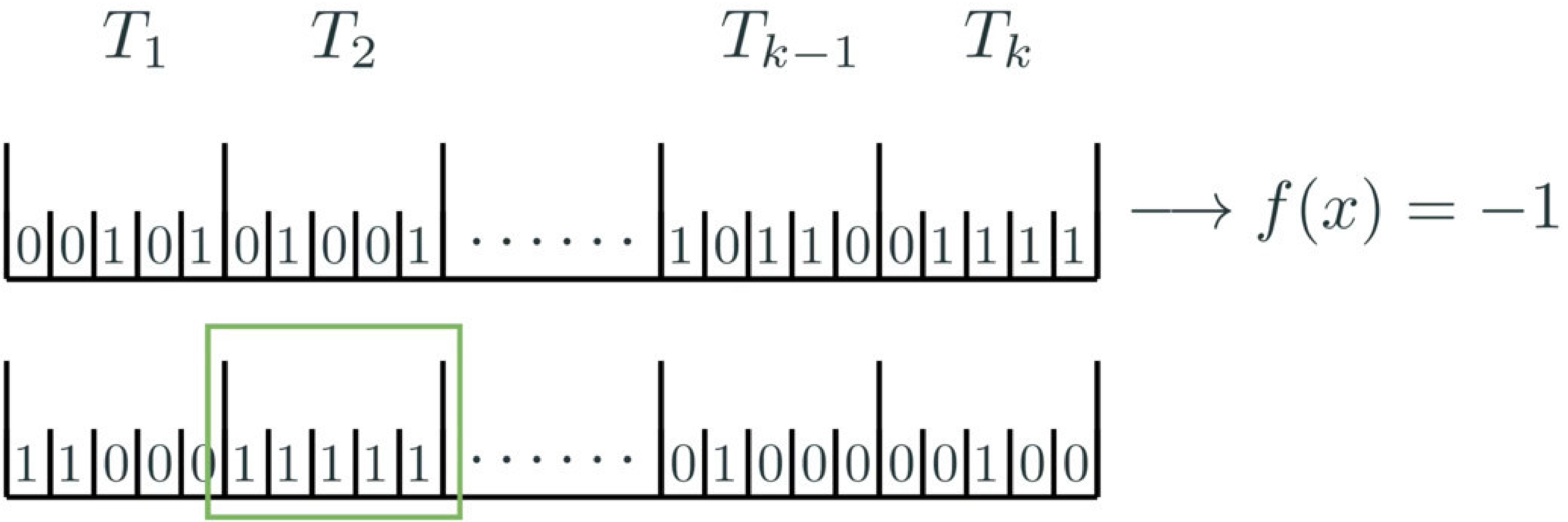

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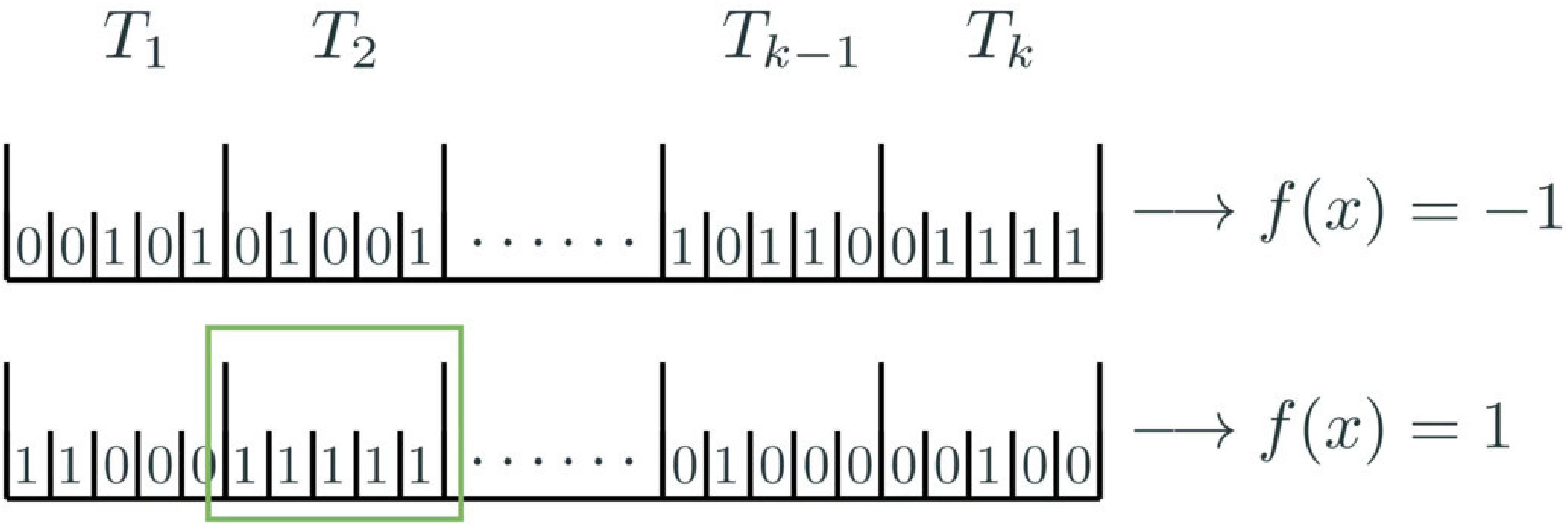

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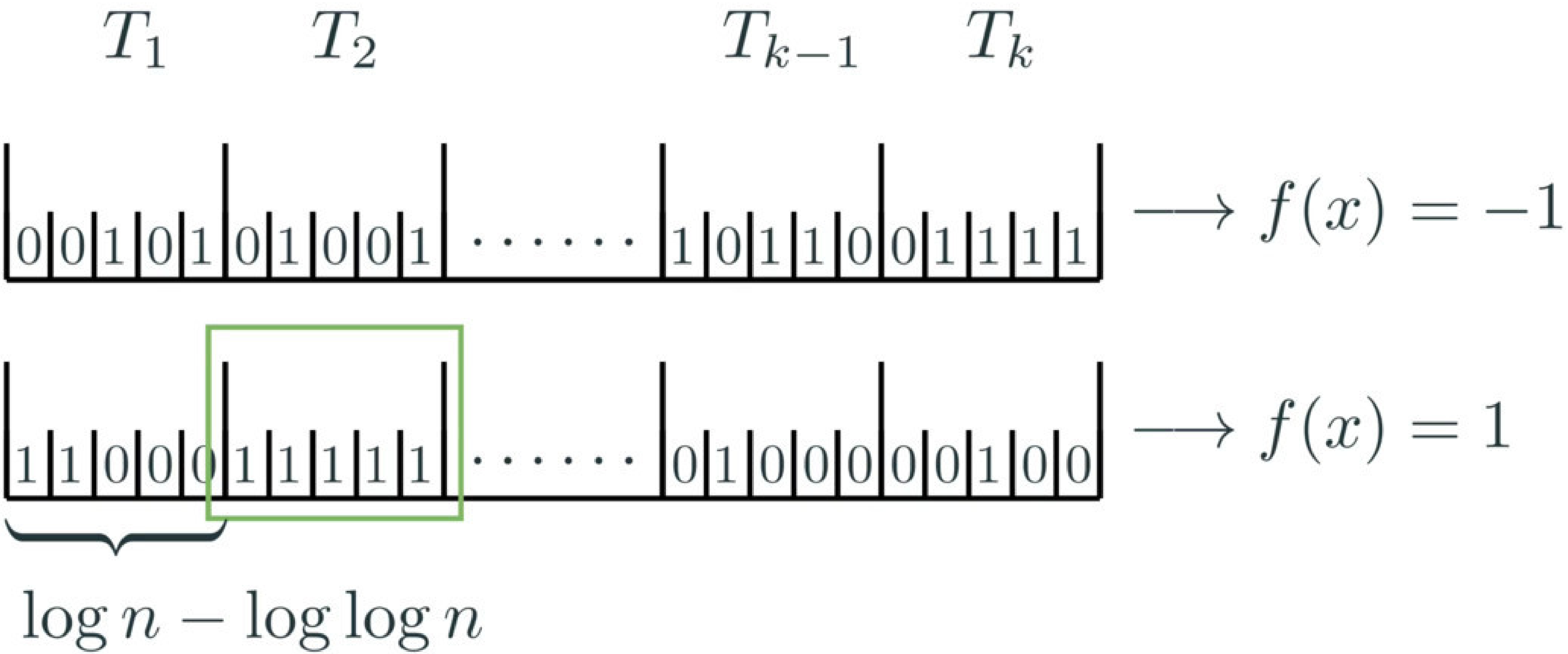

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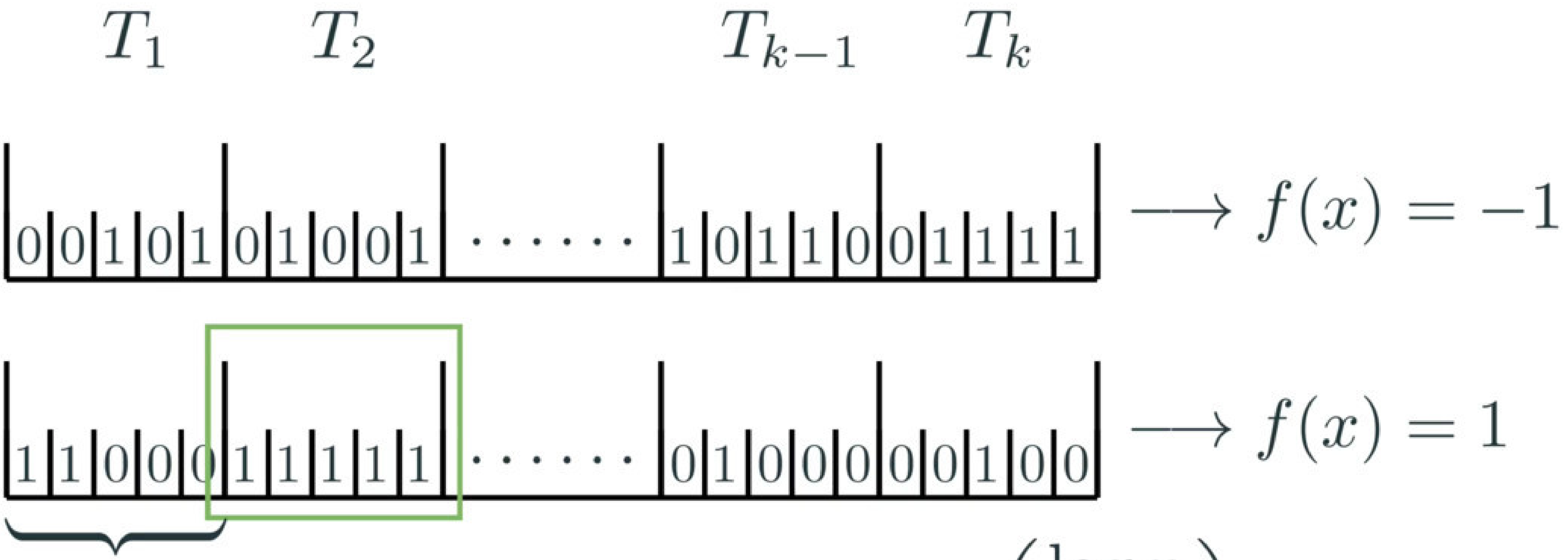

$$\log n - \log\log n \longrightarrow \mathrm{Inf}_i(\mathrm{f}) = \mathcal{O}\left(\frac{\log n}{n}\right), \mathrm{Var}(f) = \mathcal{O}(1)$$

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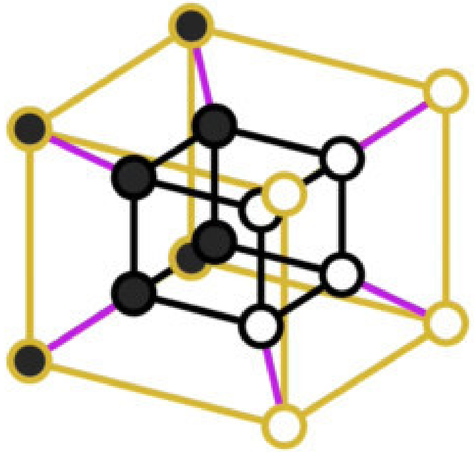

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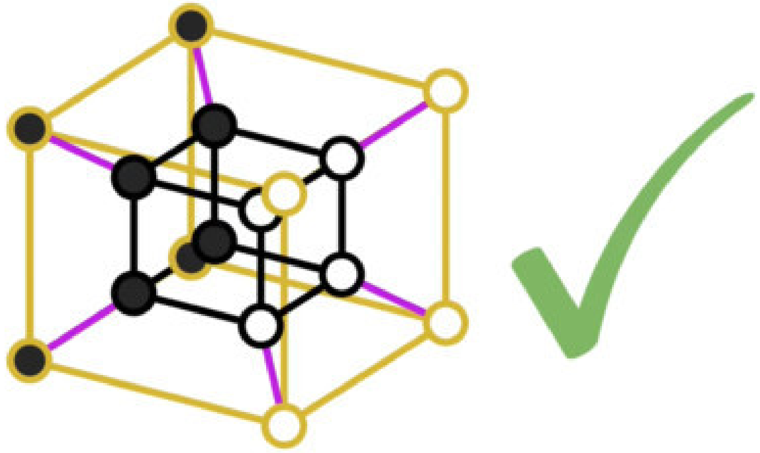

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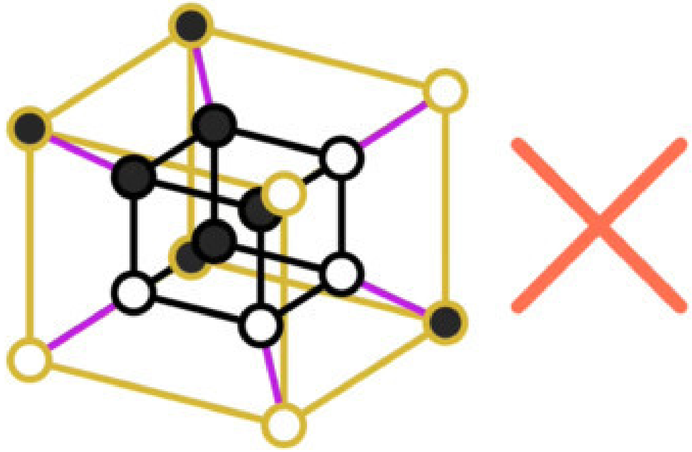
Everything else...



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KKL

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Theorem [KKL 88]: $\mathrm{Var}[f] \leq C \dfrac{\sum_{i=1}^{n} \mathrm{Inf}_i(f)}{\log(1/\max_i \mathrm{Inf}_i(f))}$

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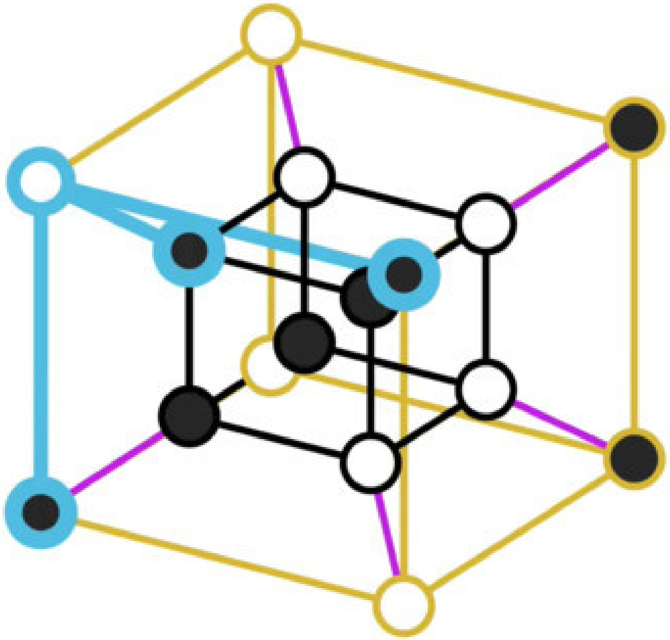

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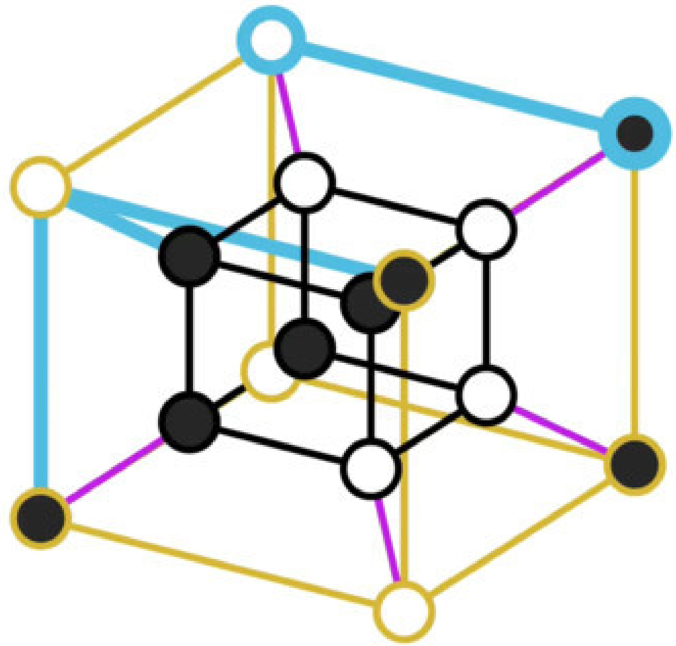

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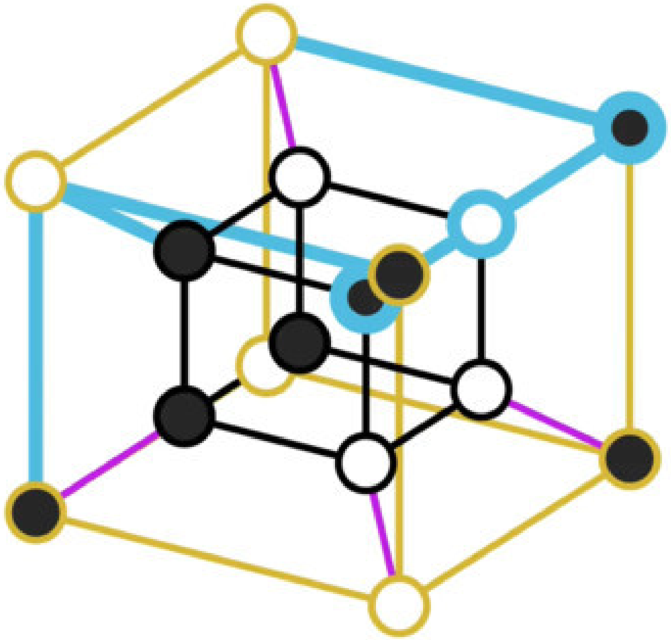

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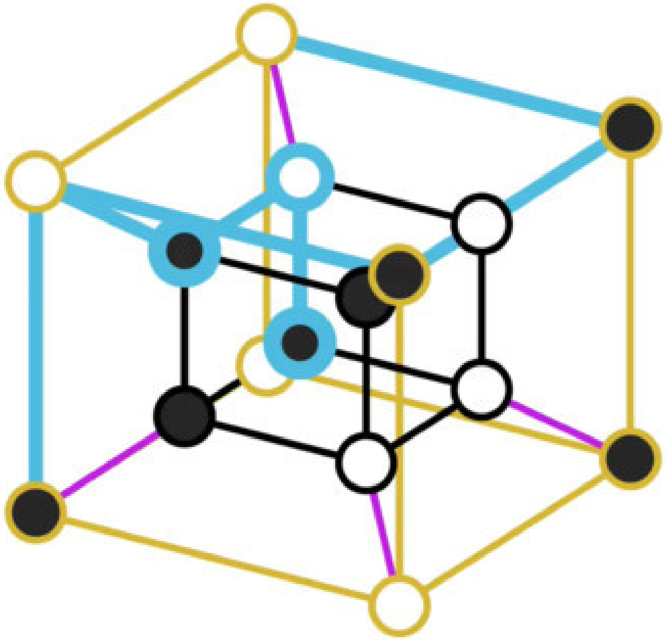

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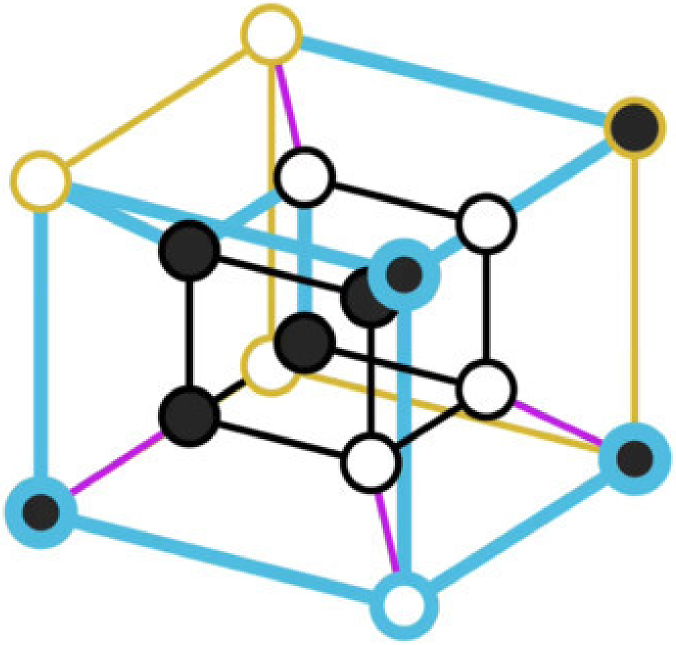

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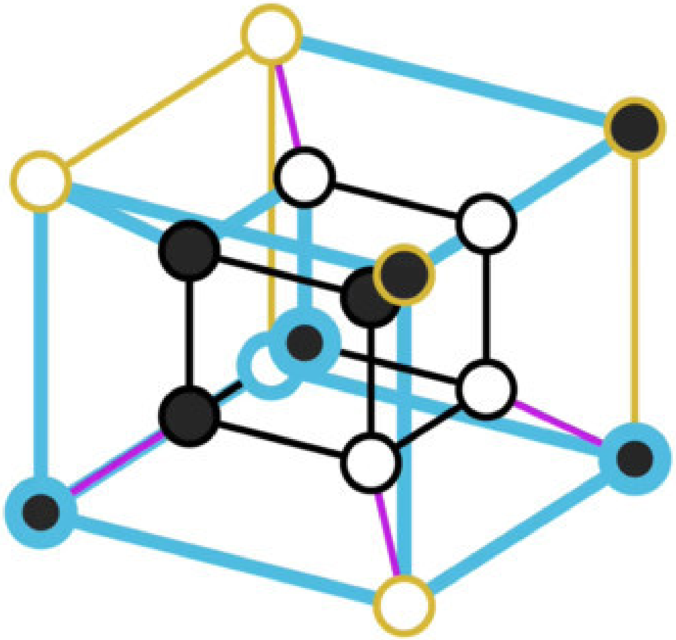

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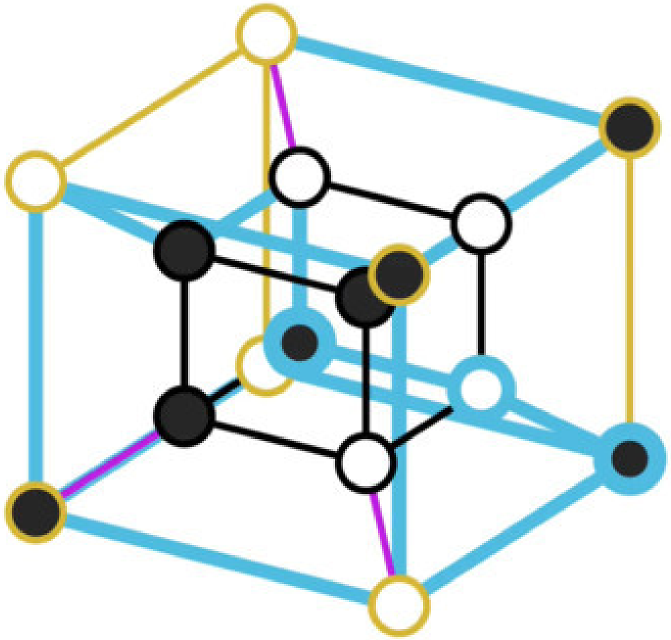

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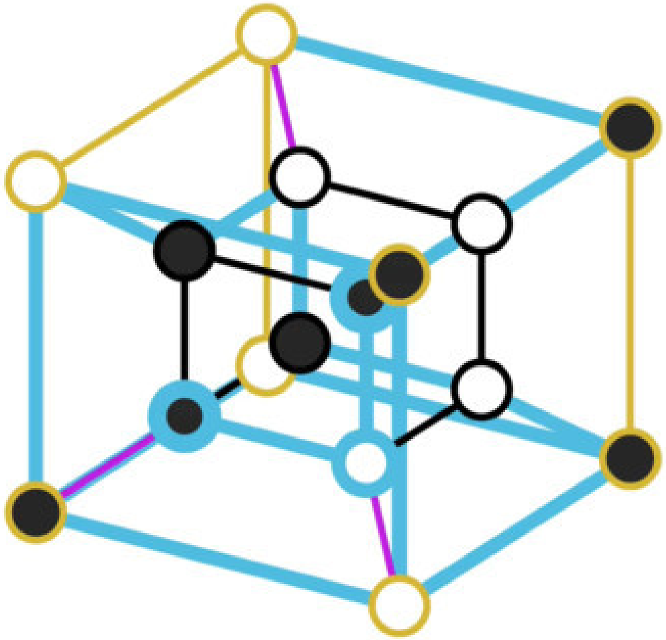

