Random Spanning Forests + Hyperbolic Symmetry.

joint w R. Bauerschmidt, N. Crawford, + A. Swan (arXiv:1912.04854)

Arboreal gas: $\beta > 0$, $G = (V, E)$ finite graph.

$$\mathbb{P}_\beta[F] \equiv \beta^{|F|} / Z_\beta \propto p_\beta^{|F|} (1 - p_\beta)^{|E \setminus F|}$$

Three origin stories:

1) p_β bond percolation, conditioned on the event that open edges are a forest.

2) q-random cluster model in the limit $q \downarrow 0$, $p = \beta q$

3) graphical representation of $\mathbb{H}^{0|2}$ spin model.

Main question: Take $G_n \uparrow G_\infty$. Is there a percolation phase transition as β varies? If so, what are the percolative properties?

Today: $G_n \uparrow \mathbb{Z}^d$. Convention: translationally invariant subseq. limits

Classical observations: 1) $\mathbb{P}_\beta \lesssim \mathbb{P}^{perc}_{p_\beta}$. Hence if $p_\beta < p_c(\mathbb{Z}^d)$, no ∞-tree, and $\mathbb{P}_\beta[0 \longleftrightarrow x] \leq Ce^{-c|x|}$

2) The # of infinite trees is a.s. 0 or 1.

<u>Example</u>: $G_n = K_n$, complete graph on n vertices.

$\mathbb{P}_\beta$ is Erdős-Rényi conditioned to be acyclic.

<u>Thm</u>: The arboreal gas has a transition on K_n at $\beta = 1/n$:

$$\mathbb{E}_{\alpha/n}\left[\,|T_0|\,\right] \sim \begin{cases} c_\alpha n & \alpha > 1 \\ c_\alpha n^{1/3} & \alpha = 1 \\ c_\alpha & \alpha < 1 \end{cases}$$

Due to Łuczak-Pittel; finer detail known, see Martin-Yeo.

<u>Thm</u> If $\alpha > 1$, 2^{nd} largest component has size $\asymp n^{2/3}$

(compare w Erdős-Rényi: 2^{nd} largest is size $\asymp \log n$)

<u>Thm</u> (BCHS) On $\mathbb{Z}^2$ for any $\beta>0$ percolation does not occur.

Moreover, $\mathbb{P}_\beta[0 \longleftrightarrow x] \leq C|x|^{-c(\beta)}$

<u>Rmk</u> (Deng-Garoni-Sokal): can predict this from $p_c(q) = \frac{\sqrt{q}}{1+\sqrt{q}}$ for q-RCM; suggests $\beta_c(q) \sim \frac{1}{\sqrt{q}} \uparrow \infty$.

<u>Challenge</u>: Find a direct probabilistic proof of this theorem

<u>Rest of talk</u> 1) Conjectural behaviour of the arbored gas on $\mathbb{Z}^d$

2) Outline of the key new tool used in our proof: a magic formula for connection probabilities.

Conjecture: On $\mathbb{Z}^2$, for all $\beta > 0$ $\exists$ $m(\beta) > 0$ s.t. $\mathbb{P}_\beta[0 \longleftrightarrow x] \lesssim e^{-m(\beta)|x|}$.

Rmk: This says the transition at $\beta_c = \infty$ is sharp. Believe $m(\beta) \simeq e^{-c\beta}$, much smaller than for approach to criticality for percolation.

Conjecture On $\mathbb{Z}^d$, $d \geq 3$, there is a percolation transition: $\exists$ $\beta_c < \infty$ s.t.

$$\mathbb{P}_\beta[0 \longleftrightarrow x] = c_\beta > 0 \qquad (\beta > \beta_c)$$

Moreover,

$$\mathbb{P}_\beta[0 \longleftrightarrow x] - c_\beta \simeq |x|^{-d+2} \qquad (\beta > \beta_c)$$

Conjecture: On any graph, $\mathbb{P}_\beta[e, f] \leq \mathbb{P}_\beta[e]\, \mathbb{P}_\beta[f]$ $\quad e \neq f$.

Very roughly, can attribute differences compared to bond percolation to very different ground states:

Percolation $\approx$ $p=1$: all edges open, unique configuration.

Arboreal gas $\approx$ $\beta = \infty$: Uniform choice of a spanning tree. Exponentially many configurations.

Rmk: Other families of $G_n \uparrow G_\infty$ interesting too; expect similar differences $\approx$ percolation.

Magic Formula

Δ graph Laplacian: $(\Delta f)(i) = \sum_{j \sim i} \beta(f(j) - f(i))$

$$\mathbb{P}_\beta[0 \longleftrightarrow x] \propto \int_{\mathbb{R}^{n-1}} e^{t_x} e^{-\sum_{i \sim j} \beta \cosh(t_i - t_j)} (\det \beta\Delta)^{3/2} \prod_{i \neq 0} e^{-3t_i} dt_i$$

Rmk If replace 3 with 1 the measure on the RHS is the 'magic formula' or 'mixing measure' for the VRJP!

