Consider two random subspaces $V, W \subseteq \mathbb{C}^d$. Then the dimension of $V \cap W$ is almost surely equal to $\max\{\dim V + \dim W - d, 0\}$ (i.e. they are in general position). Depending on the definition of “almost sure”, this easy theorem dates back 150 years or more. A more modern form could be stated thus: let $P, Q$ be random projection matrices on $\mathbb{C}^d$, and let $U_t$ be an independent Brownian motion on the unitary group $U(d; \mathbb{C})$. Conjugating $P$ by $U_t$ performs a random rotation of the subspace $P(\mathbb{C}^d)$; the general position statement can be written as $\text{Tr}[U_tPU_t^* \wedge Q] = \max\{\text{Tr}P + \text{Tr}Q - d, 0\}$ a.s. for all $t > 0$ (here $P \wedge Q$ is the projection onto $P(\mathbb{C}^d) \cap Q(\mathbb{C}^d)$). What happens when $d \to \infty$? While there is no unitarily invariant measure in infinite dimensions, it is still possible to make sense of the unitary Brownian motion and the trace for some projections on infinite-dimensional Hilbert spaces. However, the easy techniques for proving the general position theorem are unavailable. Instead, one can analyze the spectral measure $\mu_t$ of the operator $U_tPU_t^*Q$ for smoothness. The Cauchy transform $G(t, z)$ of $\mu_t$ satisfies a semilinear holomorphic PDE in the upper-half-plane:

$$\frac{\partial}{\partial t} G(t, z) = \frac{\partial}{\partial z} [z(z - 1)G(t, z)^2 + (az + b)G(t, z)].$$

I will discuss the analysis of the characteristics of this PDE, and prove not only the general position claim in infinite-dimensions but a very strong smoothing theorem that settles (a special case of) an important conjecture in free probability theory.

This is joint work with Benoît Collins.