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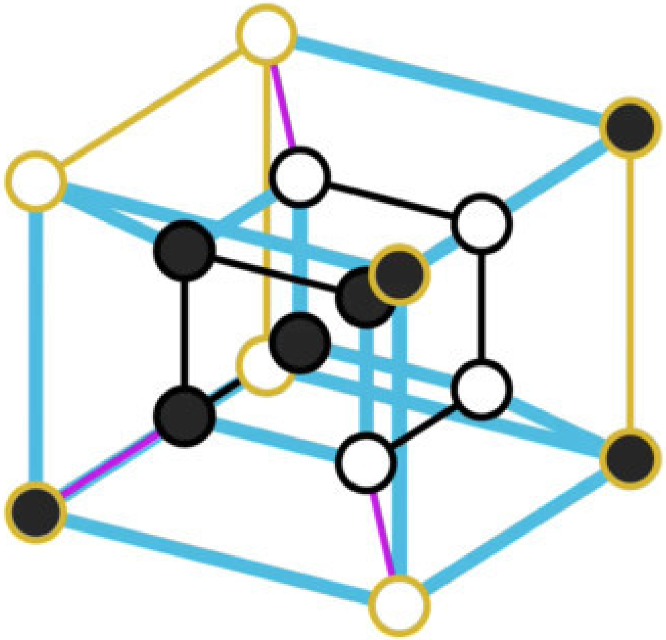

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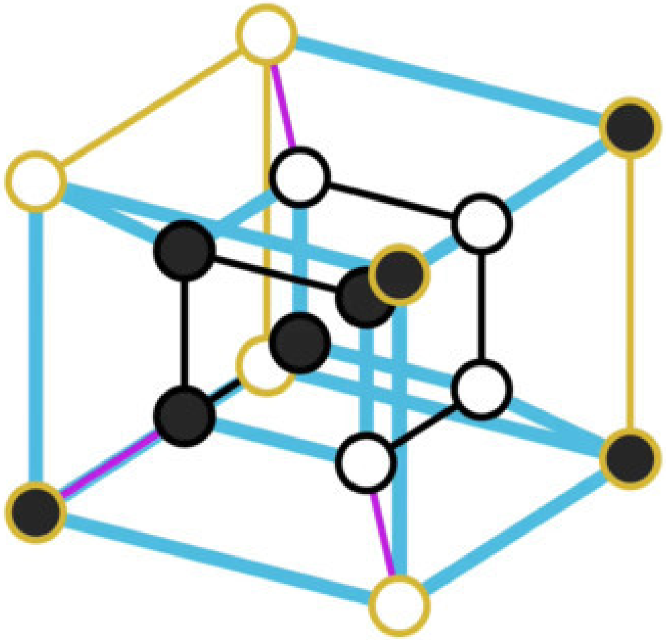

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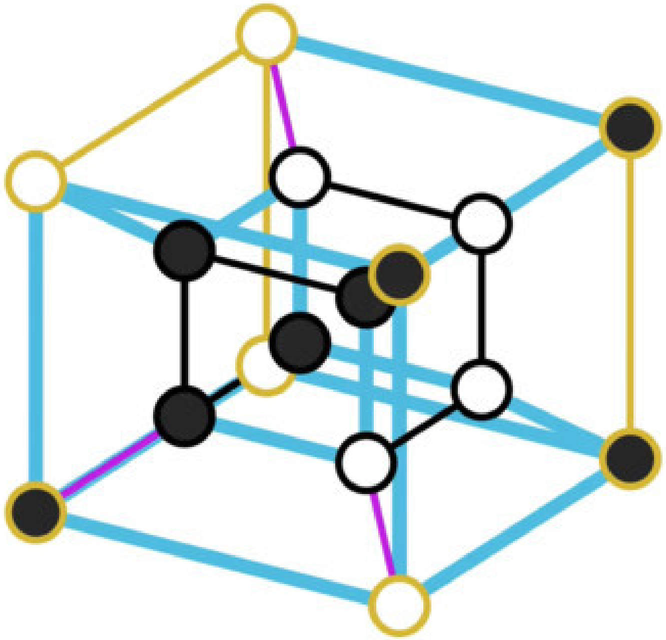

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Conj:

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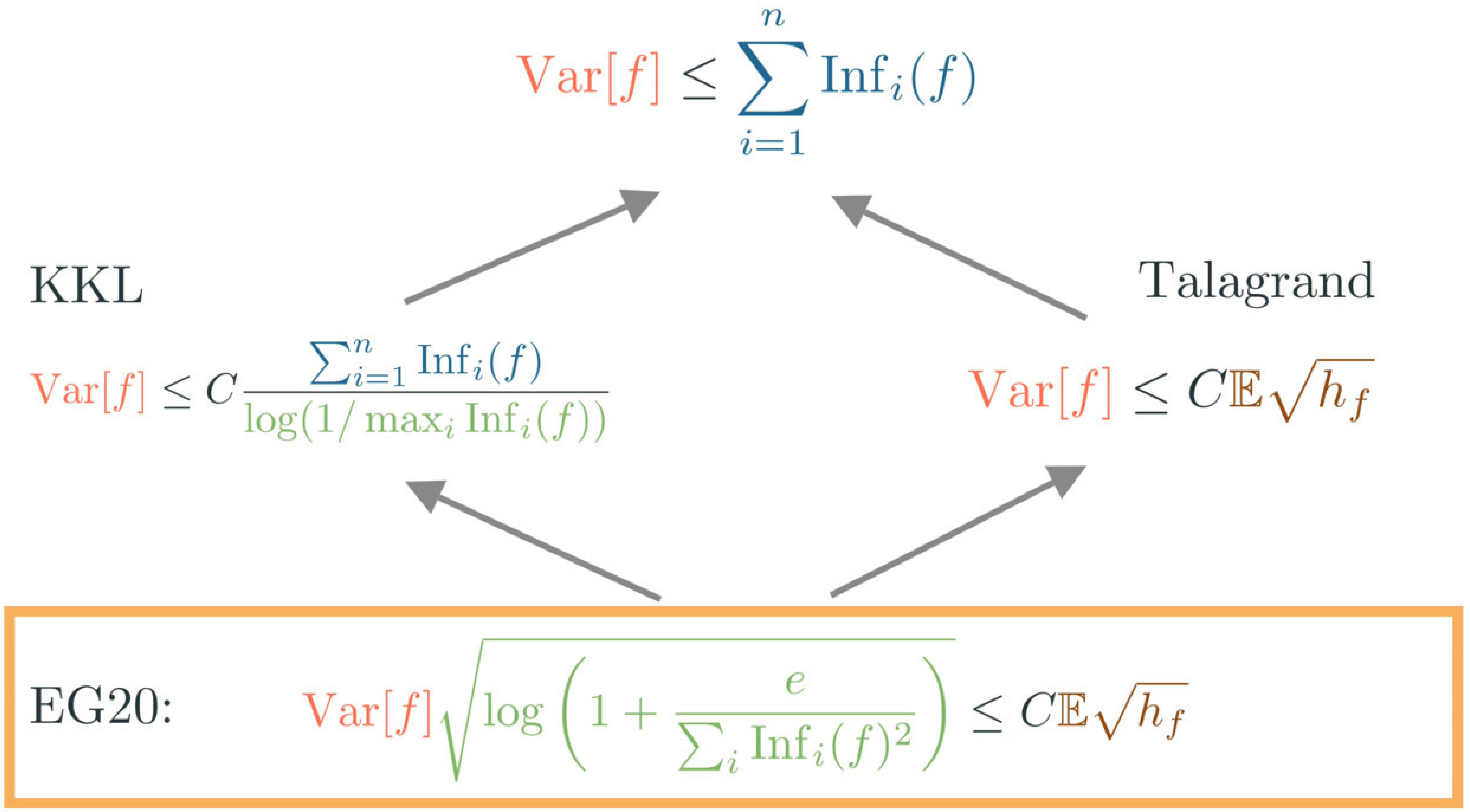

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Slight improvements

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Thm [EG20]: $\exists g_f : \{-1,1\}^n \to [0,1]$ s.t.

$\triangleright$ $\mathbb{E} g_f^2 \leq 2\mathrm{Var}[f]$

$\triangleright$ For all $1/2 \leq p < 1$

$$\mathrm{Var}[f]\left(\log\left(1+\frac{e}{\sum_i \mathrm{Inf}_i(f)^2}\right)\right)^p \leq C\ \mathbb{E}[h_f^p g_f]$$

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$\delta \sum \mathrm{Inf}_i(f) \geq \sqrt{\delta}$?

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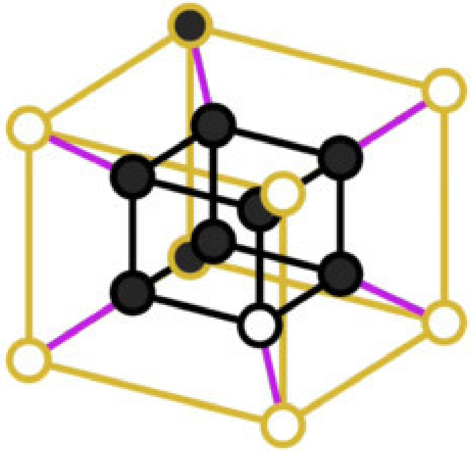

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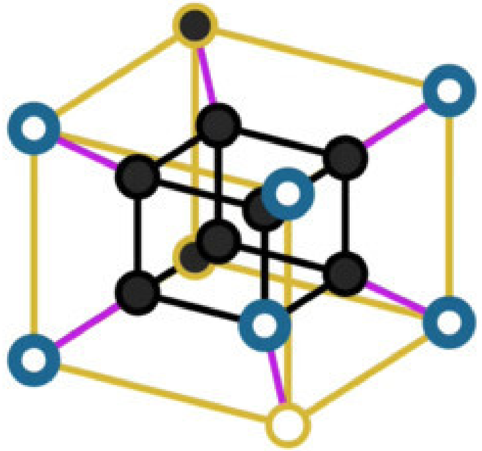

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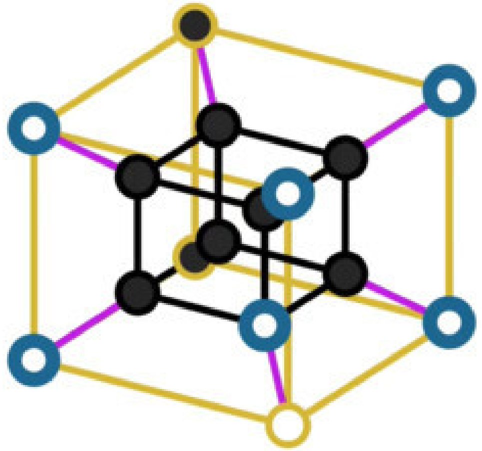

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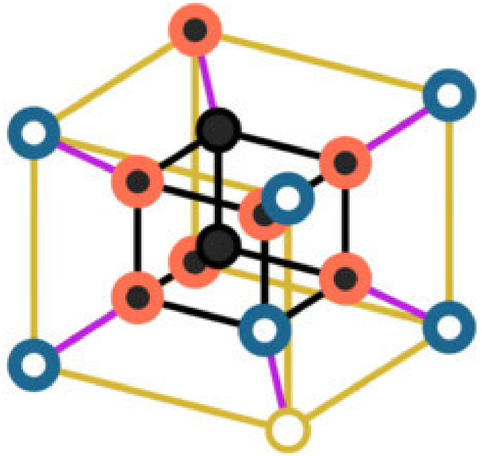

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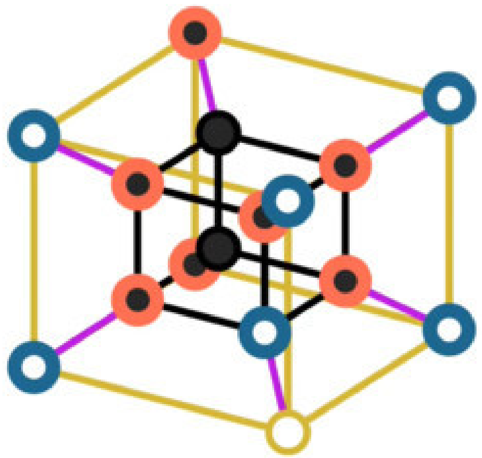

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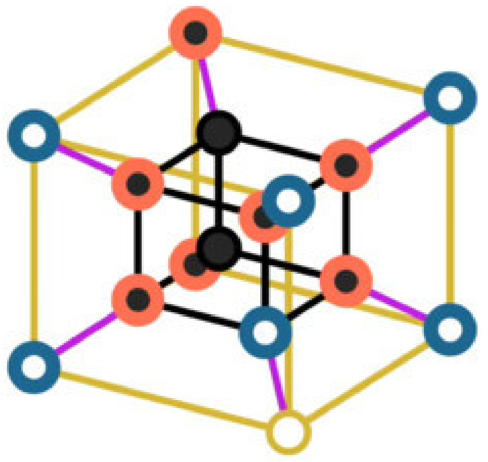

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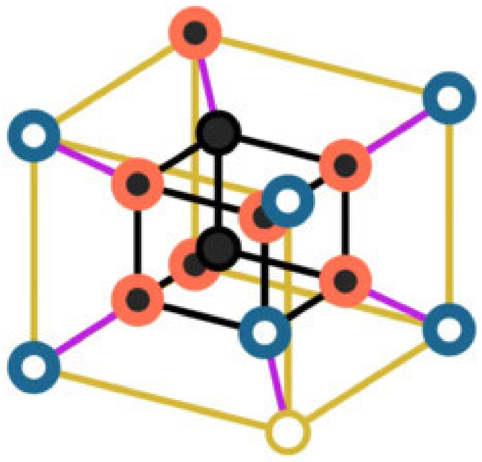

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Define the heat operator T_ρ:

$$(T_\rho f) = \mathbb{E}_y f(y), \text{ where } y_i = \begin{cases} x_i & \text{w.p } (1+\rho)/2 \\ -x_i & \text{w.p } (1-\rho)/2 \end{cases}$$

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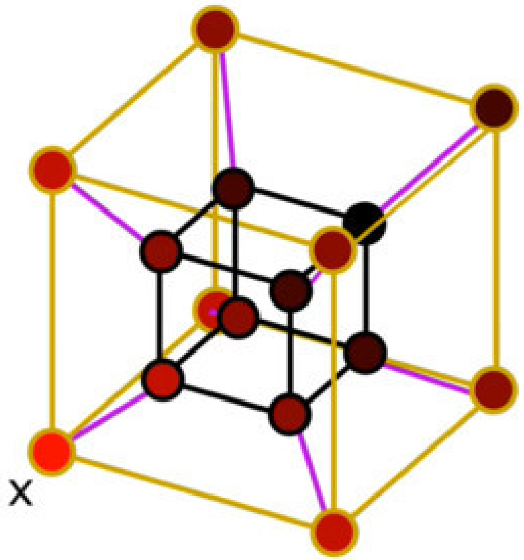

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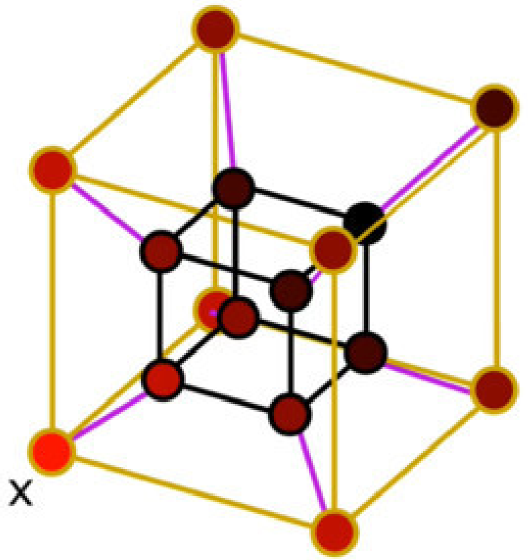
1	1	-1	1

x

ρ	ρ	$-\rho$	ρ

$\mathbb{E}y$



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$$\mathrm{Var}(f) = \sum_{S \neq \emptyset} \widehat{f}(S)^2 \qquad\qquad ||T_\rho f||_2 = \sum_{S \subseteq [n]} \widehat{f}(S)^2 \rho^{|S|}$$

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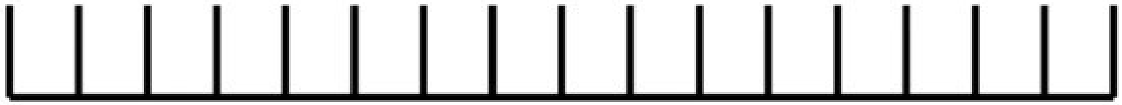

Everything is expectation

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0	0	1	0	1	0	1	0	0	1	1	0	1	1	0	0

$f(x) =$ -1

Everything is expectation

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1	1	1	1	0	0	0	0	1	1	0	1	1	0	1	0

$f(x) = 1$

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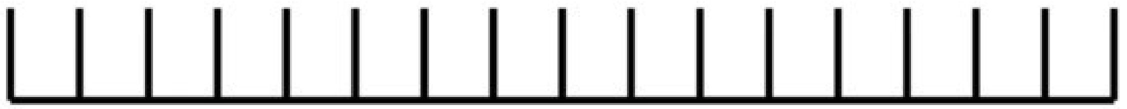

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$$X_t$$

?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?

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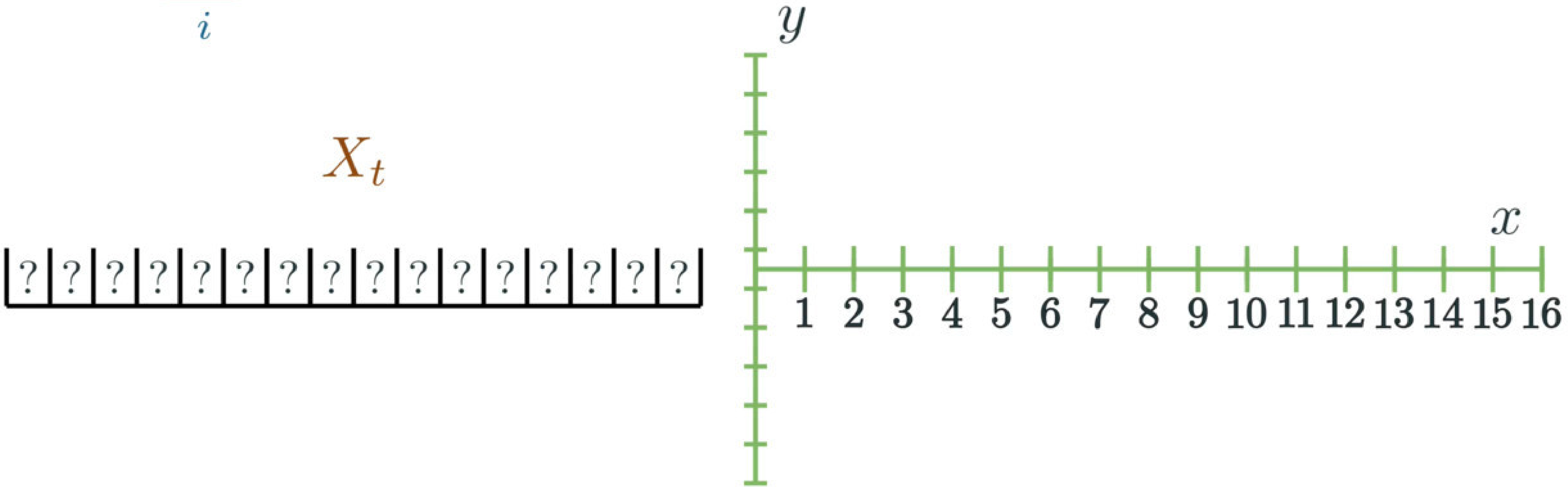

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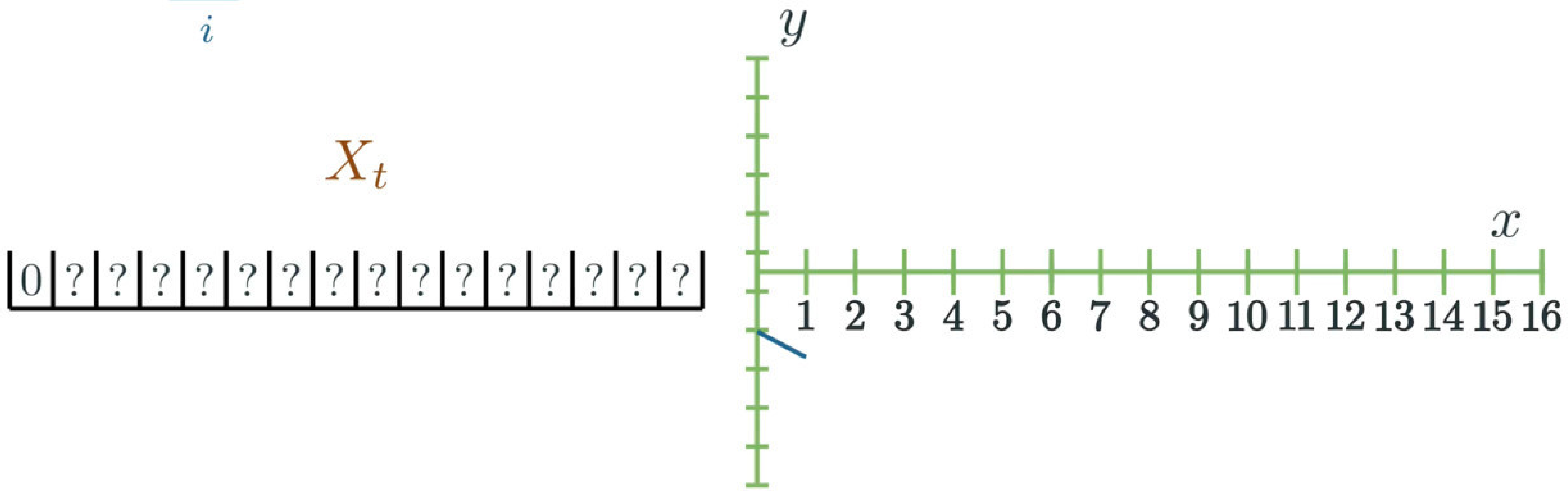

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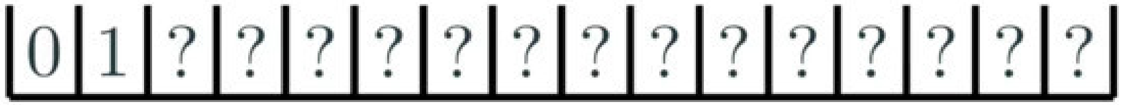

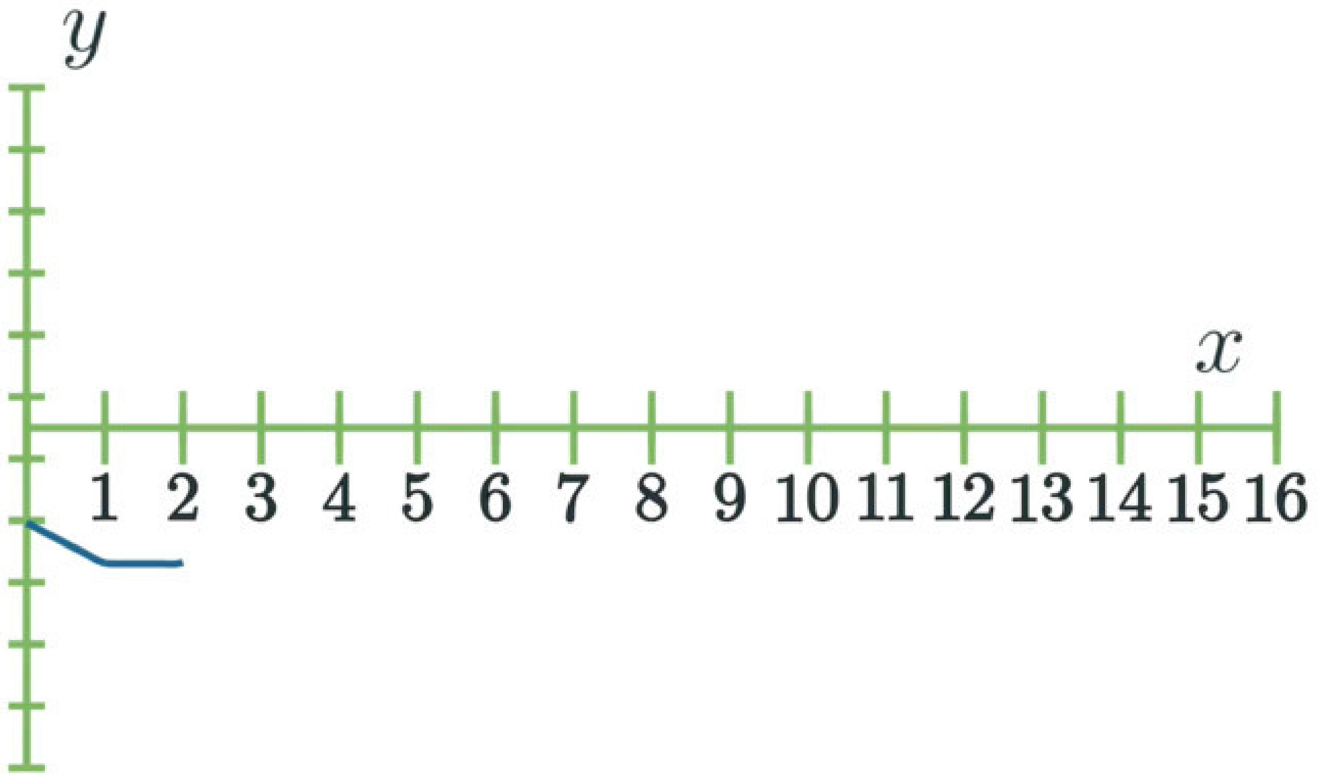

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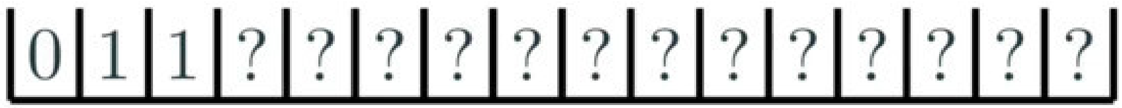

