Liberation of Random Projections / Todd Kemp, University of California, San Diego

Consider two random subspaces $V, W \subseteq \mathbb{C}^d$. Then the dimension of $V \cap W$ is almost surely equal to $\max\{\dim V + \dim W - d, 0\}$ (i.e. they are in *general position*). Depending on the definition of "almost sure", this easy theorem dates back 150 years or more. A more modern form could be stated thus: let P, Q be random projection matrices on $\mathbb{C}^d$, and let U_t be an independent Brownian motion on the unitary group $U(d;\mathbb{C})$. Conjugating P by U_t performs a random rotation of the subspace $P(\mathbb{C}^d)$; the general position statement can be written as $\mathrm{Tr}[U_t P U_t^* \wedge Q] = \max\{\mathrm{Tr}P + \mathrm{Tr}Q - d, 0\}$ a.s. for all $t > 0$ (here $P \wedge Q$ is the projection onto $P(\mathbb{C}^d) \cap Q(\mathbb{C}^d)$). What happens when $d \to \infty$? While there is no unitarily invariant measure in infinite dimensions, it is still possible to make sense of the unitary Brownian motion and the trace for *some* projections on infinite-dimensional Hilbert spaces. However, the easy techniques for proving the general position theorem are unavailable. Instead, one can analyze the spectral measure μ_t of the operator $U_t P U_t^* Q$ for smoothness. The Cauchy transform $G(t,z)$ of μ_t satisfies a semilinear holomorphic PDE in the upper-half-plane:

$$\frac{\partial}{\partial t} G(t,z) = \frac{\partial}{\partial z}\left[z(z-1)G(t,z)^2 + (az+b)G(t,z)\right].$$

I will discuss the analysis of the characteristics of this PDE, and prove not only the general position claim in infinite-dimensions but a very strong smoothing theorem that settles (a special case of) an important conjecture in free probability theory.

This is joint work with Benoit Collins.