ERRW: edge weights $\{\gamma + n_e\}_{e \in E}$. Diaconis-Coppersmith proved

$$\mathbb{P}_0^{ERRW(\gamma)} \propto \int \mathbb{P}_0^{SRW(C)} \, d\mu_\gamma(C)$$

Schematic proof of non-percolation on $\mathbb{Z}^2$

$$
\begin{tikzcd}
\text{Arboreal gas} \arrow[rr, leftrightarrow, "\text{`nonlinear'}", "\text{matrix-tree thm}"'] & & \mathbb{H}^{0|2} \text{ spin model} \arrow[d, leftrightarrow, "\text{Dimensional reduction.}"] \\
& & \mathbb{H}^{2|4} \text{ spin model} \arrow[d, "\text{Horospherical coordinates and Gaussian integrals}"] \\
\text{Absence of percolation} & & \text{Magic formula} \arrow[ll, "\text{Mermin-Wagner}", "\text{theorem}"']
\end{tikzcd}
$$

<u>Rmk</u>: 'magic' is the two vertical steps: <u>same</u> magic as for VRJP/ERRW

Matrix-tree theorem and $\mathbb{H}^{0|2}$

Uniform spanning tree = determinantal
= free fermion

$\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_n$ anticommuting variables:

$$\xi_i \eta_j = -\eta_j \xi_i, \quad \eta_i \eta_j = -\eta_j \eta_i, \quad \xi_i \xi_j = -\xi_j \xi_i,$$

Grassmann algebra $\mathbb{R}$-linear combinations, e.g., $3\xi_1\eta_2 + 7 + \xi_1\xi_3\eta_5$

Derivatives : F a monomial not containing ξ_i.

$$\partial_{\xi_i}(\xi_i F) = F, \quad \partial_{\xi_i} F = 0.$$

Extend ∂_{ξ_i} to full algebra by linearity.

Exercise: For any $n \times n$ matrix A,

$$\det A = \partial_{\eta_1}\partial_{\xi_1} \cdots \partial_{\eta_n}\partial_{\xi_n} \exp\Big(\sum_{i,j} A_{ij} \xi_i \eta_j\Big)$$

Cf A positive definite, $\frac{(2\pi)^n}{\det A} = \int_{\mathbb{R}^{2n}} e^{-(\varphi, A\varphi)} d\varphi$

Matrix-tree theorem: take $A = -\beta\Delta + h$

$$\int e^{-(\eta, A\xi)} = \sum_{(F,R)}' \prod_{(T,r)} \beta^{|T|} h_r$$

Uniform spanning trees: take $h = \mathbb{1}_{i=0}$. $\{\text{Forests with one root}\} \simeq \{\text{spanning trees}\}$.

Fundamental observation (Caracciolo et al) if $h = \mathbb{1}$

$$\int e^{-(\eta, A\xi) - \sum_{i\sim j} \beta \xi_i \eta_i \xi_j \eta_j} = \sum_{F}{}' \beta^{|F|}$$

Reformulation ($\mathbb{H}^{0|2}$ model)

$z_i \equiv 1 - \xi_i \eta_i$

$u_i \equiv (\xi_i, \eta_i, z_i)$

$u_i \cdot u_j \equiv -\xi_i \eta_j - \xi_j \eta_i - z_i z_j$

$$\int \left(\prod \frac{1}{z_i}\right) e^{-\frac{1}{2}\beta (u, -\Delta u)} = \sum_{F}{}' \beta^{|F|}$$

Moreover: $\mathbb{P}_\beta [i \longleftrightarrow j] = -\langle u_i \cdot u_j \rangle$

Step 2: Dim. Reduction: $\mathbb{H}^{2|4}$ has spins $u = (x, y, z, \xi_1, \eta_1, \xi_2, \eta_2)$

with $x^2 + y^2 - z^2 - 2\xi_1\eta_1 - 2\xi_2\eta_2 = -1$, $z \geq 0$,

where $x, y \in \mathbb{R}$, $\xi_1, \eta_1, \xi_2, \eta_2$ anticommute.

Formally, working with algebra generated by ⟶ with coefficients smooth fns of x, y.

Berezin integral: $\displaystyle\int_{\mathbb{H}^{2|4}} F \equiv \int_{\mathbb{R}^2} dx\,dy \int_{\mathbb{H}^{0|4}} \frac{1}{z} F$.

On finite graph G, $\mathbb{H}^{2|4}$ model defined as $\mathbb{H}^{0|2}$. Integrate on $(\mathbb{H}^{2|4})^\Lambda$.

Thm (dimensional reduction/SUSY localization)

$$\langle u_i \cdot u_j \rangle_{\mathbb{H}^{2|4}} = \langle u_i \cdot u_j \rangle_{\mathbb{H}^{0|2}}$$

Step 3: Horospherical coordinates $\mathbb{H}^2$ model has $u = (x, y, z)$ w

$$x^2 + y^2 - z^2 = -1, \quad z \geq 0$$

$$u_i \cdot u_j = x_i x_j + y_i y_j - z_i z_j$$

Good parametrization of 2-dim'l mfld $\mathbb{H}^2$ is given by:

$$x = \sinh t - \tfrac{1}{2} s^2 e^t$$

$$y = s e^t \qquad\qquad s, t \in \mathbb{R}$$

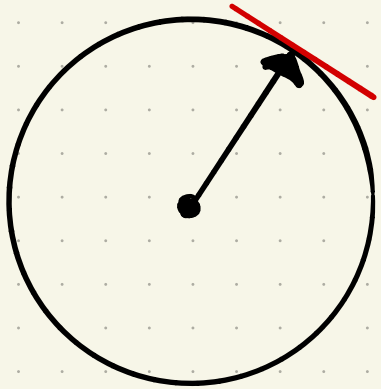
$$z = \cosh t + \tfrac{1}{2} s^2 e^t$$

then $$u_i \cdot u_j = \cosh(t_i - t_j) + \tfrac{1}{2} s_i s_j e^{t_i + t_j} - 1$$

Similarly good coordinates for $\mathbb{H}^{2|4}$, also get terms quadratic in ξs and ηs.

Step 4: Mermin–Wagner

This is a classical argument that says long-range order is incompatible with continuous symmetries in $d=1$ and $d=2$:



LRO $\simeq$ $\perp$ fluctuations are Gaussian,

but GFF has unbounded fluctuations in $d=1,2$.

Made rigorous by Mermin–Wagner, many others.

We adapt a version of Sabot for VRJP ($a=1$)