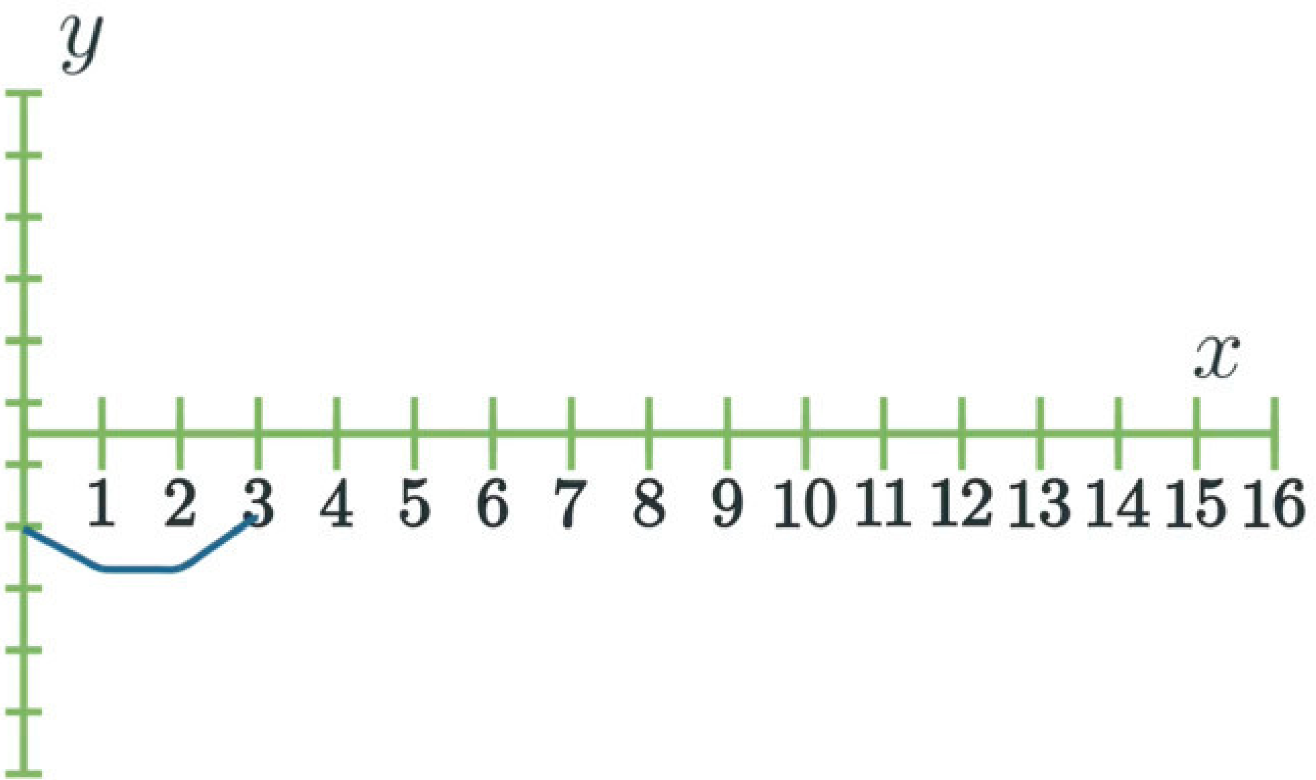

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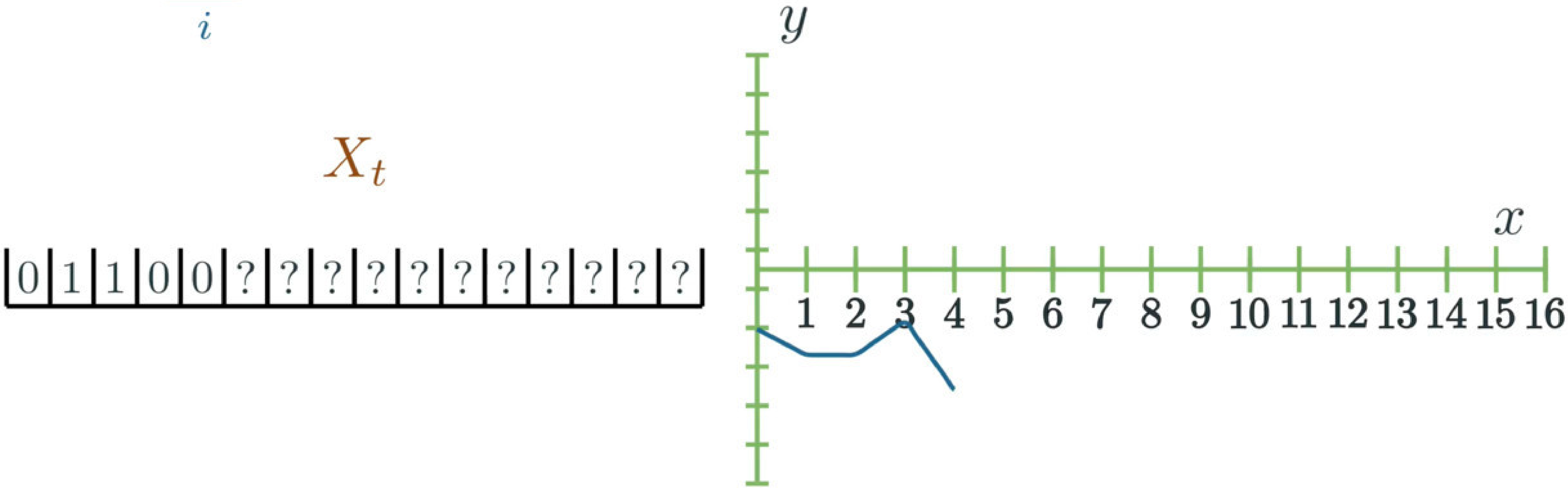

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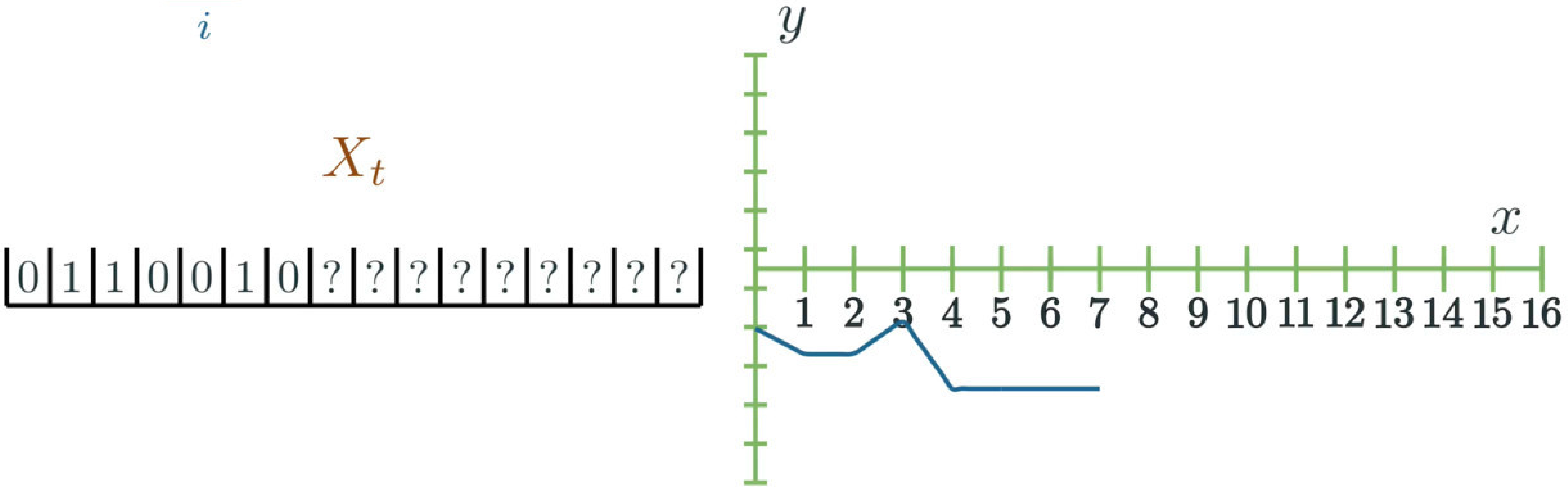

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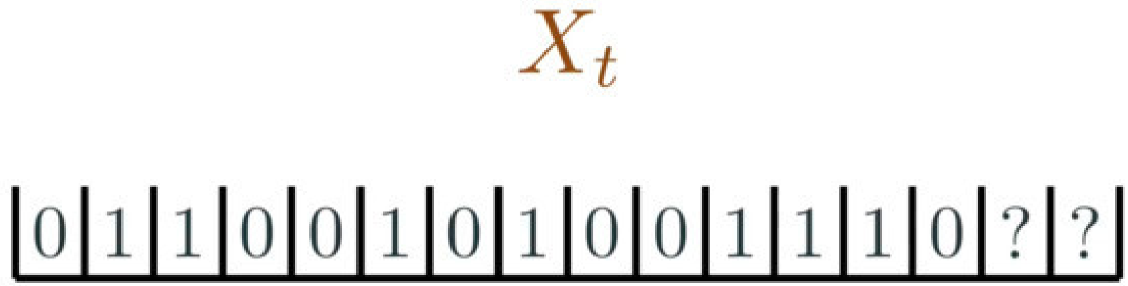

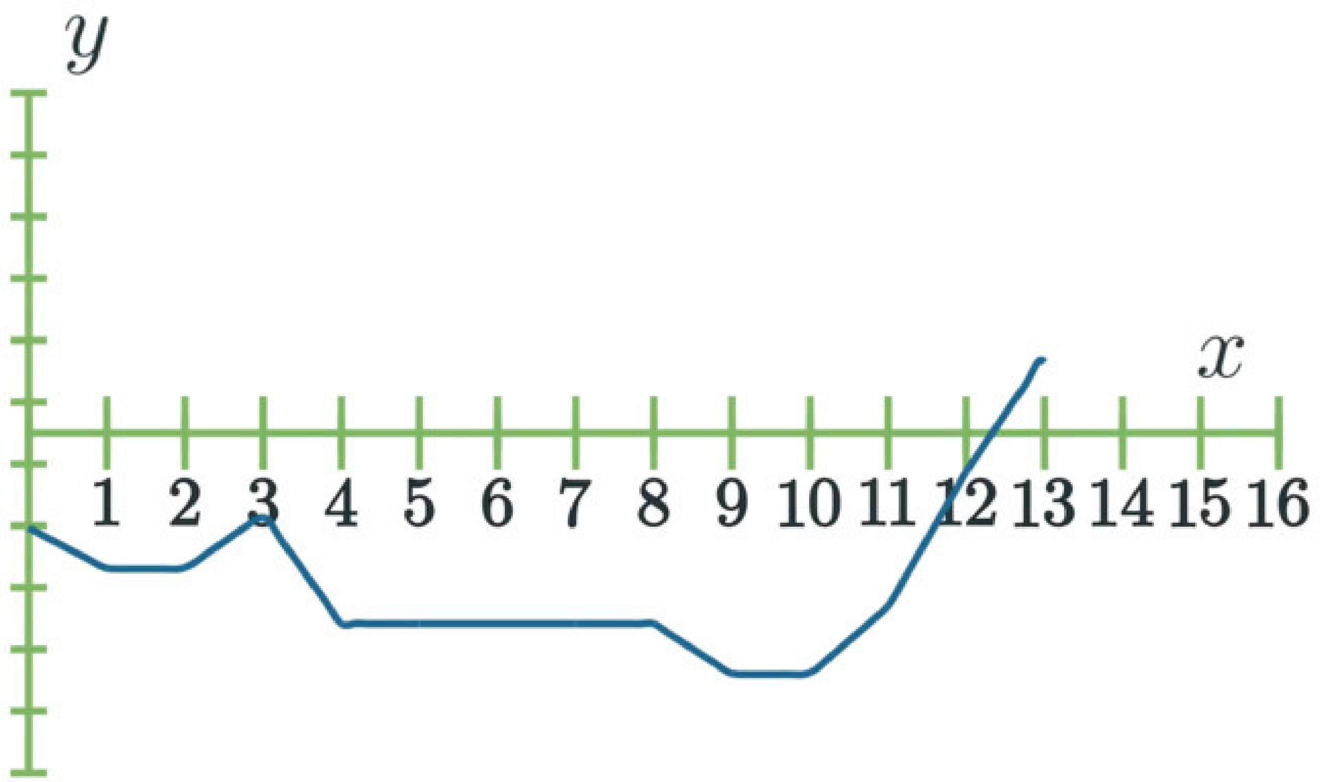

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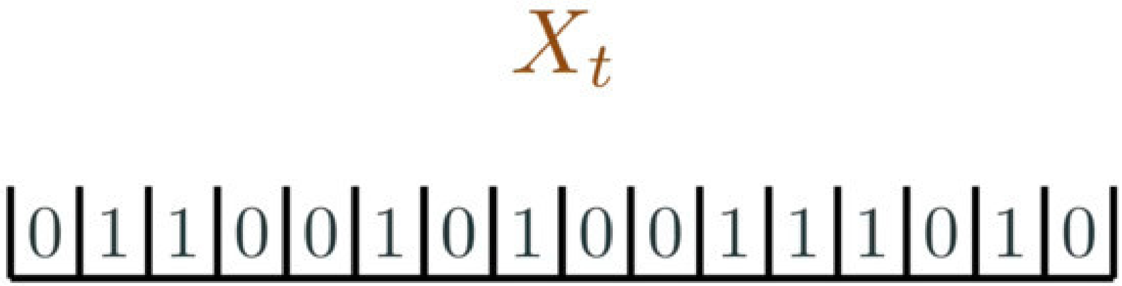

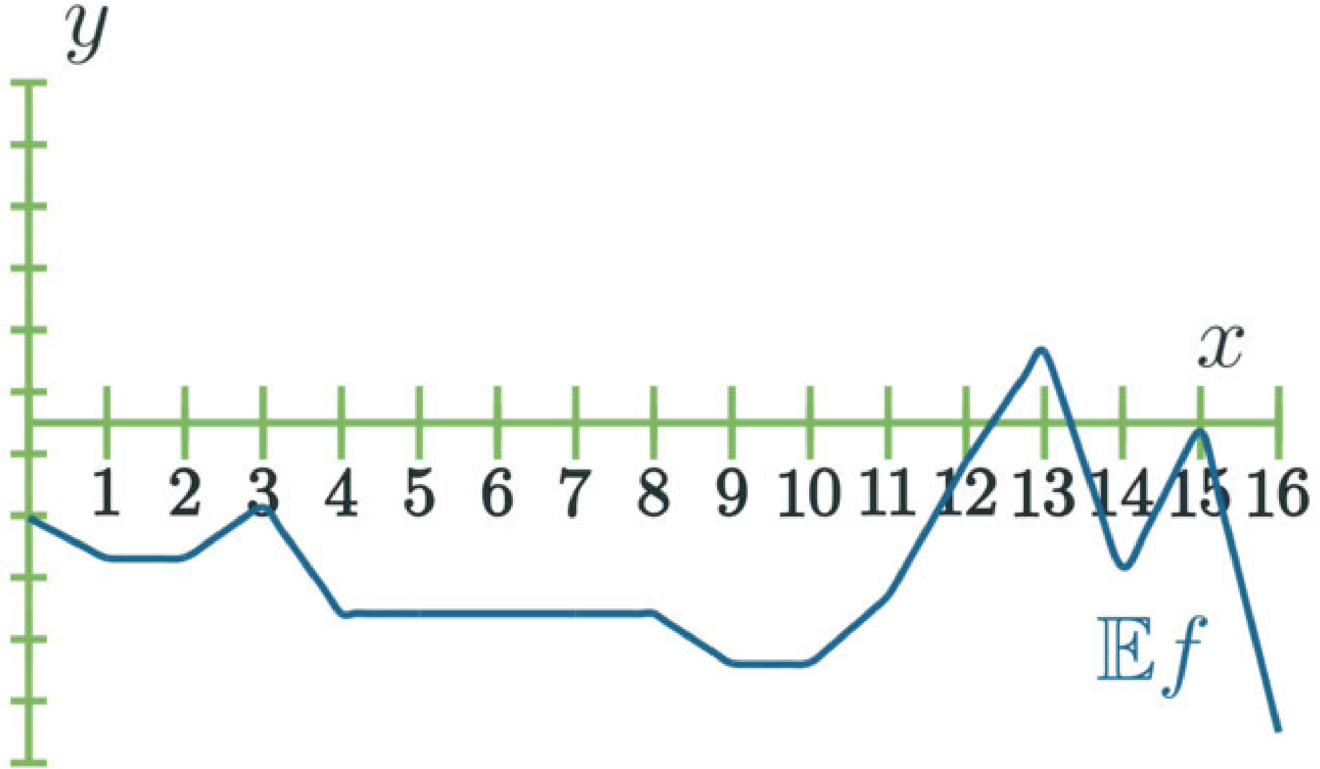

A martingale

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Let $B_t = \left(B_t^{(1)}, \ldots, B_t^{(n)}\right) \in \mathbb{R}^n$ be a stochastic process s.t:

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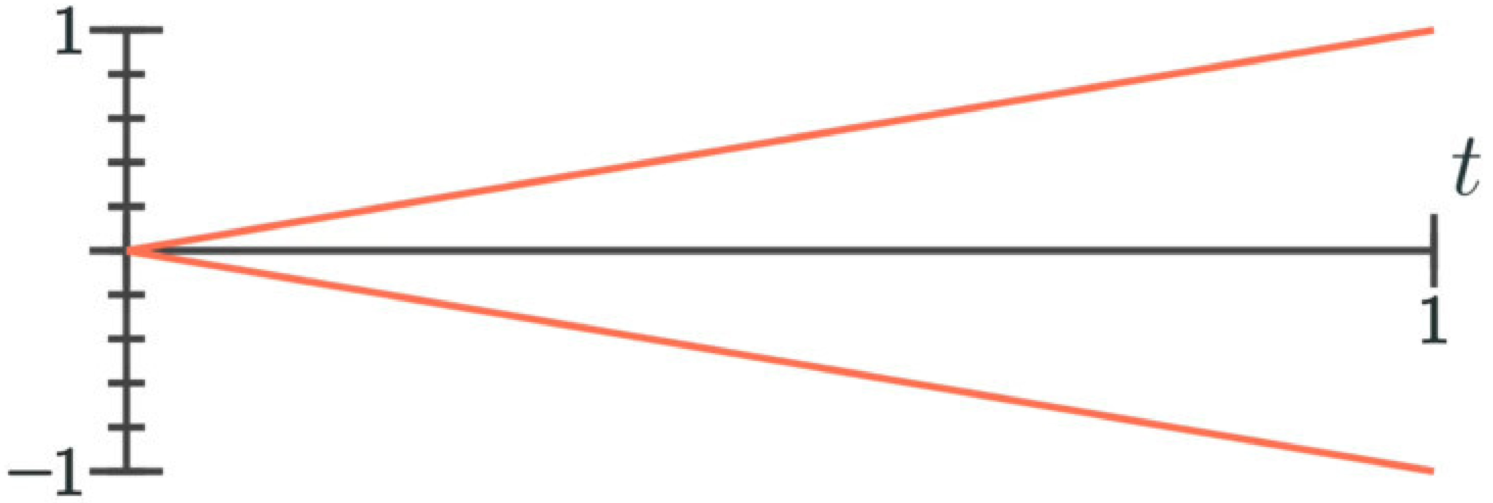

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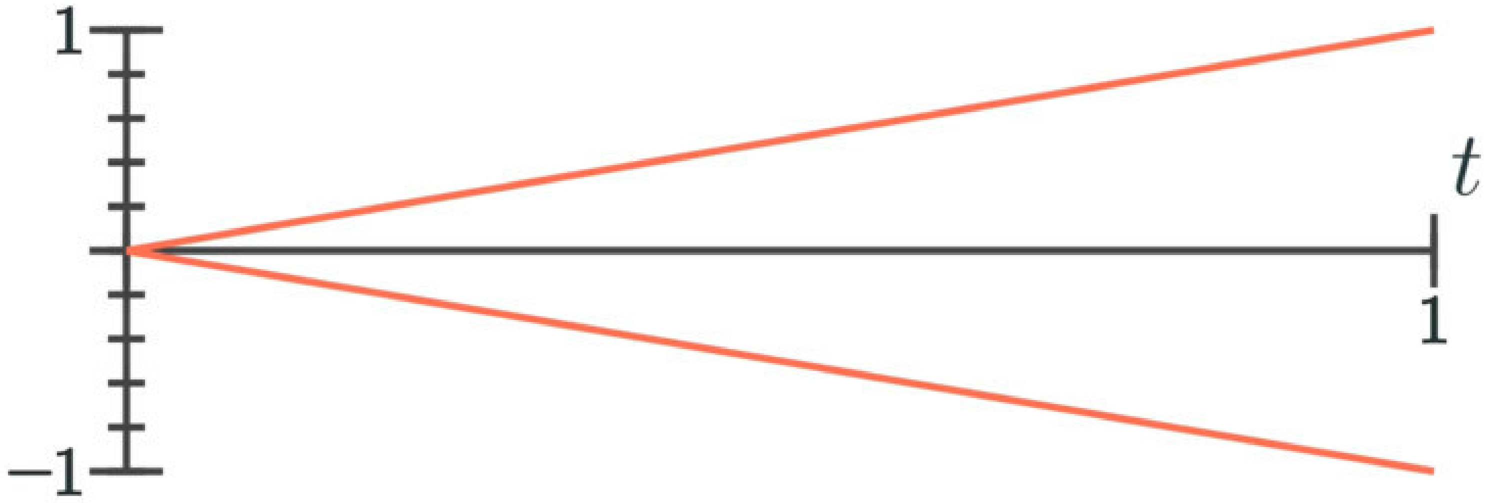

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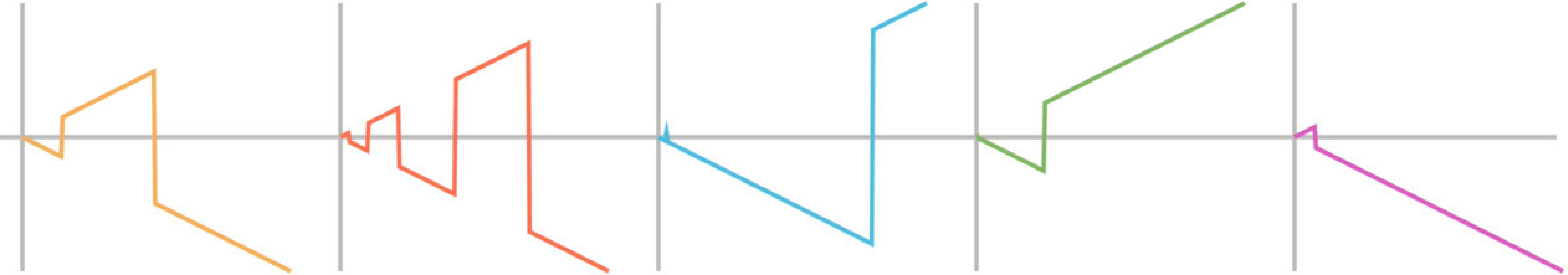

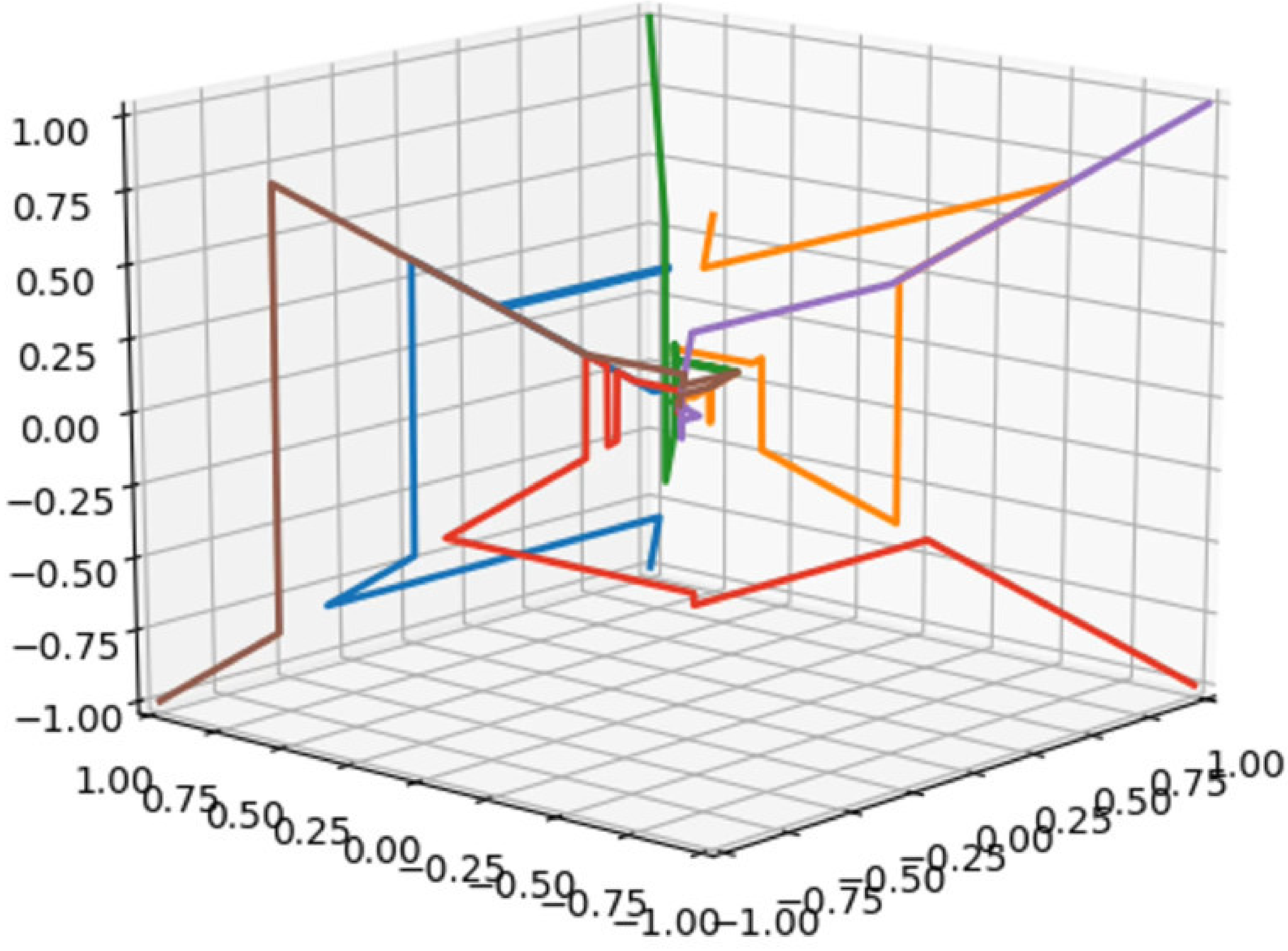

1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
−1.00
−0.75
−0.50
−0.25
0.00
0.25
0.50
0.75
1.00

Jump probability

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$$\begin{aligned} B_t^{(i)} &= \mathbb{E}\left[B_{t+\varepsilon}^{(i)} \mid B_s^{(i)}\right] \\ &= -(t+\varepsilon)\text{Sign}\left(B_t^{(i)}\right) p \\ &\quad +(t+\varepsilon)\text{Sign}\left(B_t^{(i)}\right)(1-p) \end{aligned}$$

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$$t\text{Sign}\left(B_t^{(i)}\right) = B_t^{(i)} = \mathbb{E}\left[B_{t+\varepsilon}^{(i)} \mid B_s^{(i)}\right]$$
$$= -(t+\varepsilon)\text{Sign}\left(B_t^{(i)}\right) p$$
$$+(t+\varepsilon)\text{Sign}\left(B_t^{(i)}\right)(1-p)$$

$p = \dfrac{\varepsilon}{2(t+\varepsilon)} \longrightarrow B_t^{(i)}$ is a Poisson process with rate $\dfrac{1}{2t}$

Jump processes

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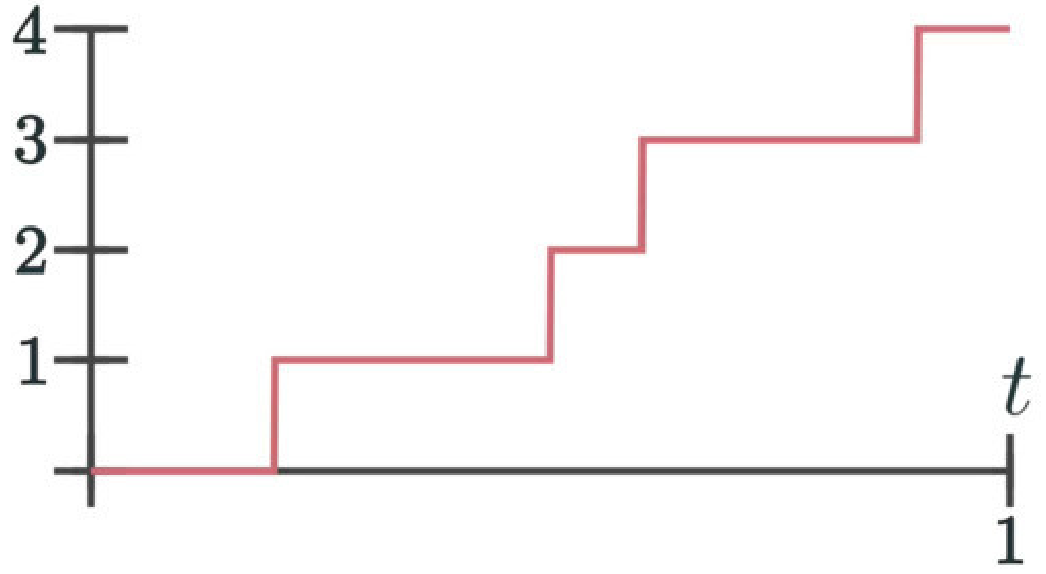

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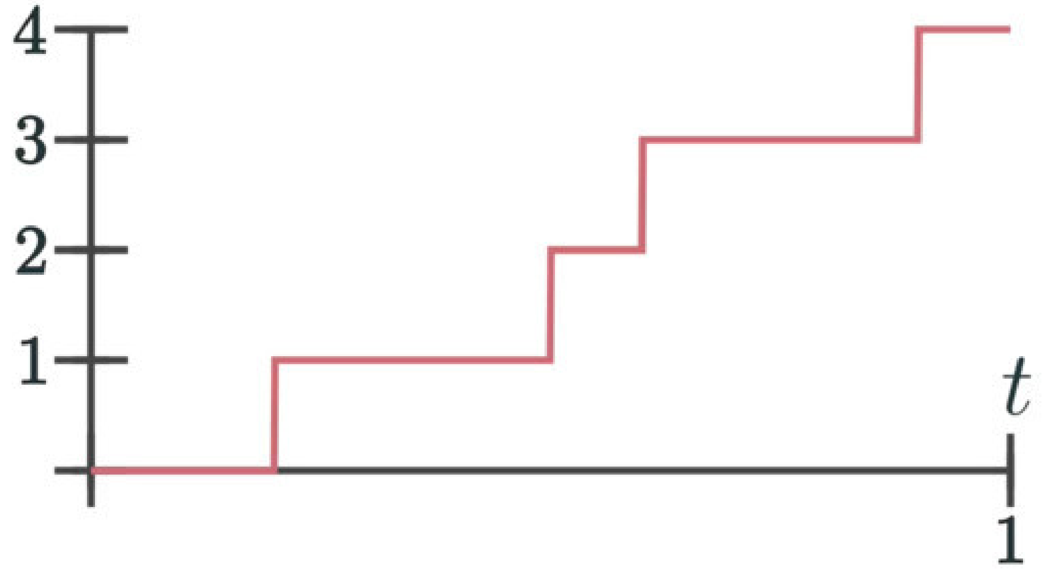
Quadratic variation:



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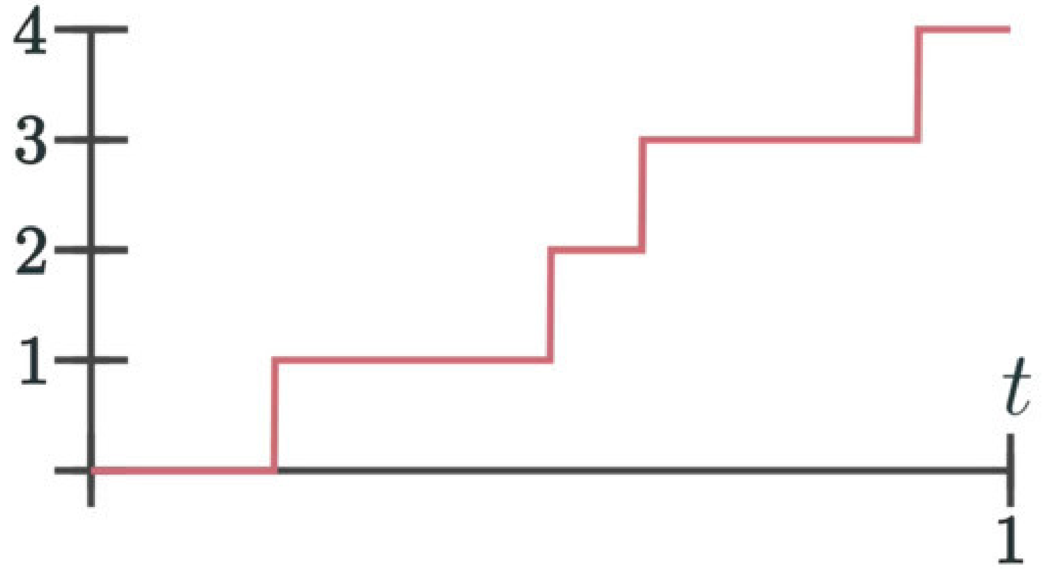

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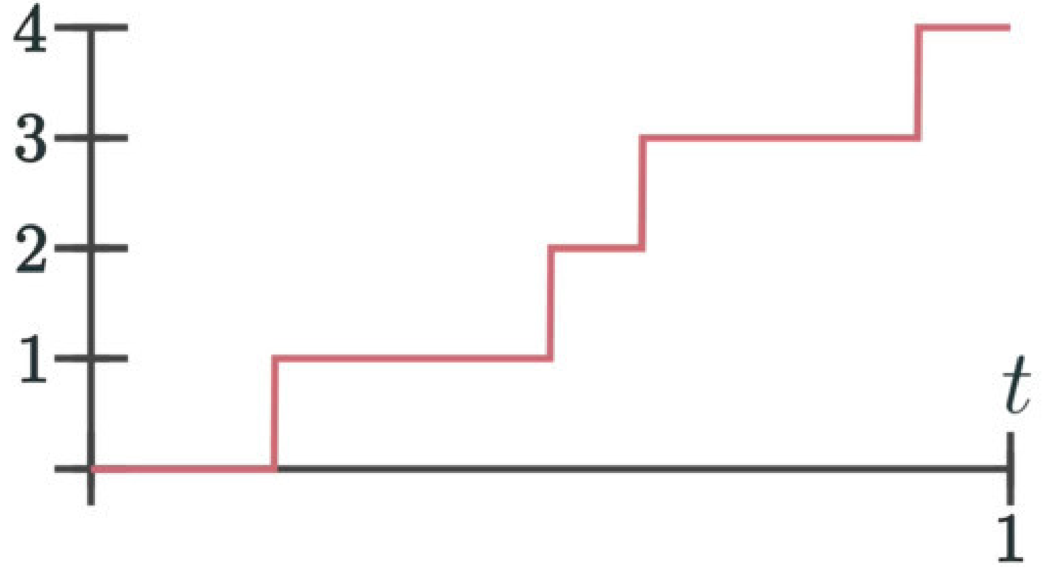
For jump prcs: $[N]_t = \sum_{s \leq t} (\Delta N_s)^2$



How to build

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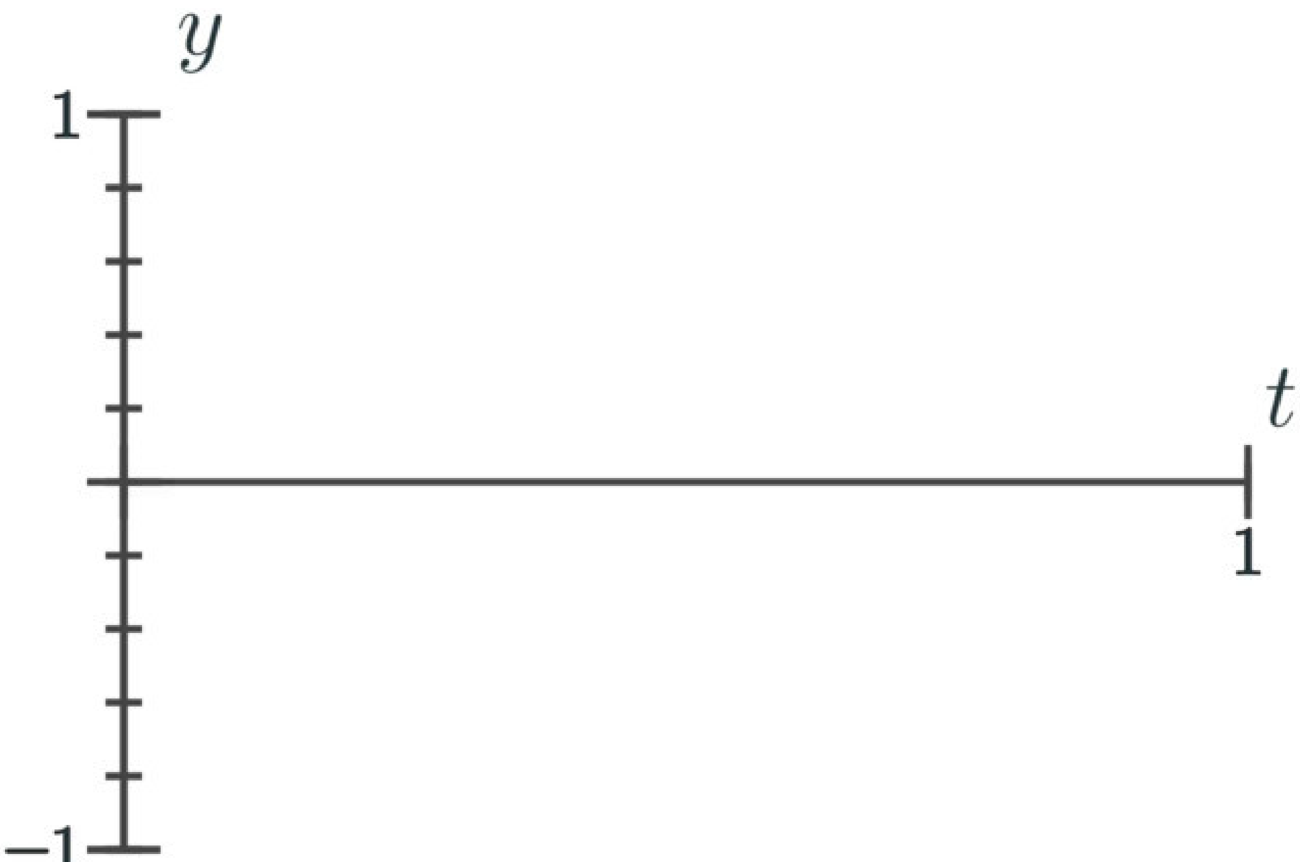

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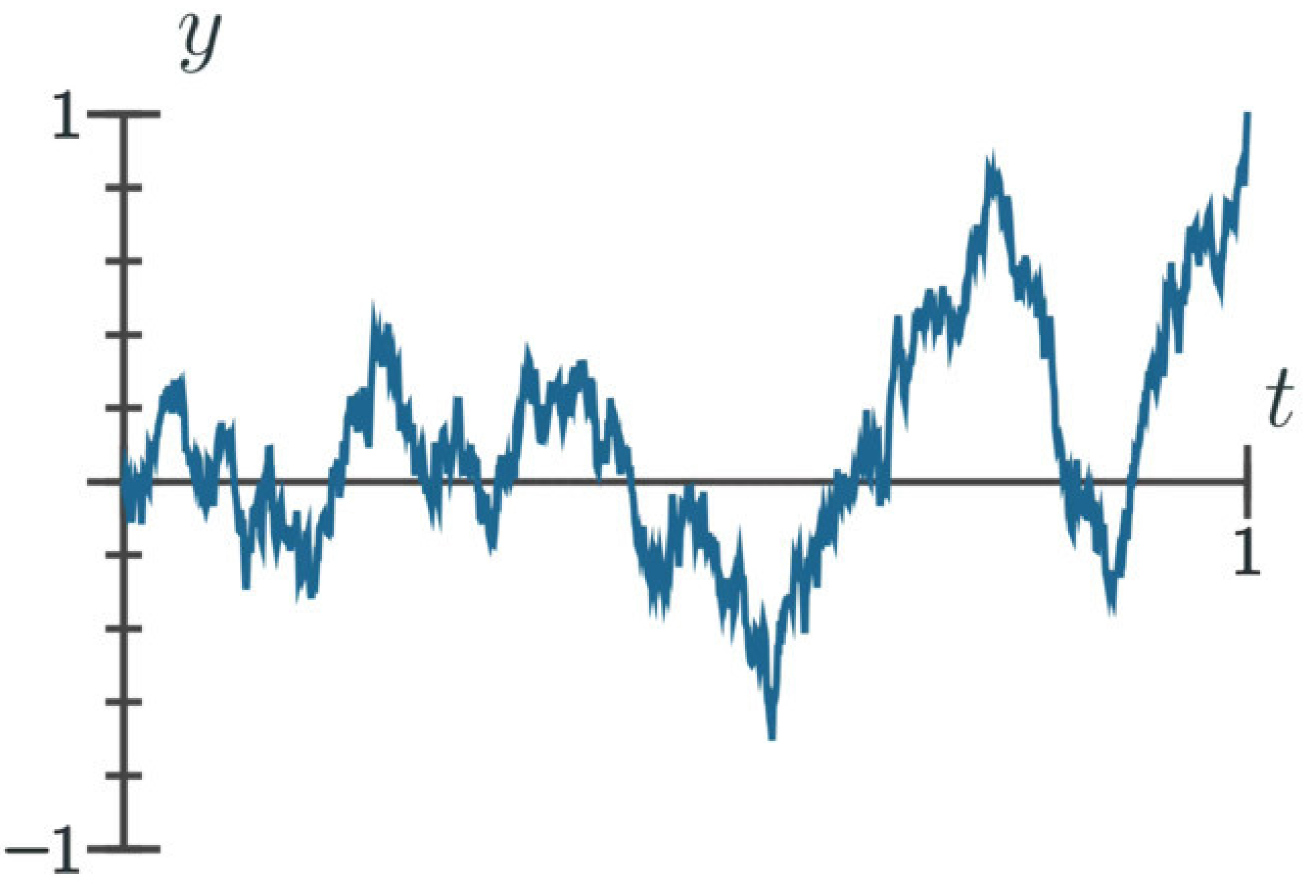

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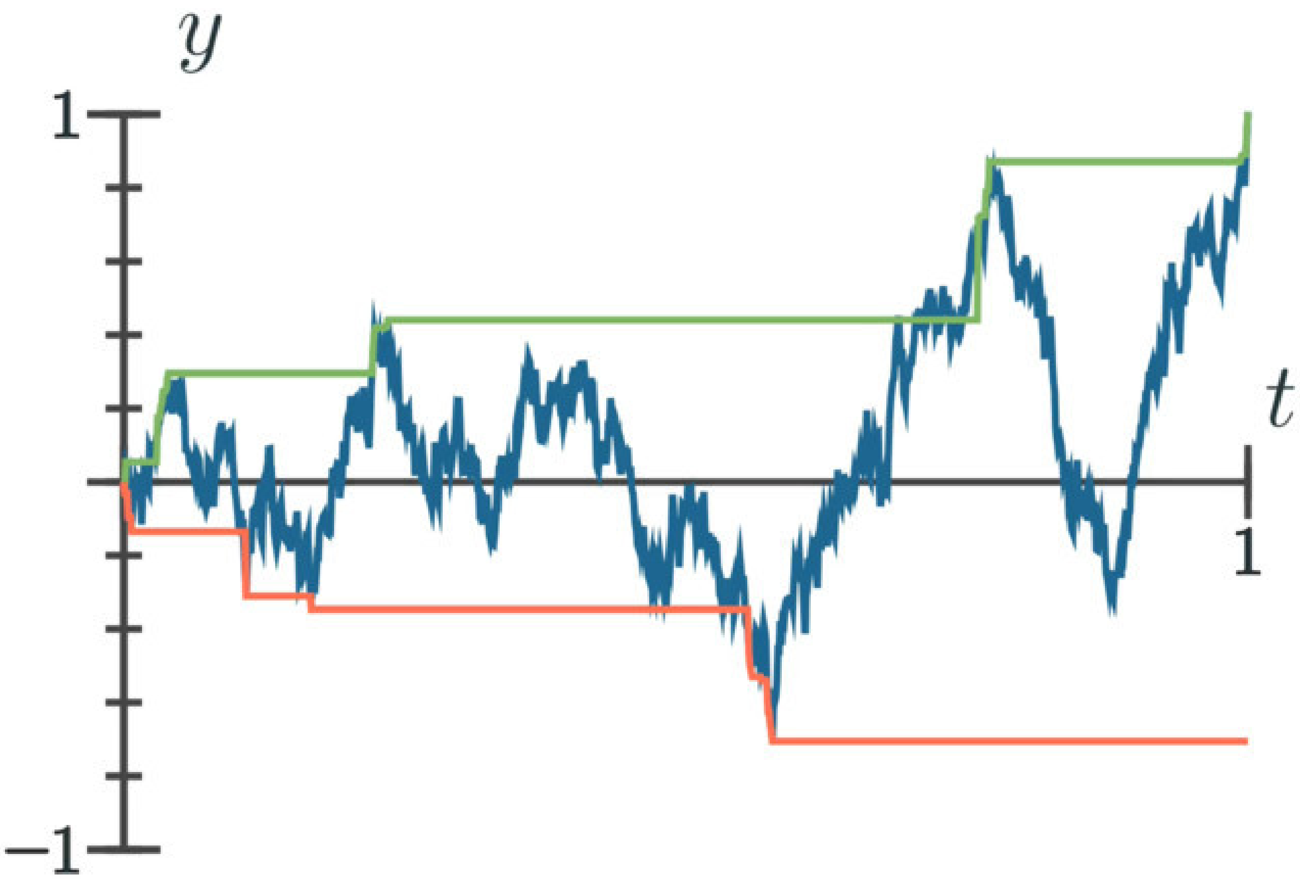

$$\tau(t) = \inf\{s > 0 \,\big|\, |W_s| > t\}$$

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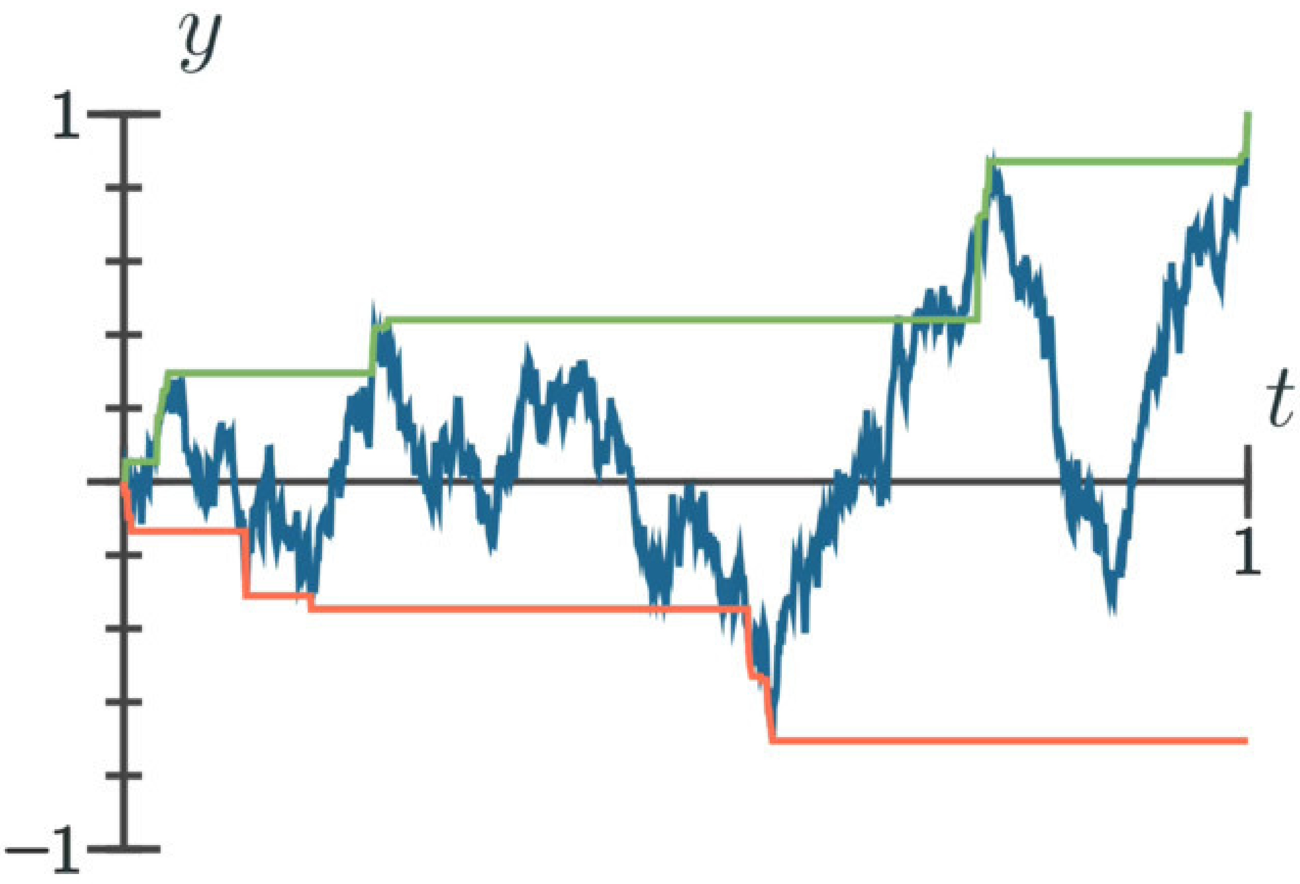

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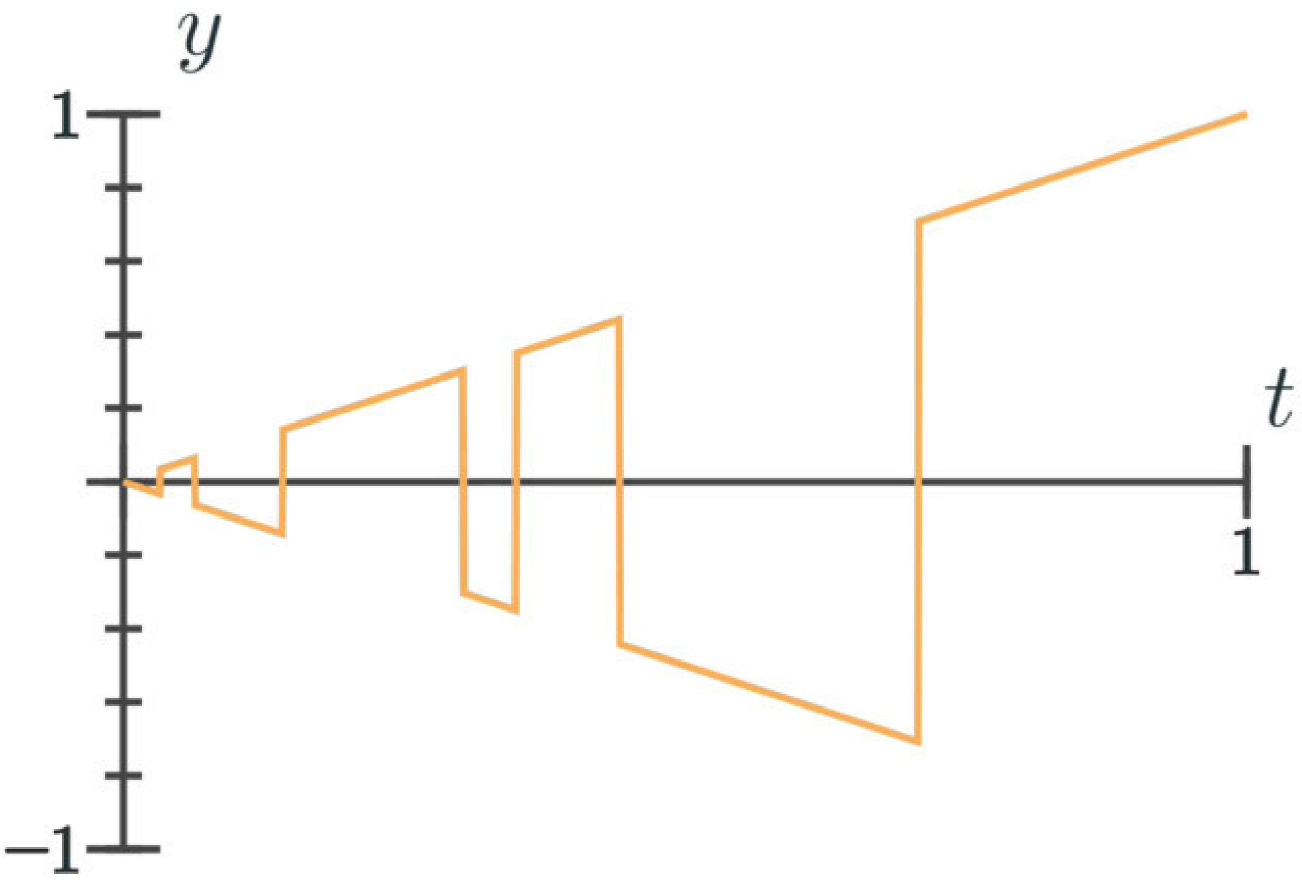

$$X_t = W_{\tau(t)}$$

How to use:

How to use: $$f(x) = \sum_{S \subseteq [n]} \widehat{f}(S) \prod_{i \in S} x_i$$

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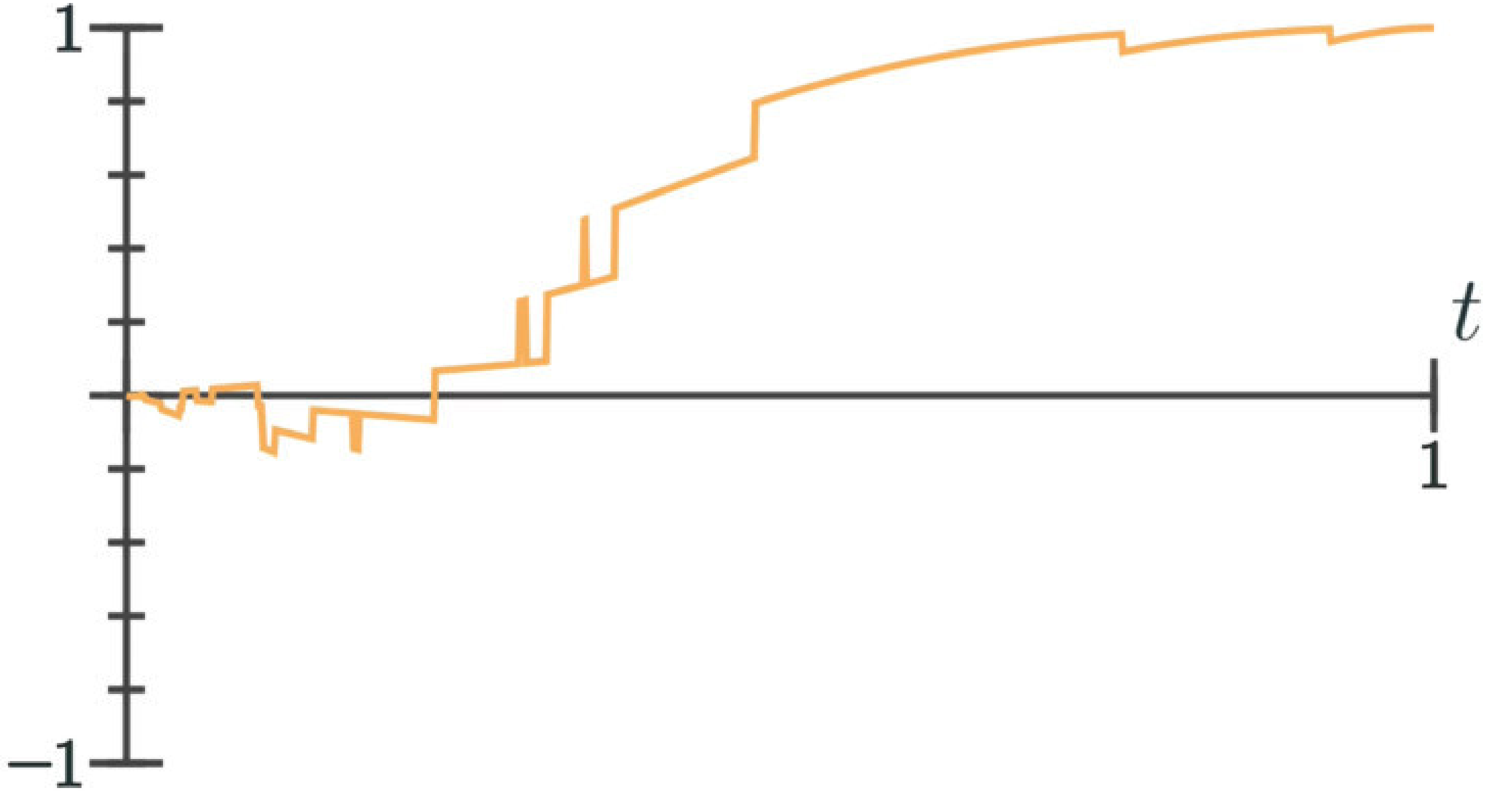

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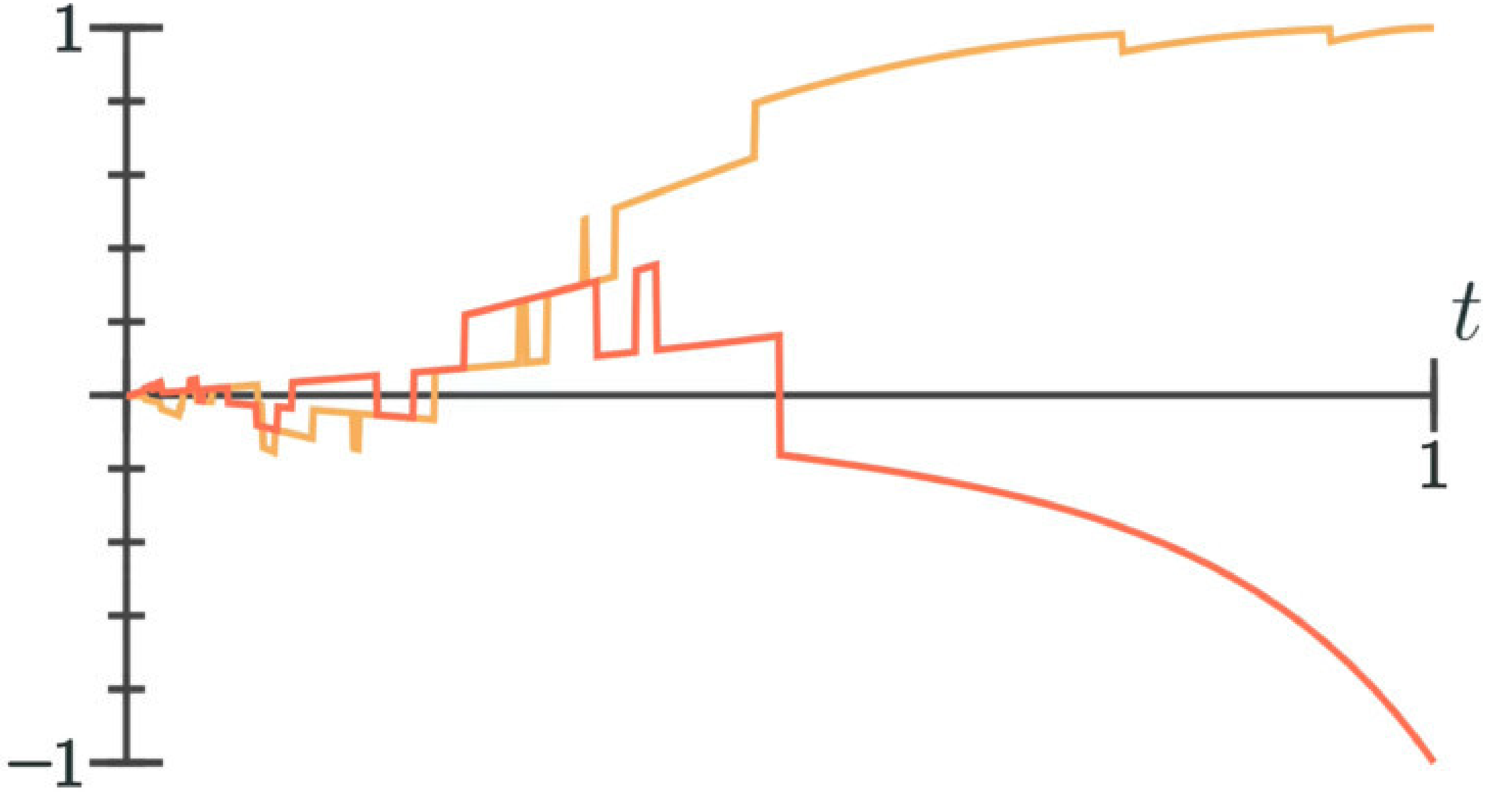

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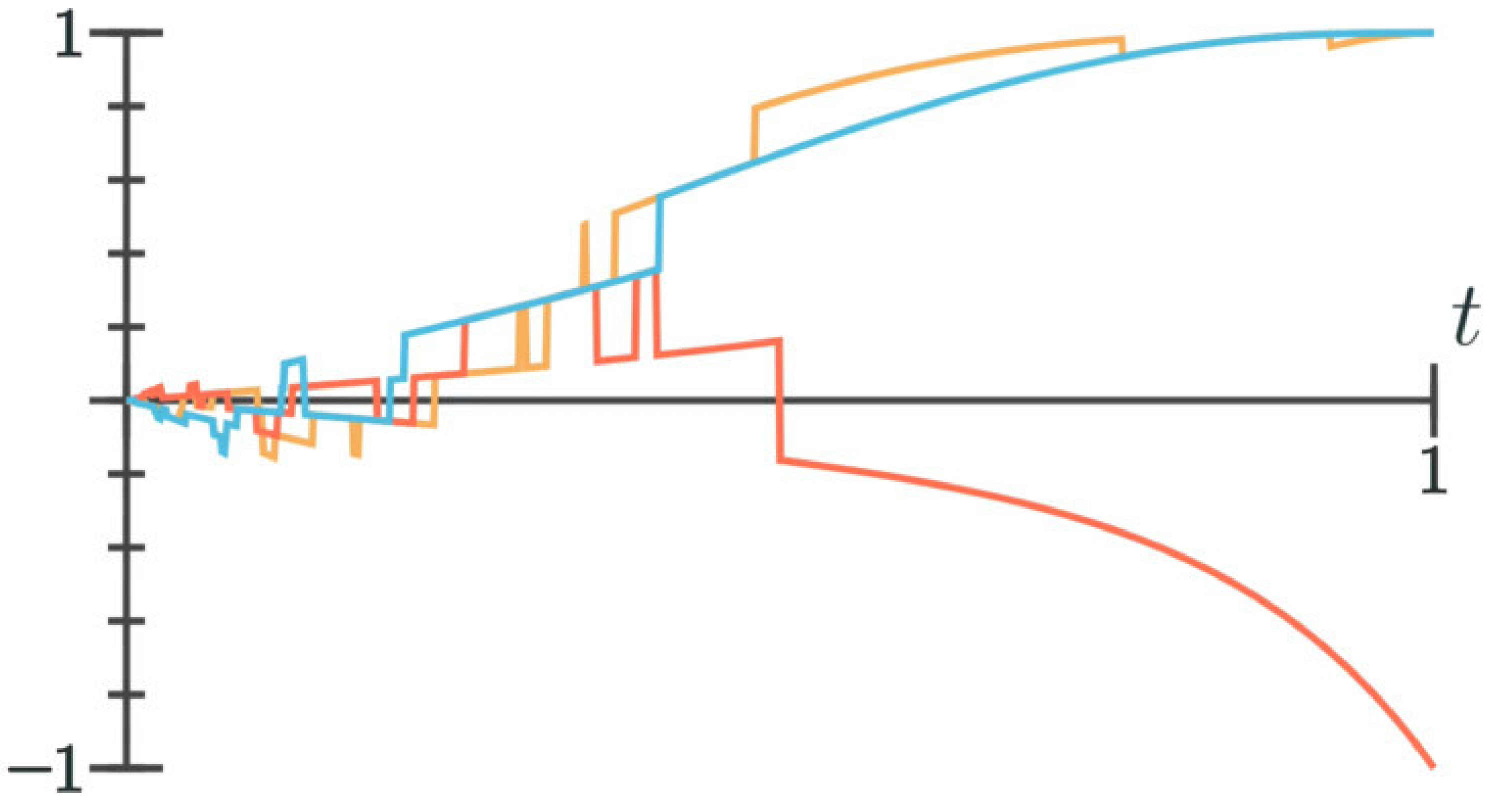

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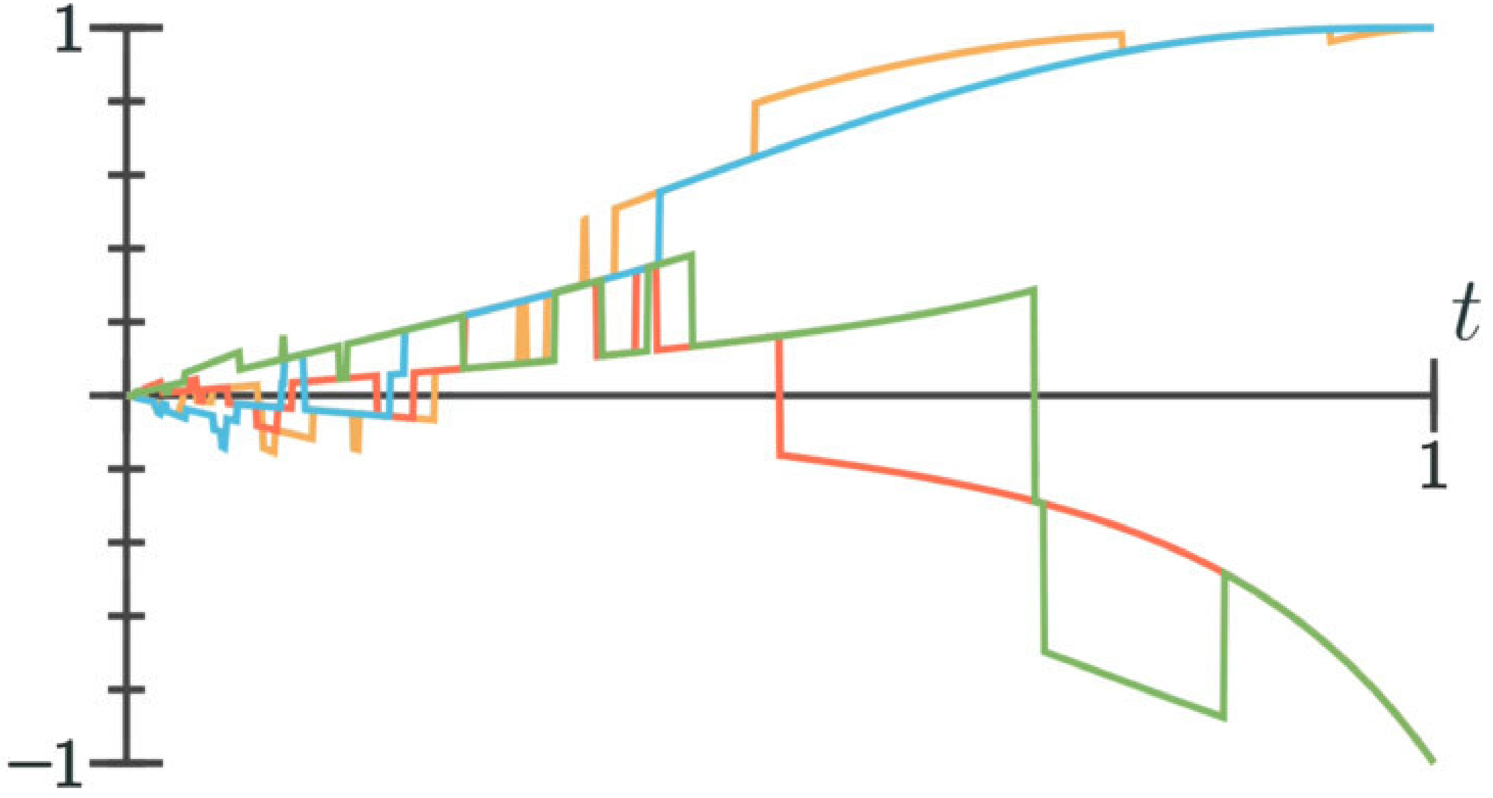

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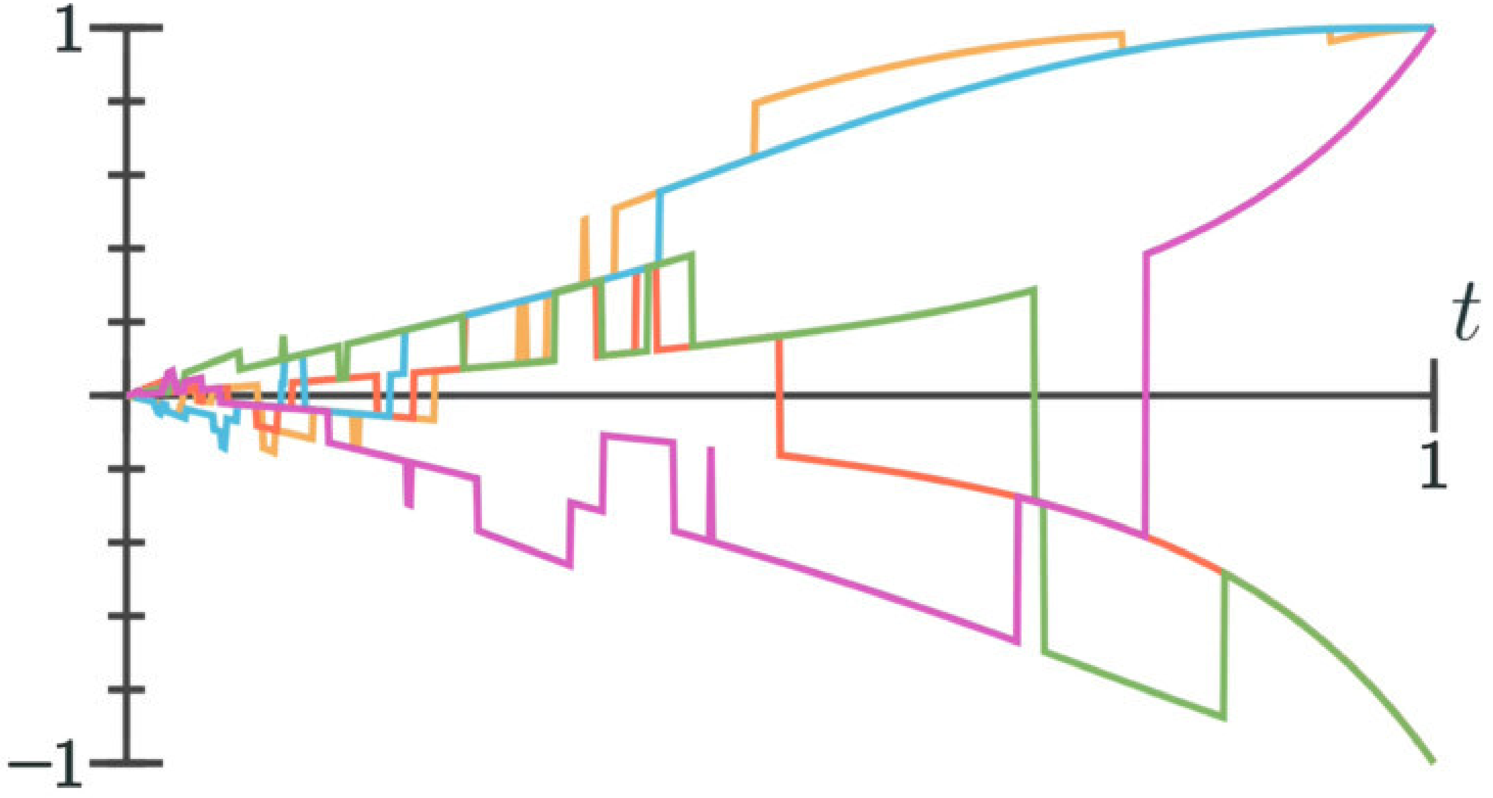

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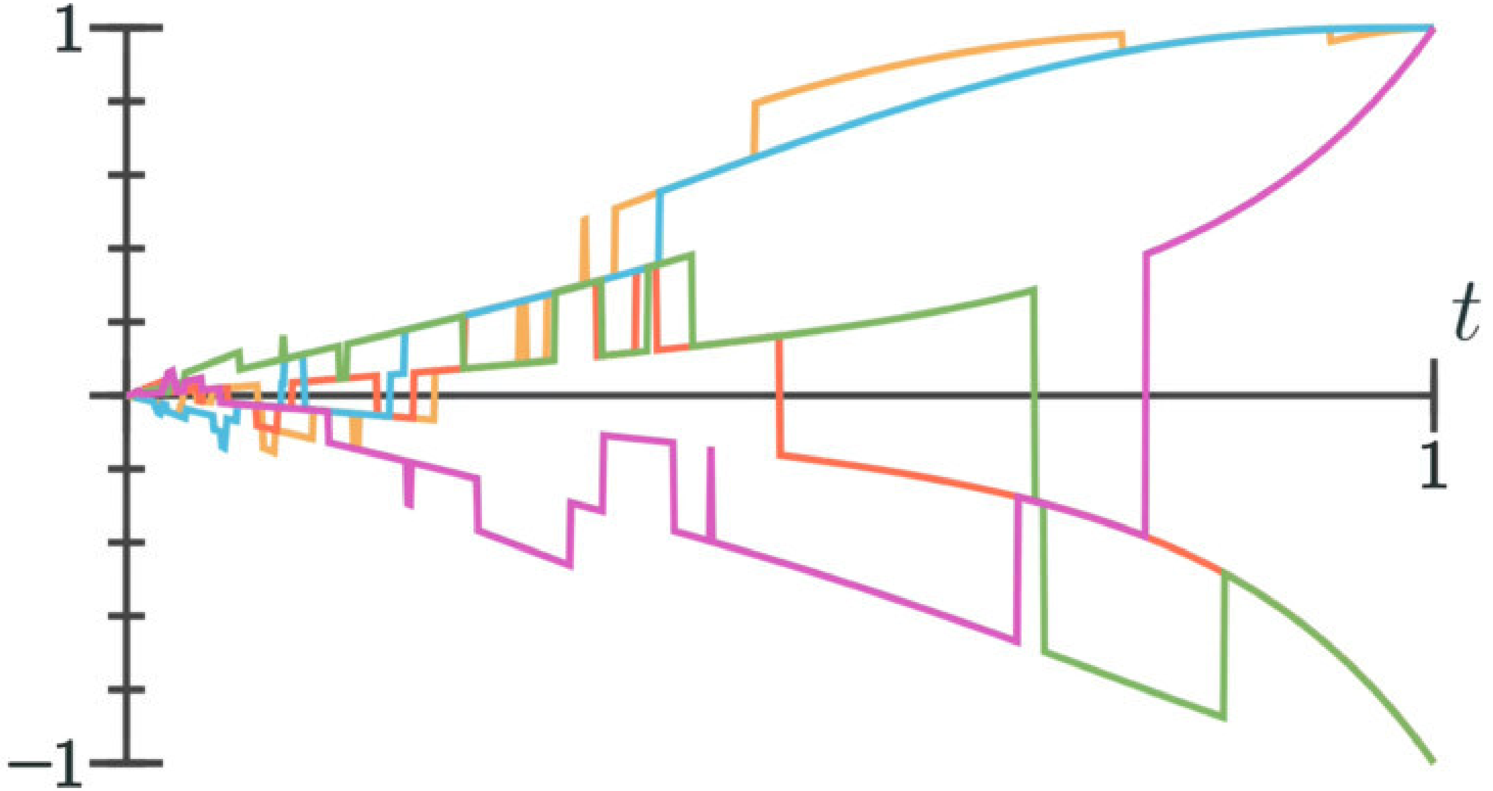
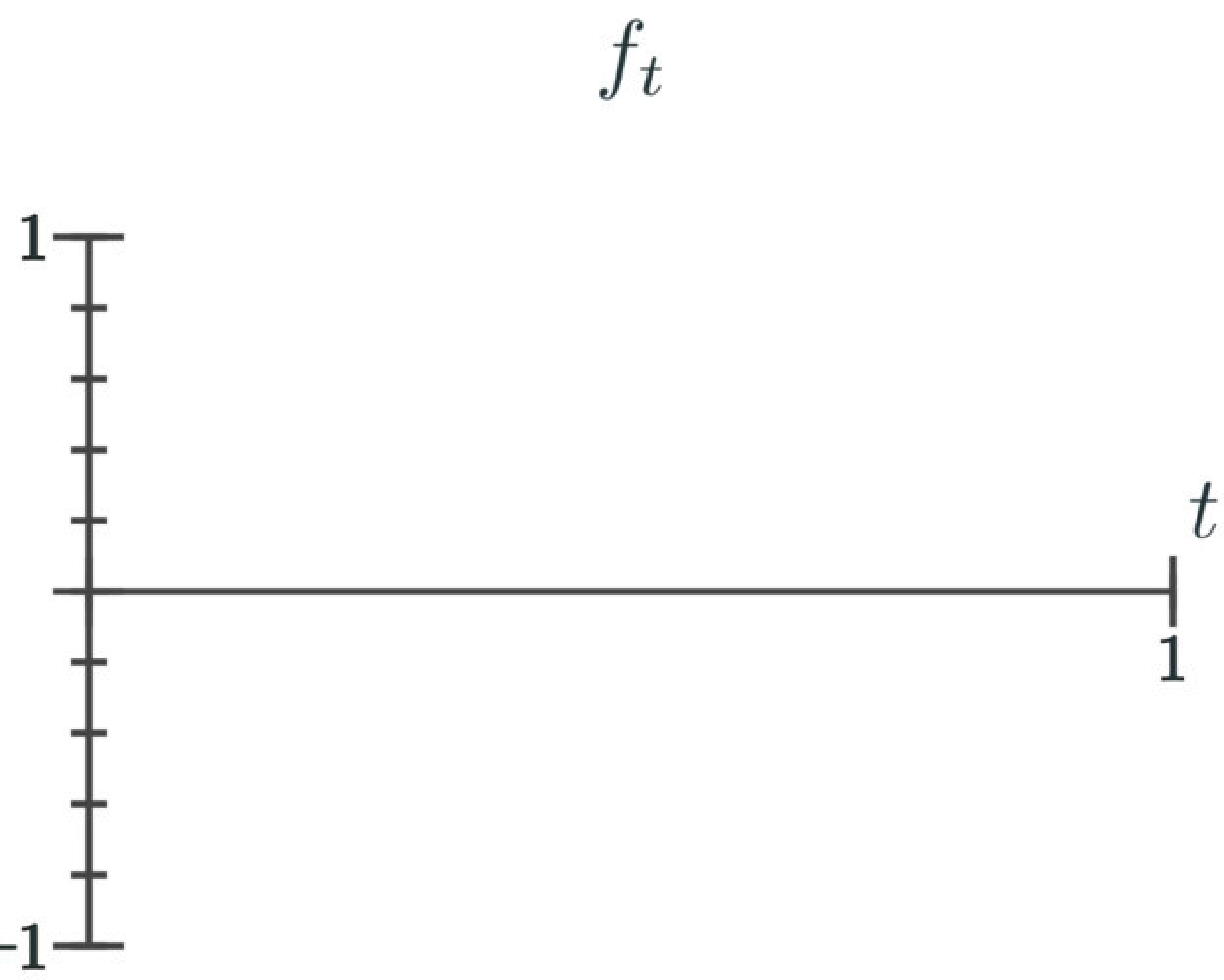
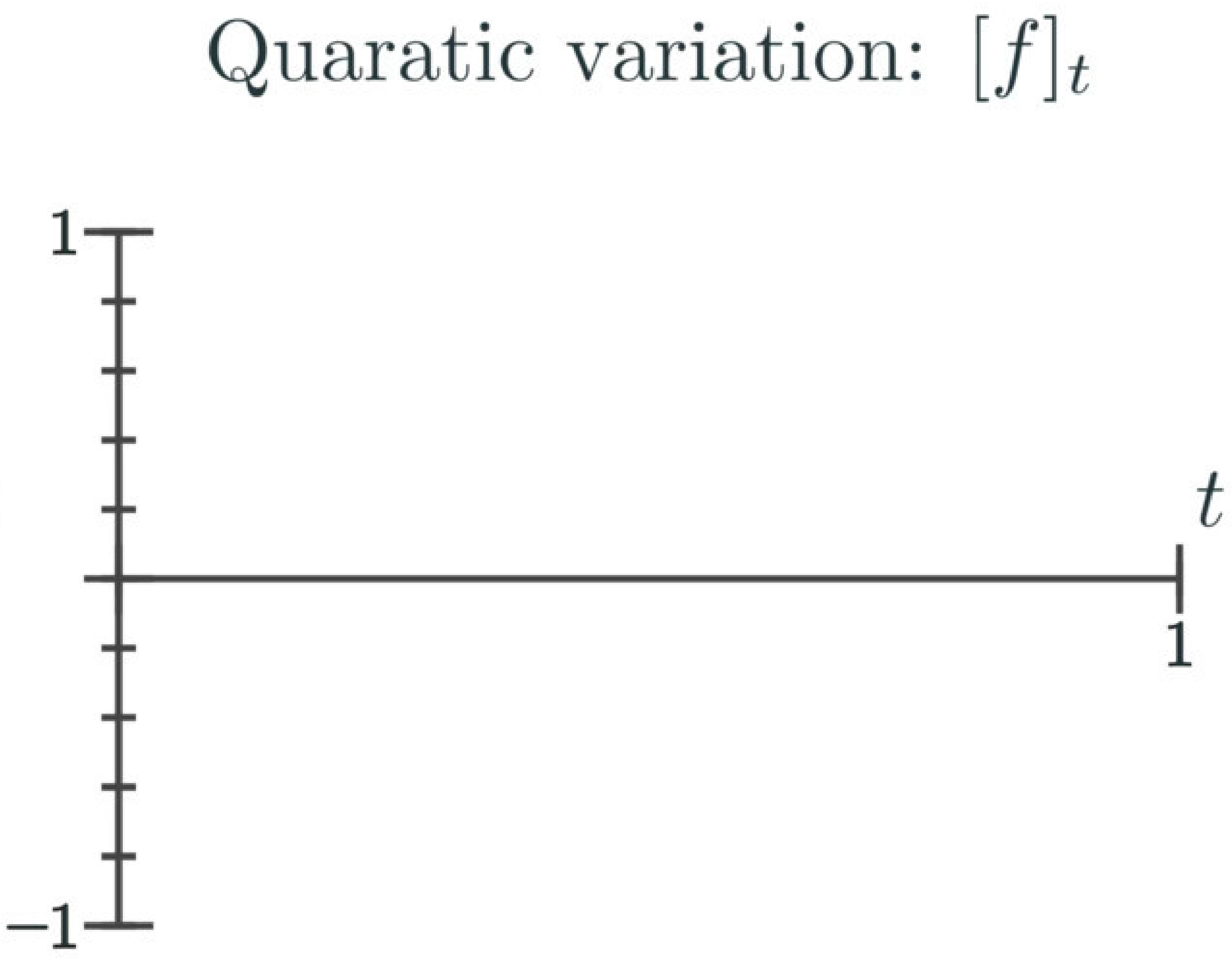
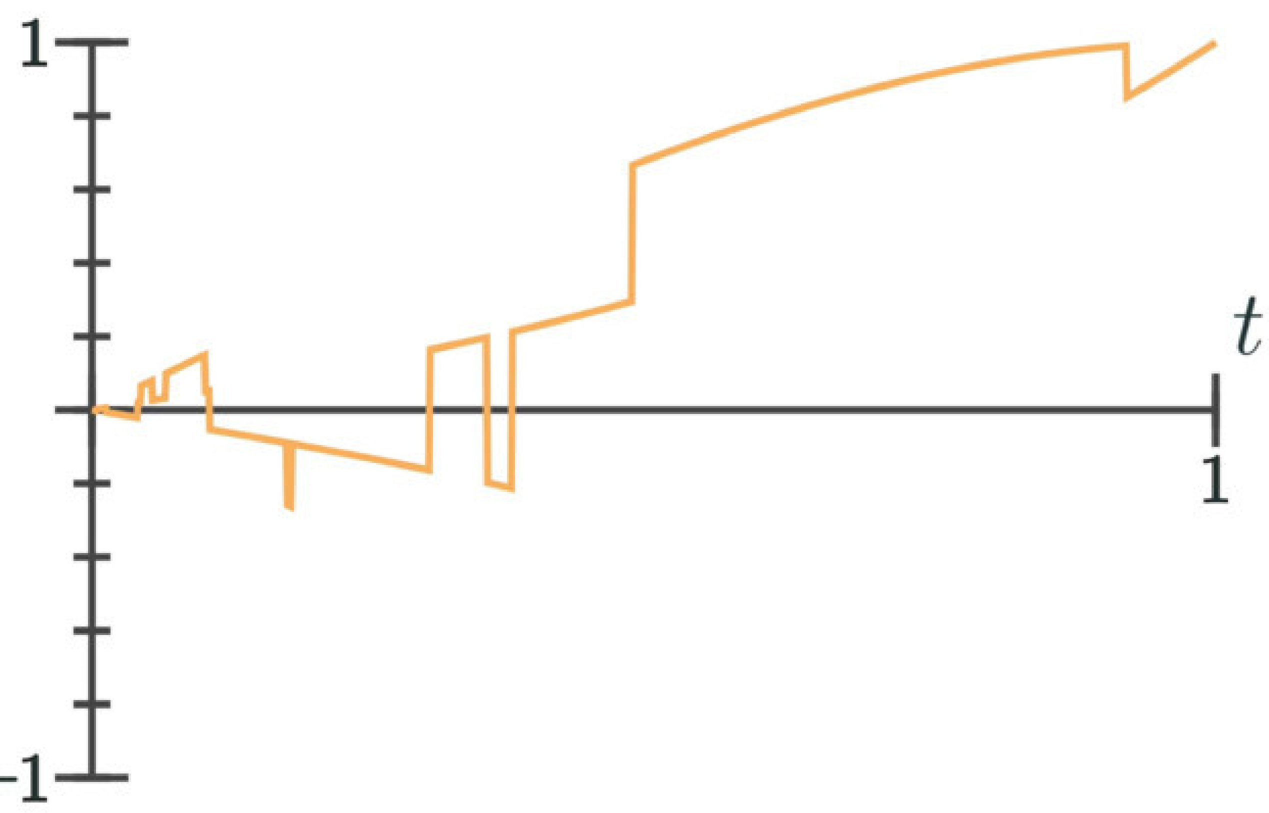


Jump at t: $\Delta f(B_t) = 2t\partial_i f(B_t)$

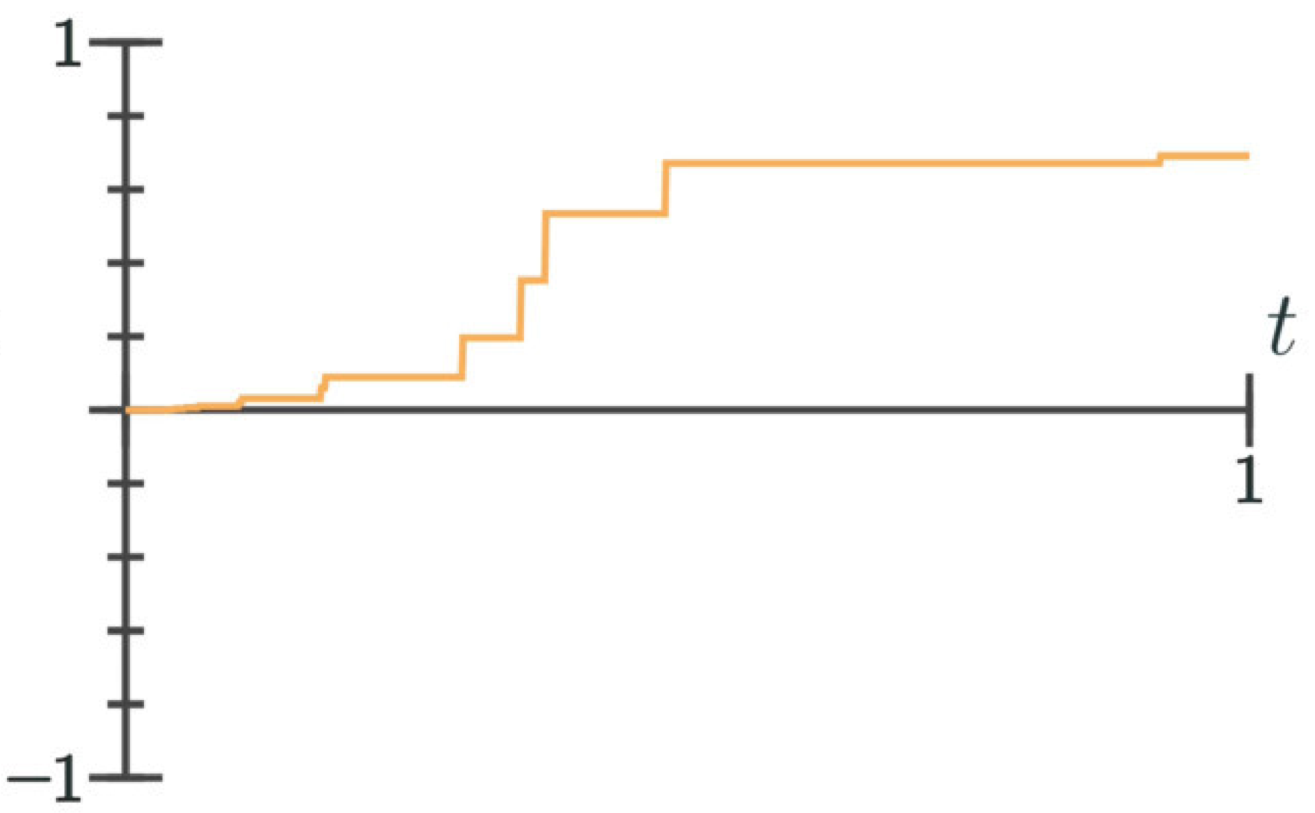

f_t
1
-1
t
1


Quaratic variation: $[f]_t$
1
−1
t
1

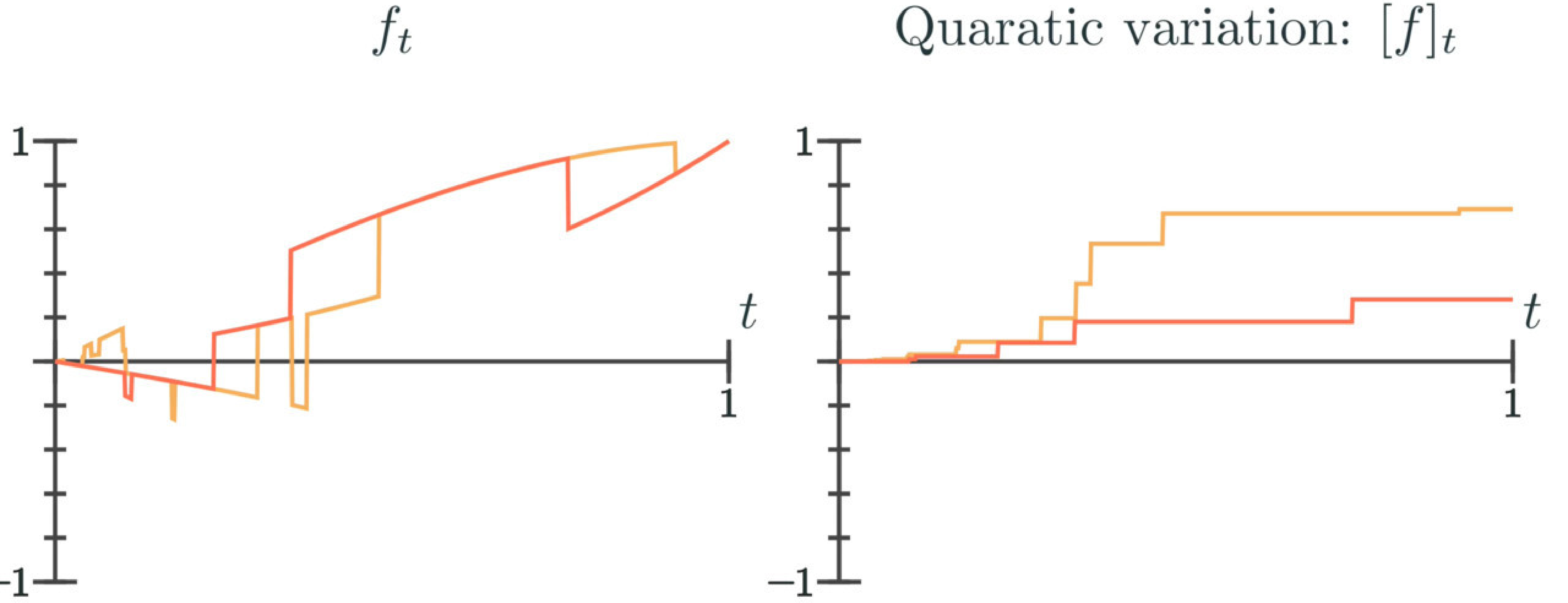
f_t



Quaratic variation: $[f]_t$



- ▷ $[f]_t = \sum_{s \leq t \text{ jumps}} (\Delta f_s)^2$
- ▷ $\mathrm{Var}[f_t] = \mathbb{E}[f]_t$



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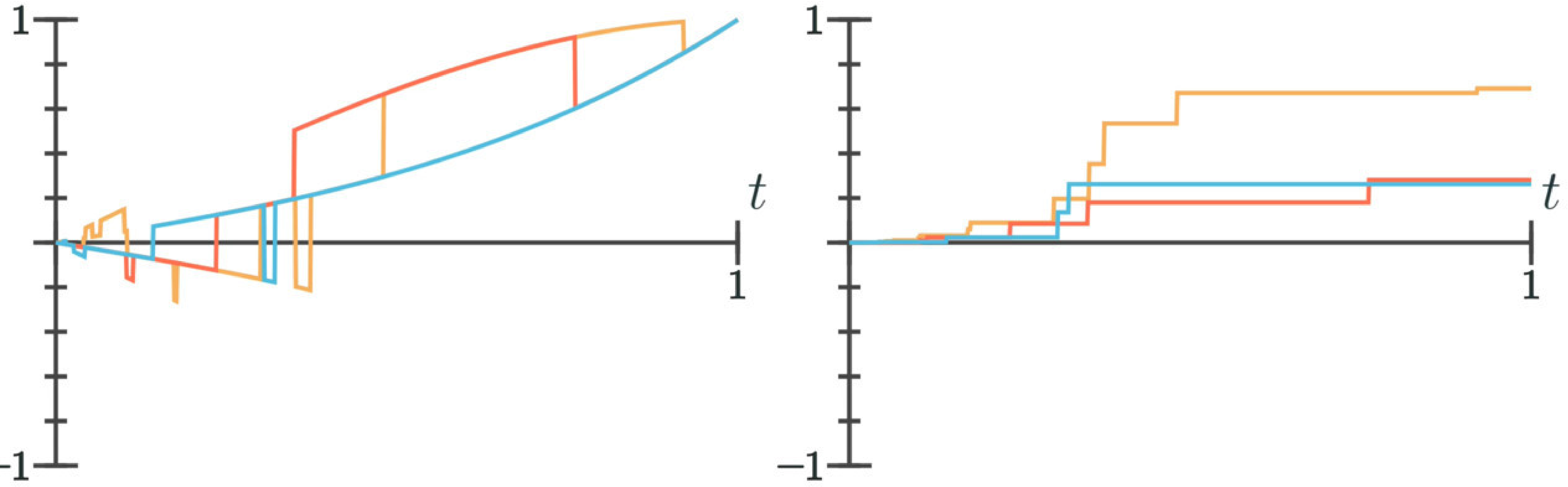

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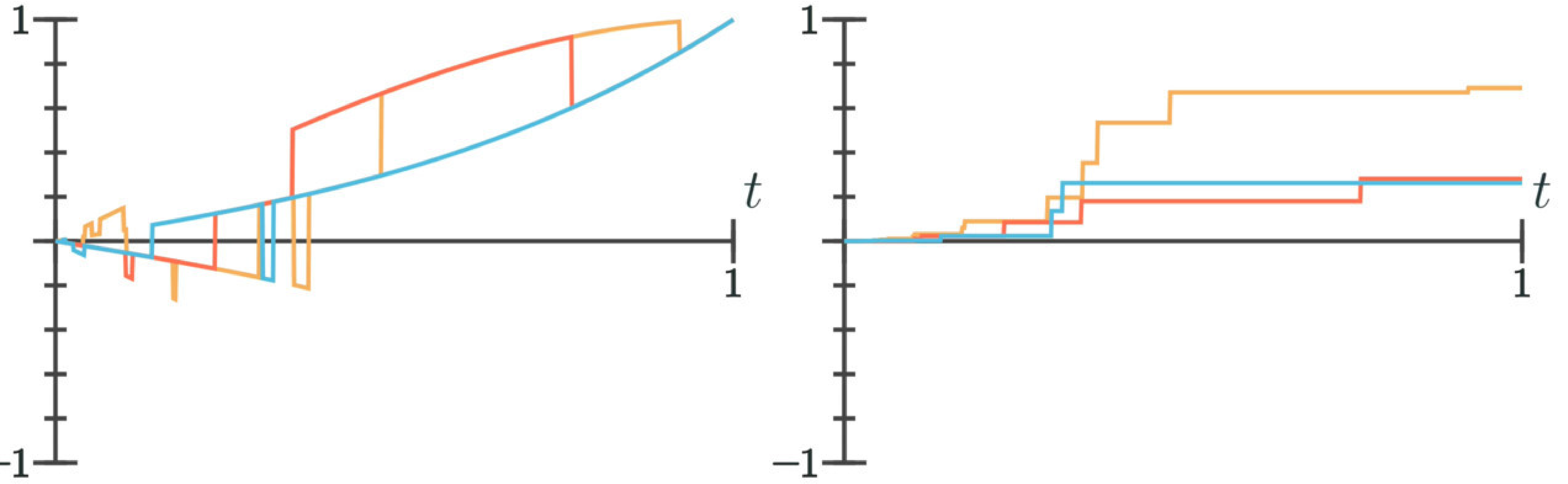

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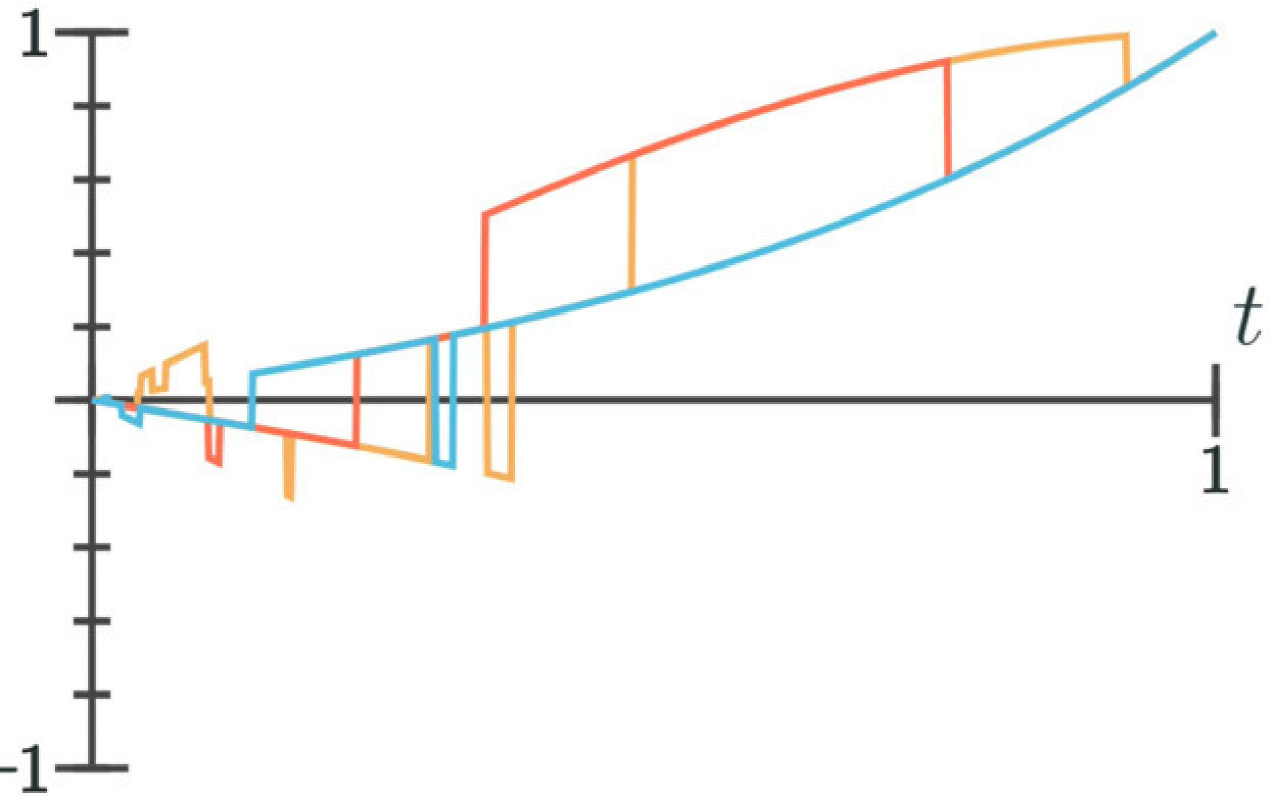

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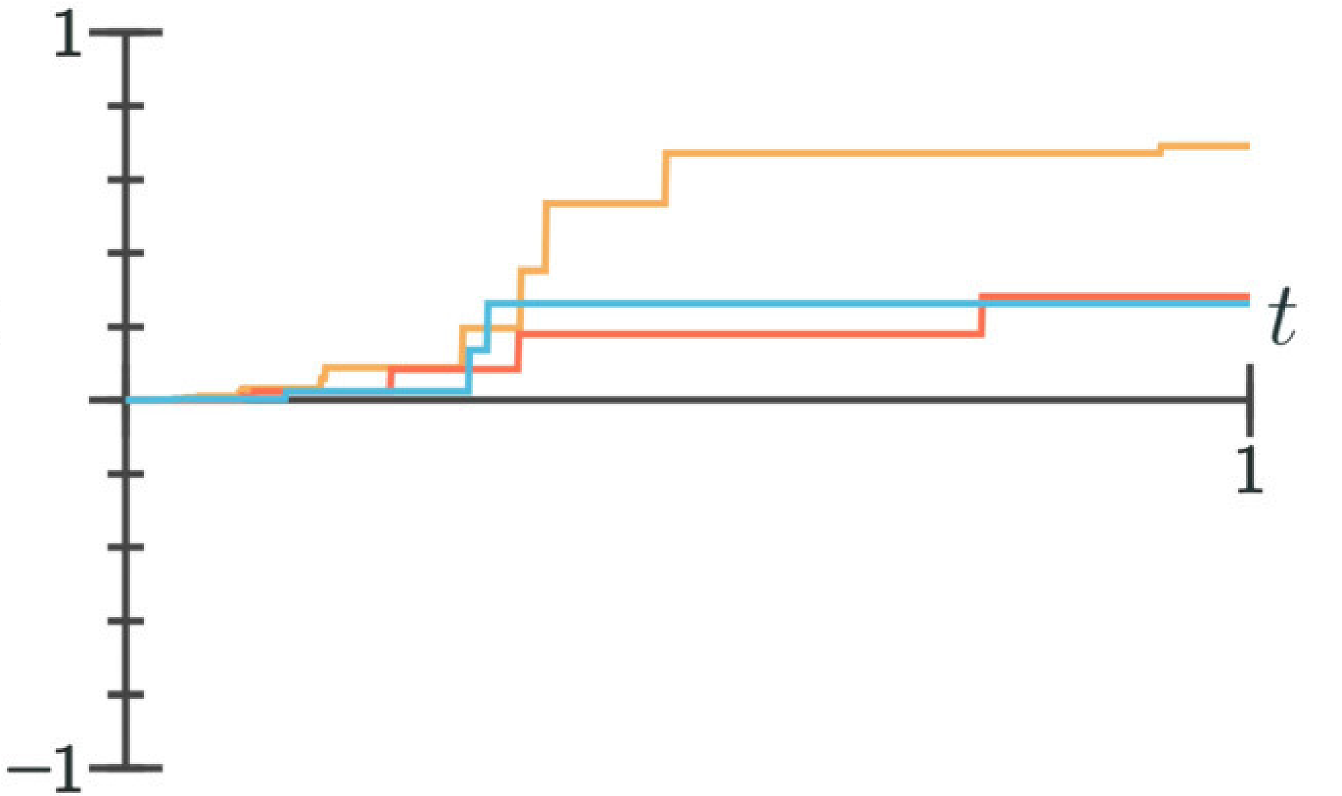

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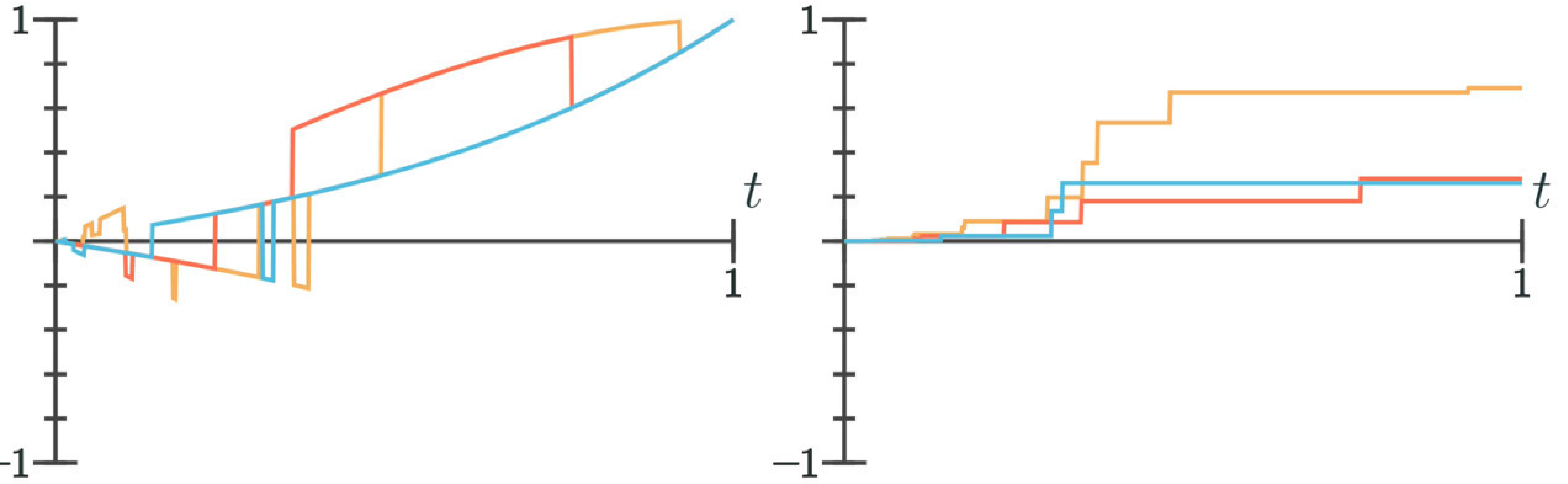

Jump rate for each $i = 1/2t$

Jump at t: $\Delta f(B_t) = 2t\partial_i f(B_t)$

$$\mathrm{Var}[f] = \mathrm{Var}[f(B_1)]$$
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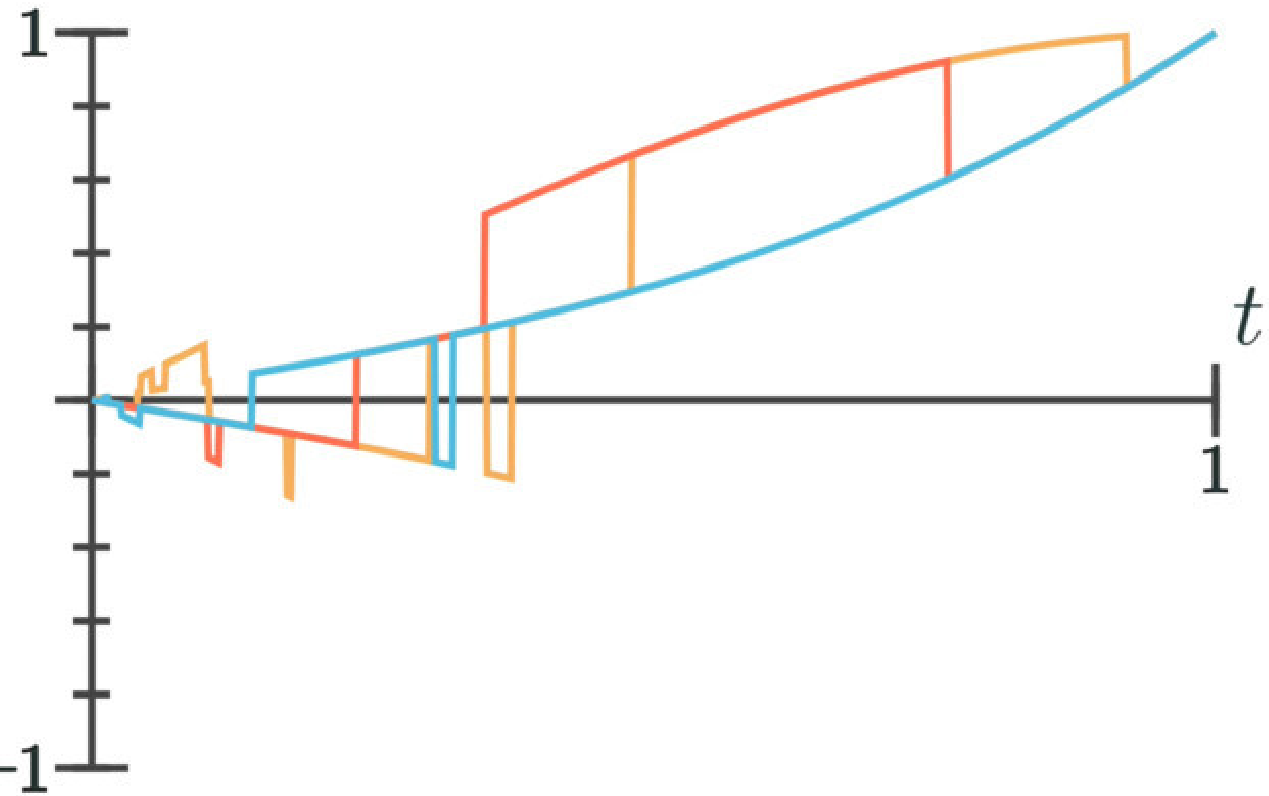
f_t



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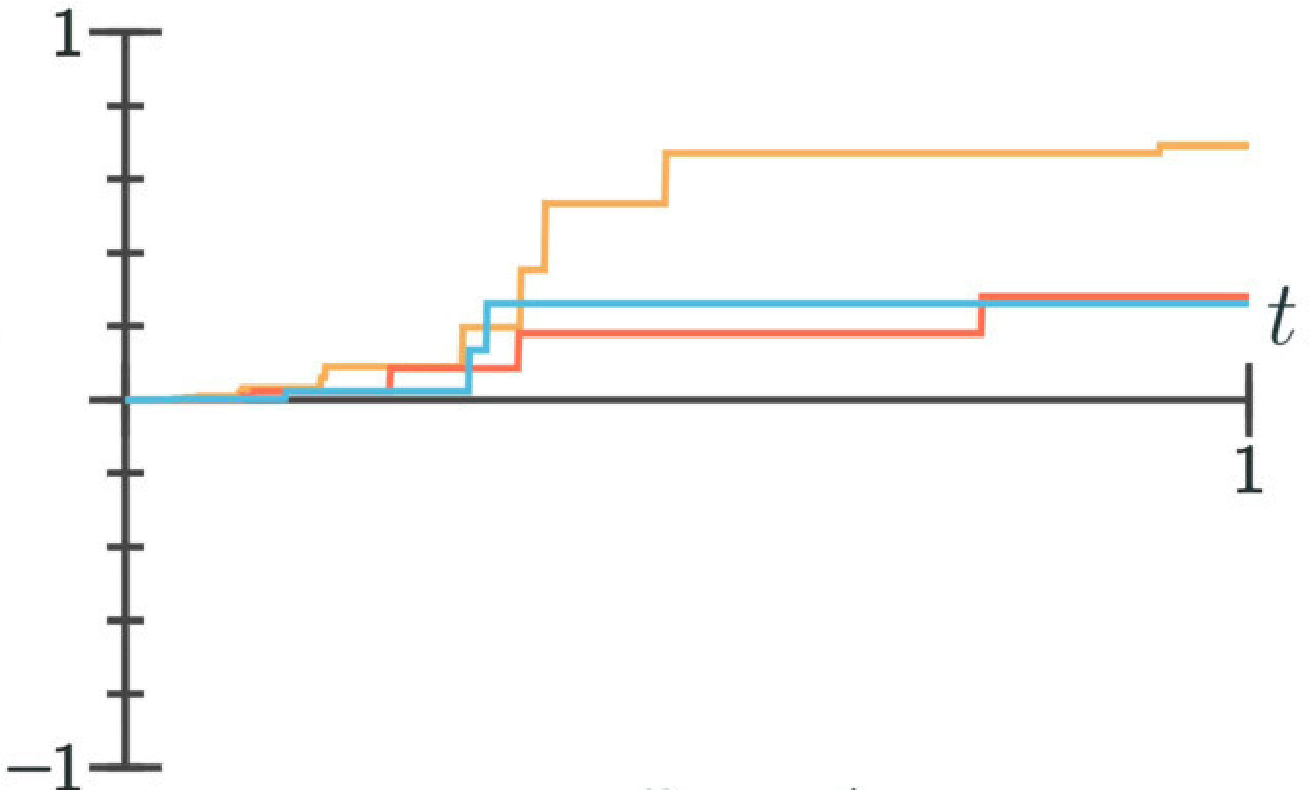

Jump rate for each $i = 1/2t$

$$\mathrm{Var}[f] = 2\mathbb{E}\sum_{i=1}^{n}\int_0^1 t(\partial_i f_t)^2 dt$$

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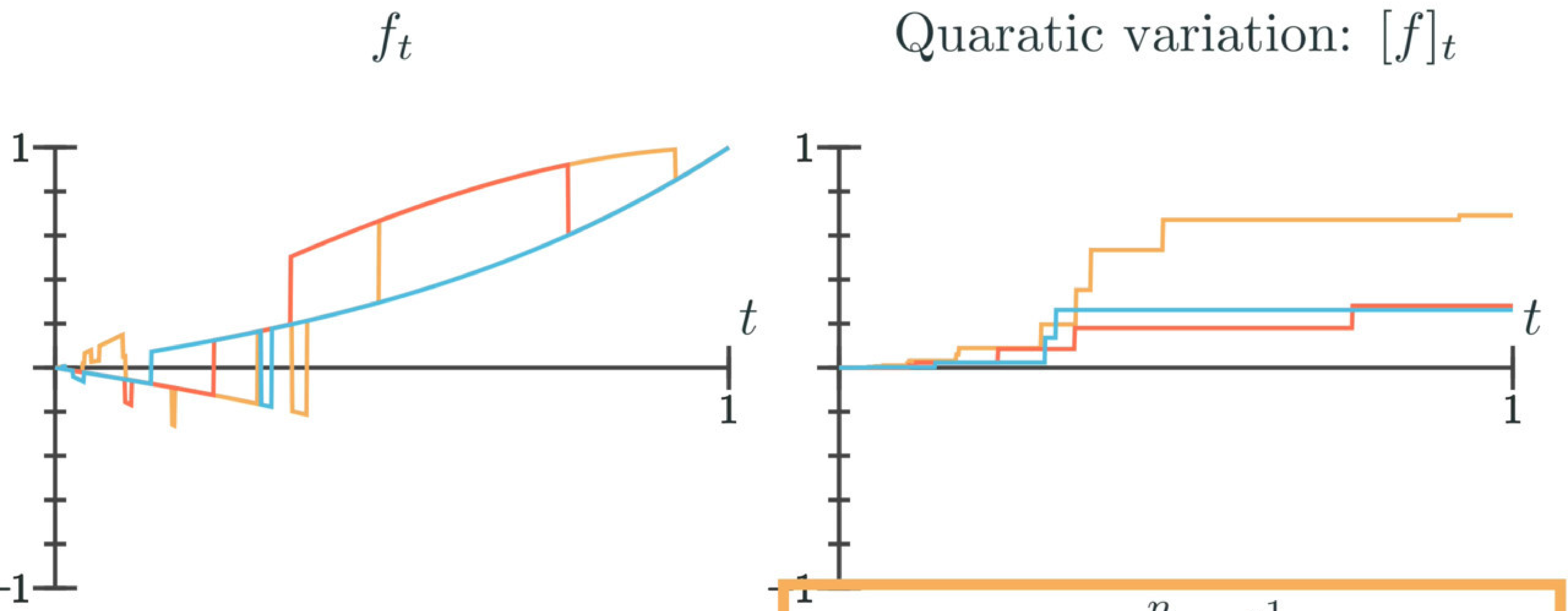

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Stopping times

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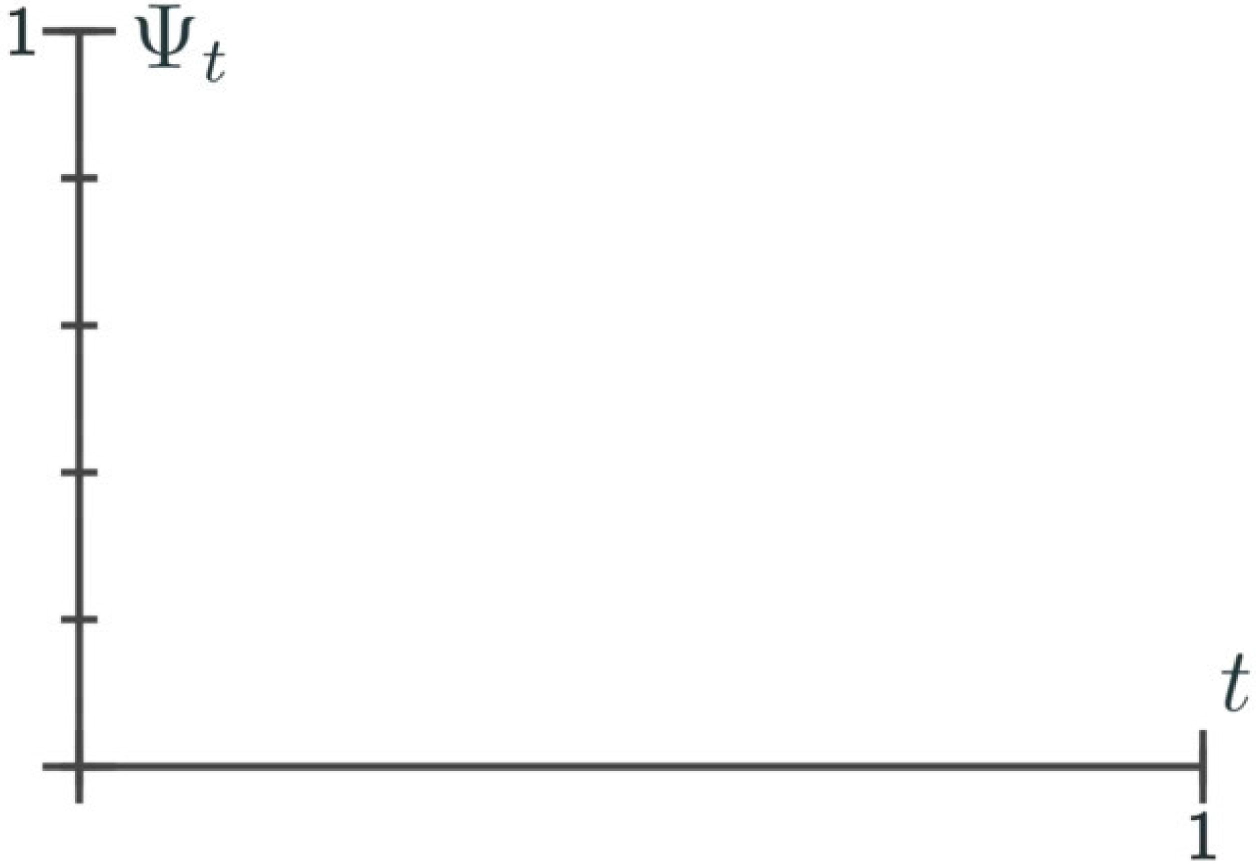

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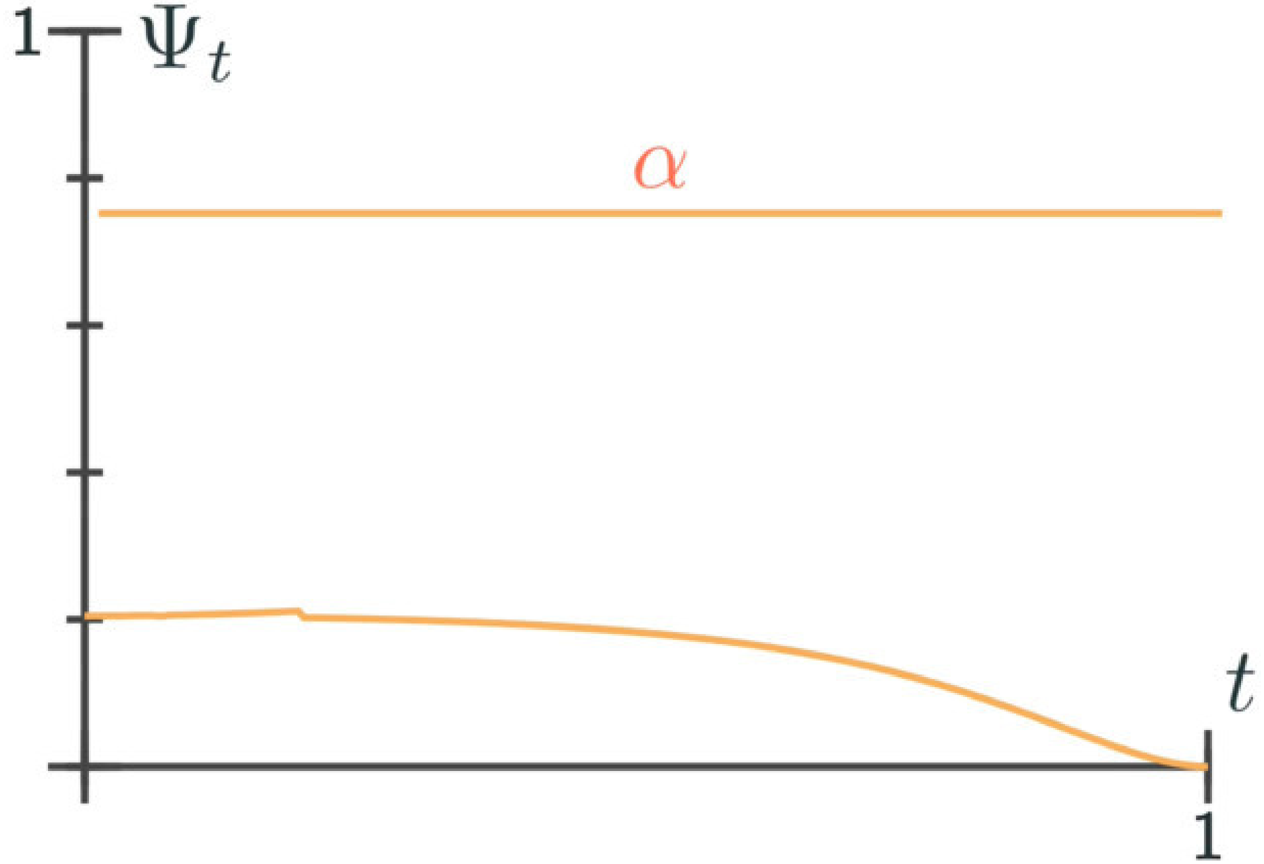

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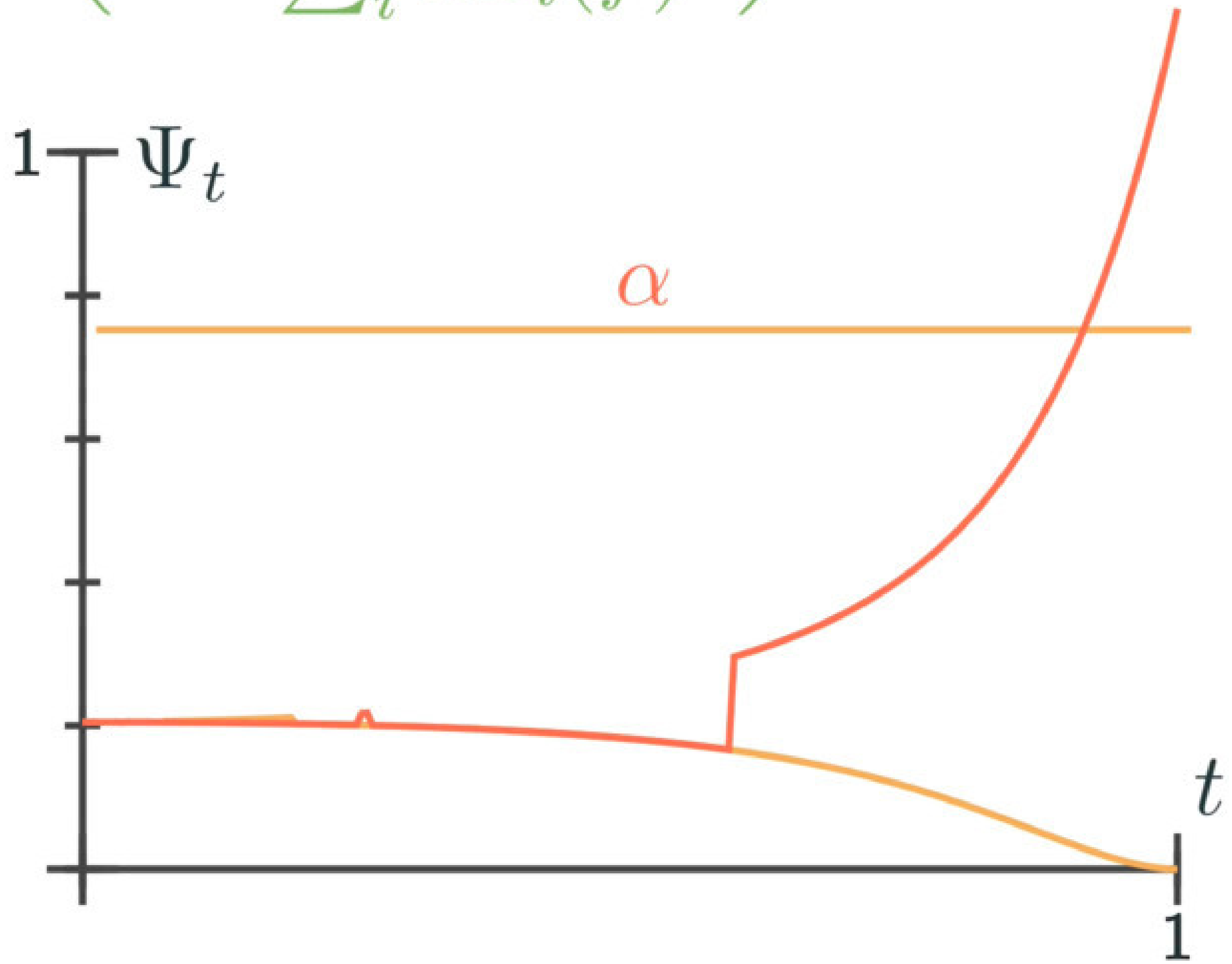

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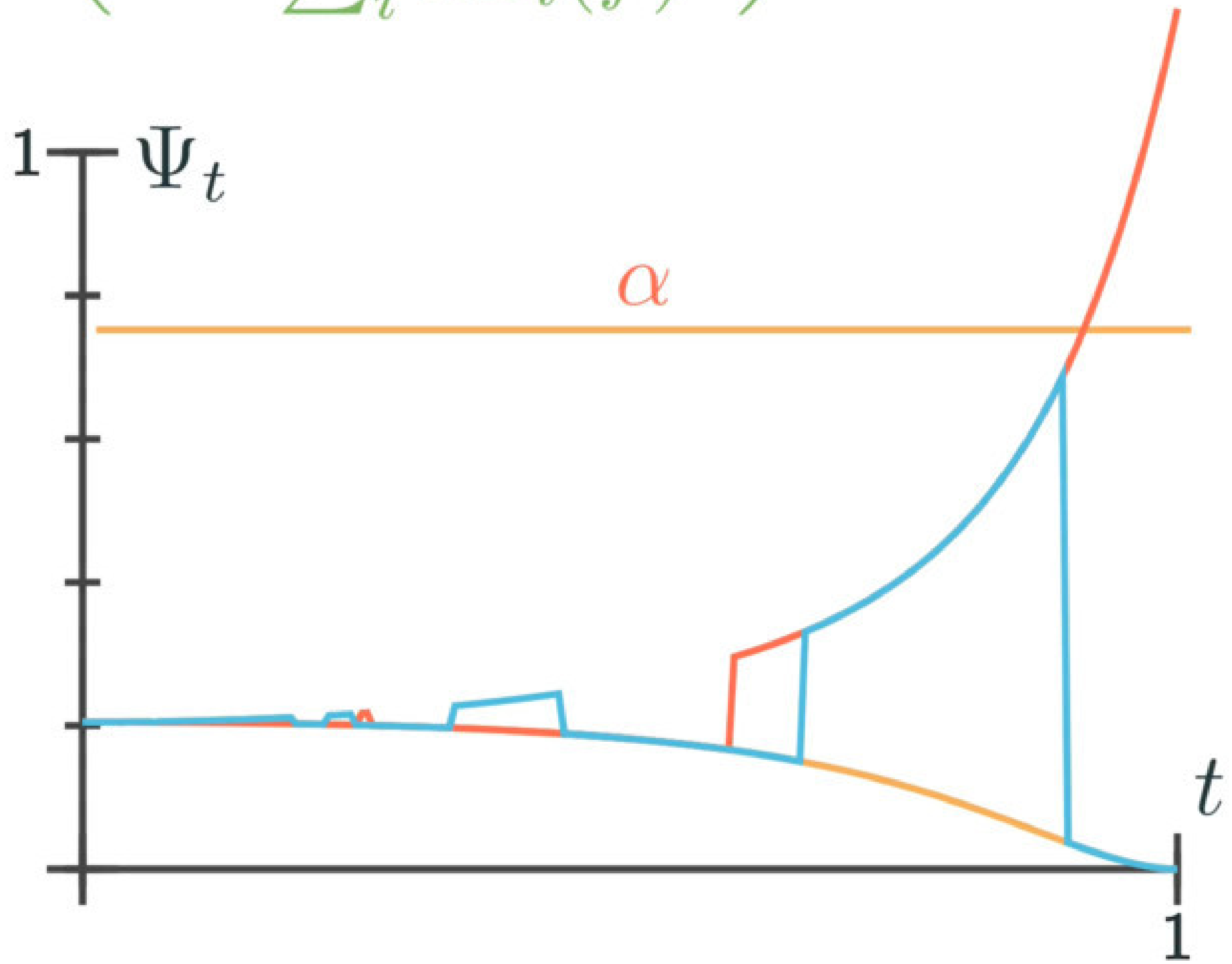

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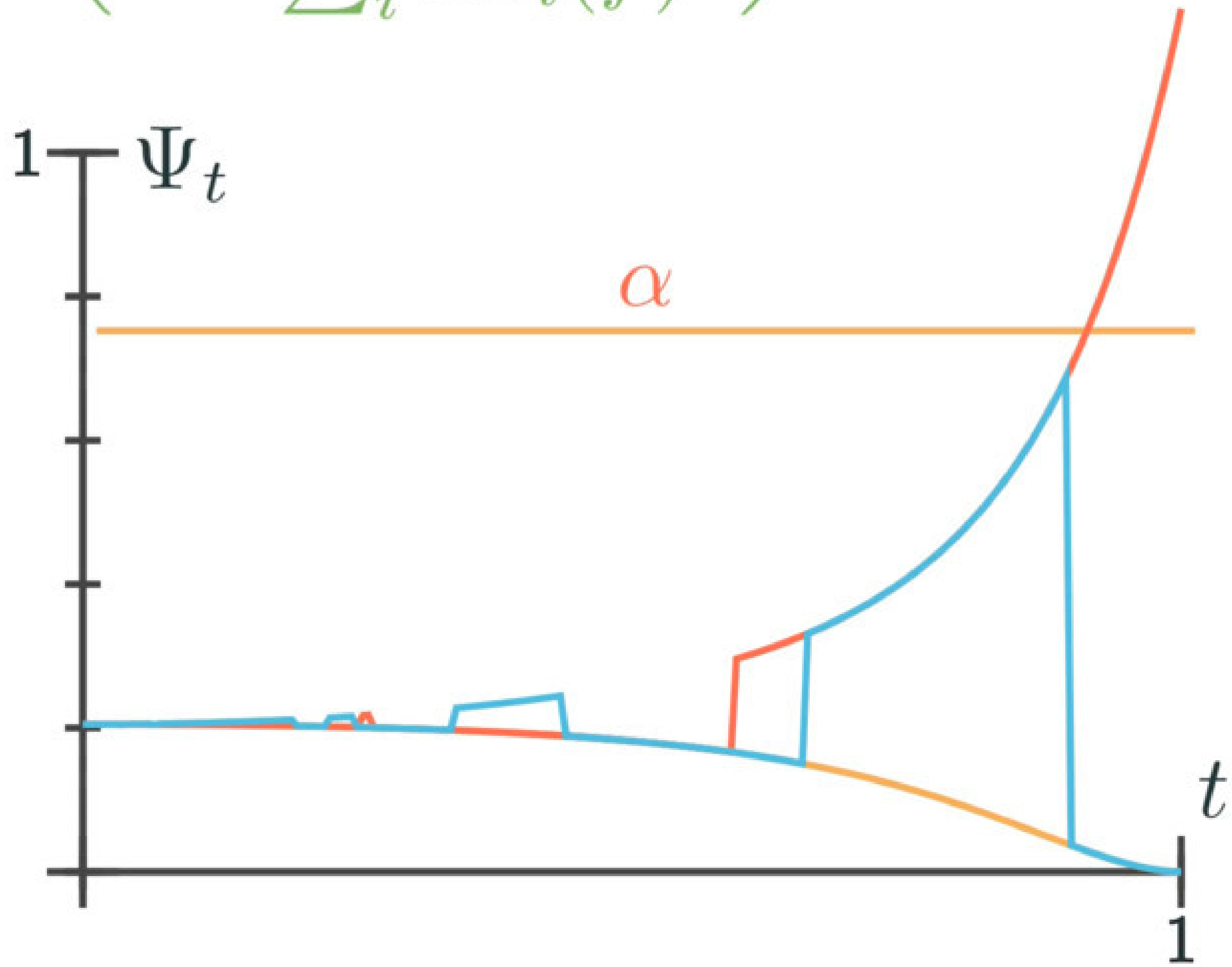
$\tau_\alpha = \inf\{t \mid \Psi_t > \alpha\} \wedge 1$



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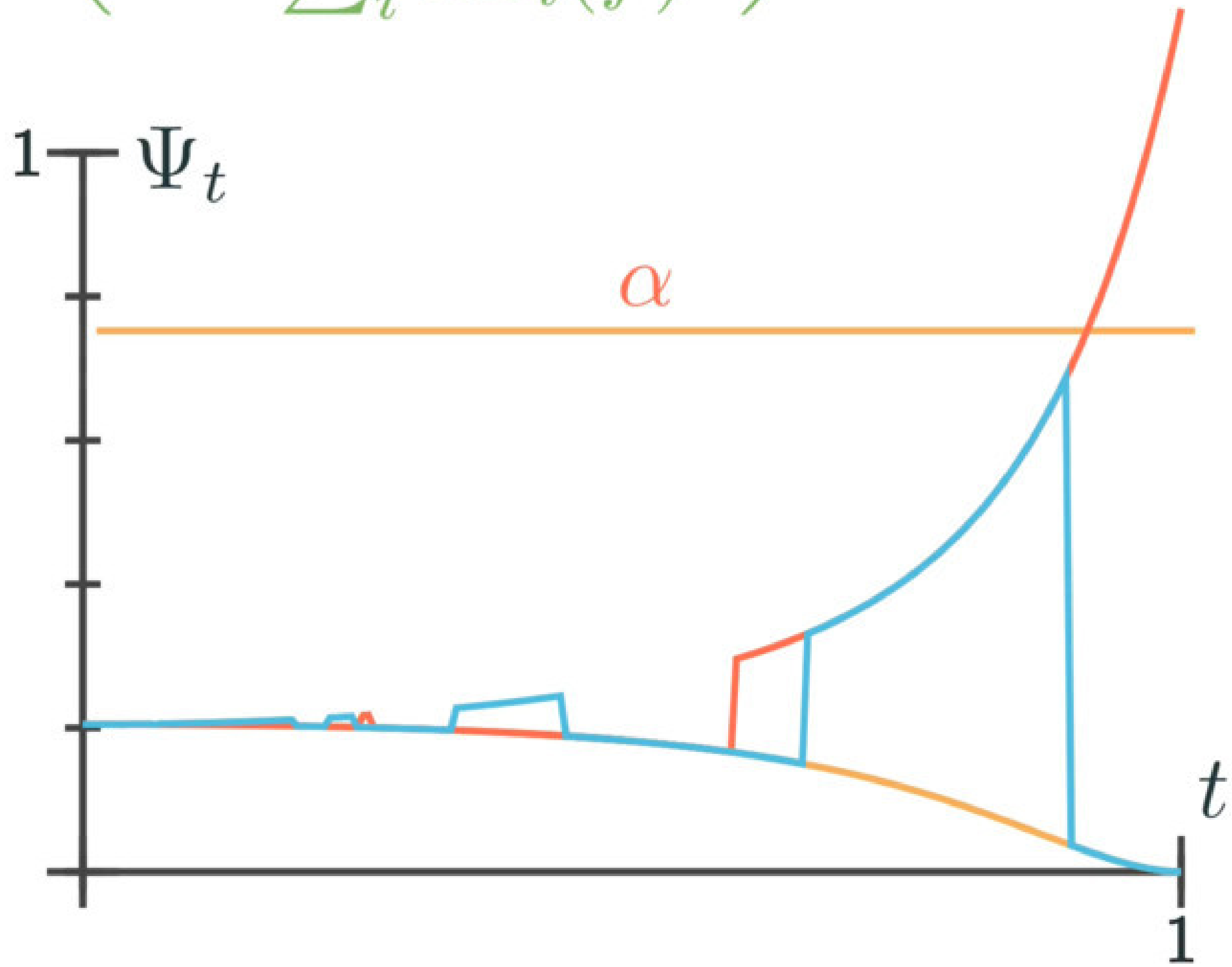
$\tau_\alpha = \inf\{t \mid \Psi_t > \alpha\} \wedge 1$

$\mathbb{E}\Psi_1 \geq \mathbb{E}\Psi_{\tau_\alpha} \geq \mathbb{P}[\tau_\alpha < 1]\alpha$



Level 2 ineq: $\frac{\mathrm{d}}{\mathrm{d}t} \sum_i (\partial_i f_t)^2 \ \leq \ C(t) \ ||\nabla f_t||_2^2 \cdot \log\left(\frac{C(t)}{||\nabla f_t||_2^2}\right)$

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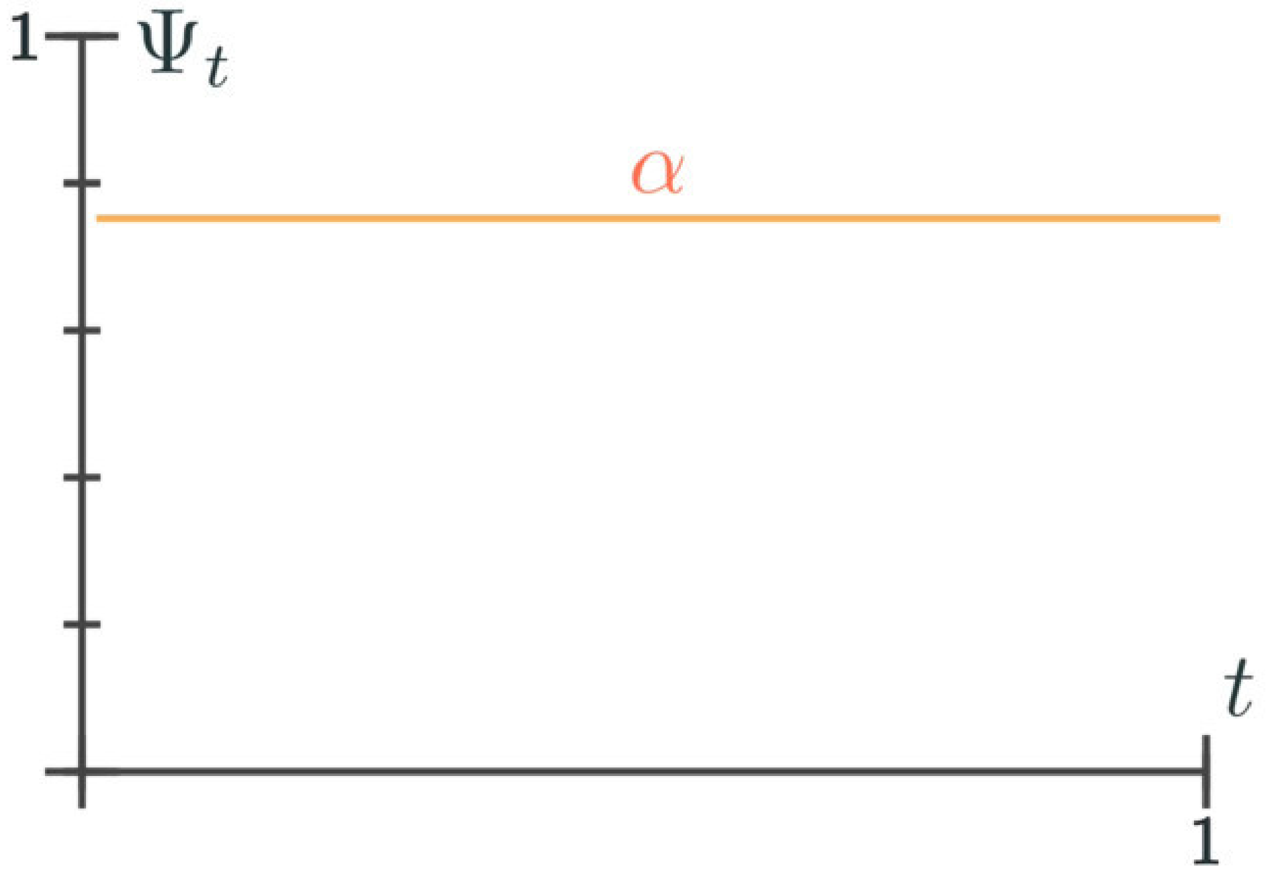

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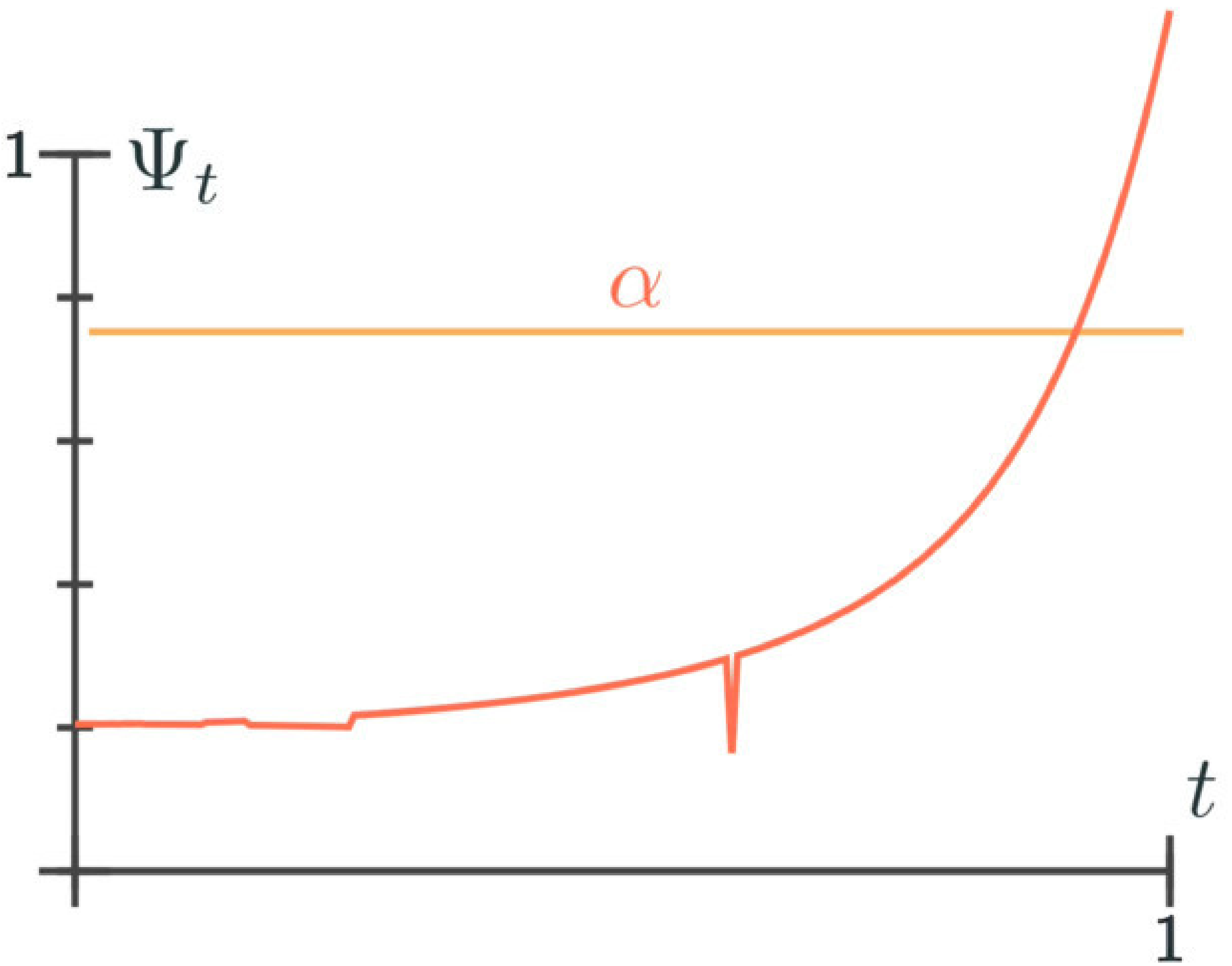

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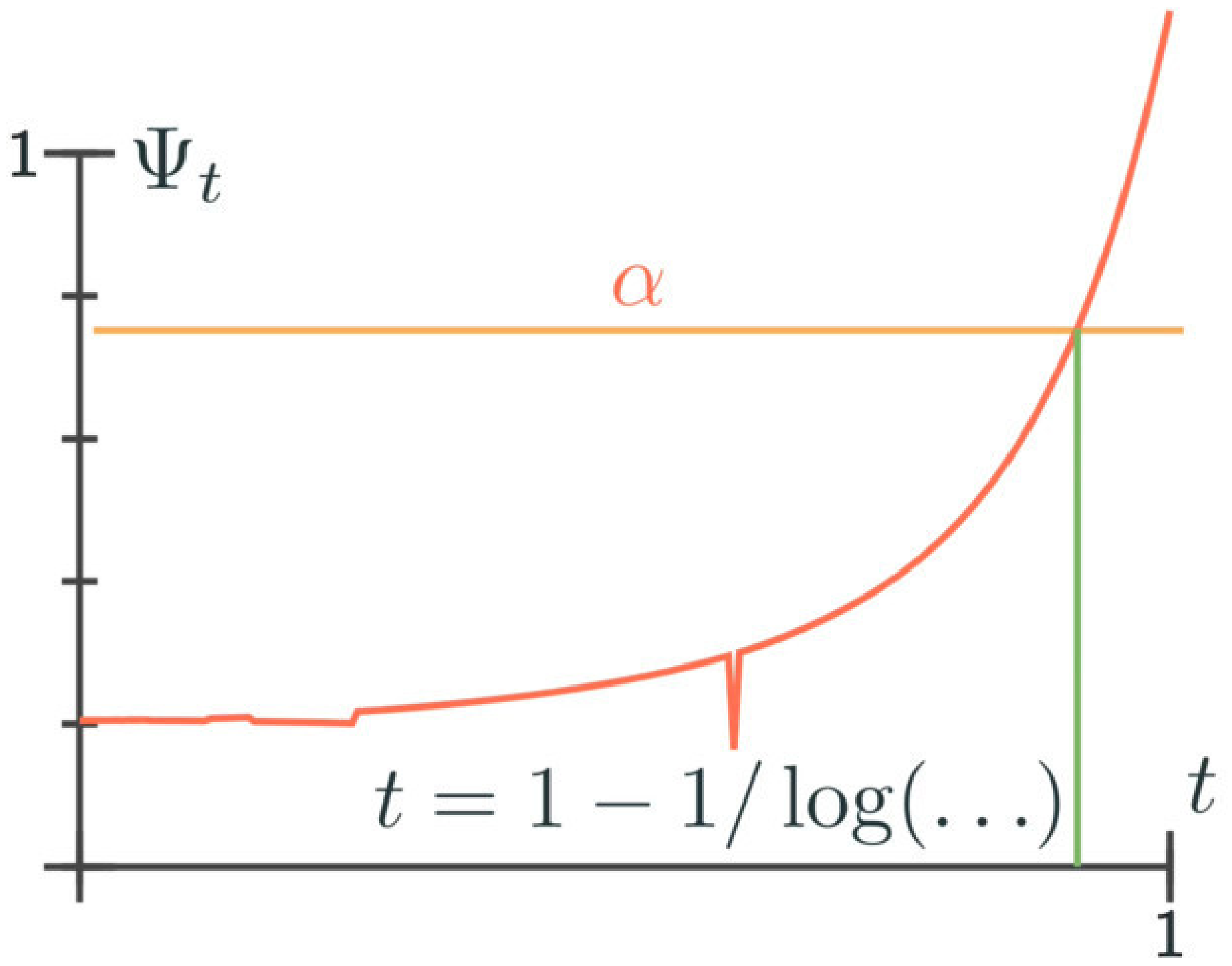

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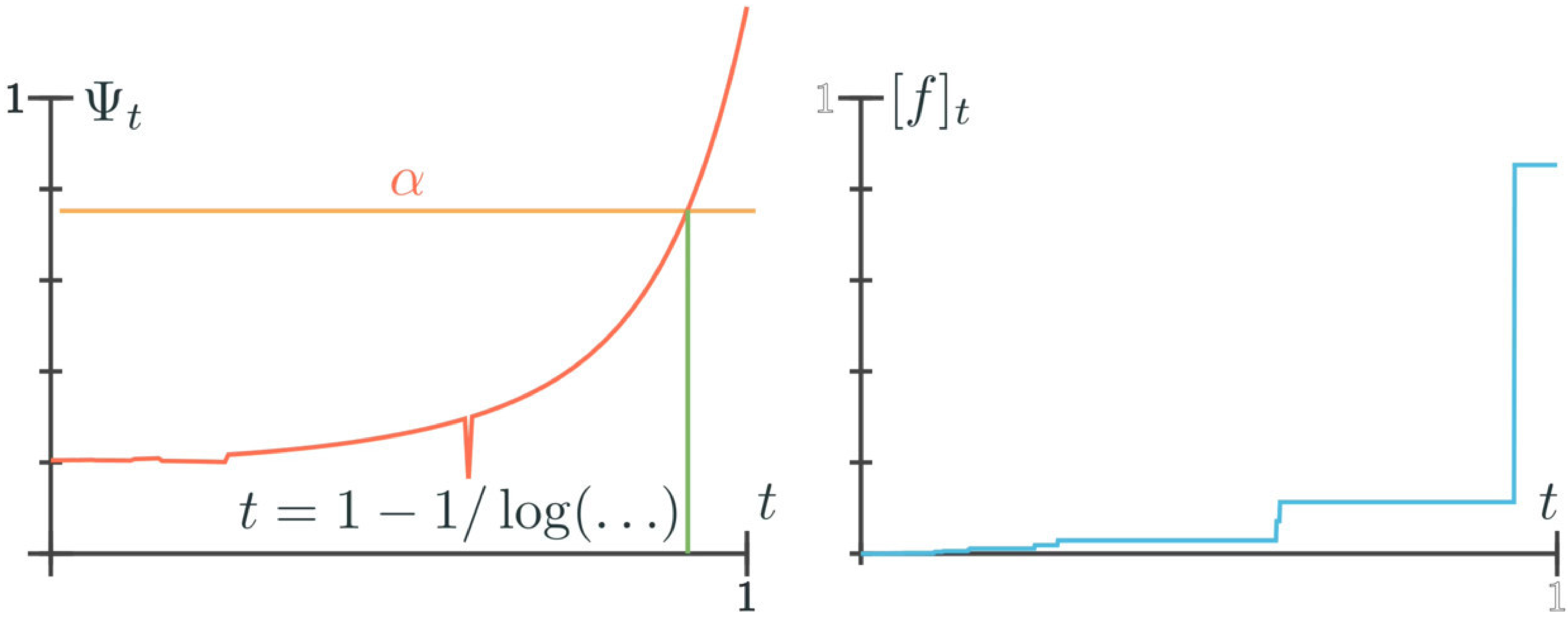

The plan

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(1) is solved using $\mathbb{E}\left[[f]_1 \cdot \mathbf{1}_{\{\tau=1\}}\right] = 2\mathbb{E}\sum\int t(\partial_i f_t)^2 \mathbf{1}_{\{\tau=1\}} dt$

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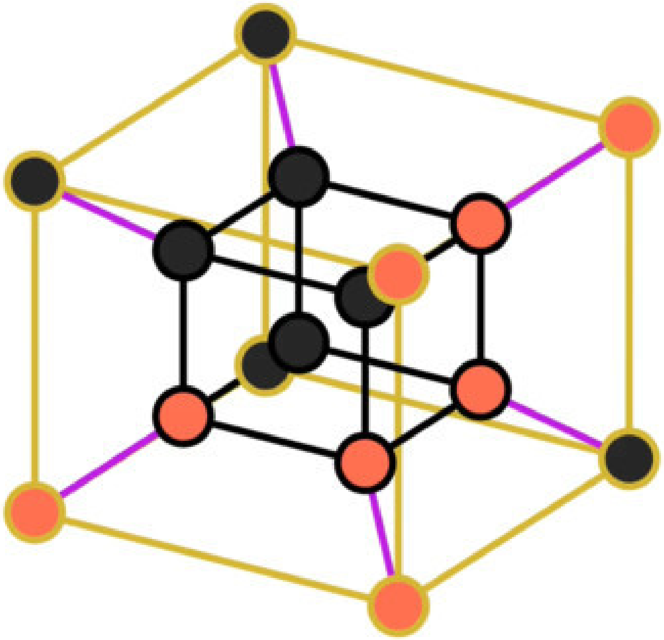
(2) is a calculation, martingale with known endpoint

The end, for now.

Ronen Eldan



Renan Gross

No bits were harmed in the making of this presentation