**Advanced topics in Probability**

**Thermodynamics and statistical physics:**

**Idea:** To determine the behavior of a large system from a description of its microscopic constituents and their interactions. A focus on specific models.

**Random Walk:**

1. **One-dimensional:**
   - Simple RW: prob. \( \frac{1}{2} \) to walk to each side.
   - Location: time

2. **Two dimensions:**
   - Location: time

**Scaling limit:** Brownian motion

**Ising Model:** \( \Lambda \subseteq \mathbb{Z}^2 \)

\[ \theta : \Lambda \rightarrow \{-1, 1\} \]

**Probability measure:**

**Energy of \( \theta \):**

\[ H(\theta) = -\sum_{u \sim v} \theta_u \theta_v \]

\( u \) adjacent to \( v \)

**Temperature:** \( T > 0 \)

inverse Temp: \( \beta = \frac{1}{T} \)
\[ P(\mathbf{s}) = \frac{1}{Z} e^{-\beta H(\mathbf{s})} \]

Configurations with lower energy are favored, moreso if the temperature is low.

\[ Z = \sum_{\mathbf{S}} e^{-\beta H(\mathbf{S})} \quad \text{is termed partition function.} \]

Total magnetization:

\[ m_z = \left( \sum_{\mathbf{S}} \mathbf{S} \cdot \mathbf{e}_z \right) \cdot \frac{1}{\Lambda_z} \]

Phase transition: \( \exists T_c \in (0, \infty) \) s.t.

\[
\begin{align*}
1 & \left( \frac{\langle m_z \rangle}{\Lambda_z} \right) \quad \text{as } T \to \infty \quad \\
\to 0 & \quad T > T_c \\
\to & \quad 0 \quad T < T_c
\end{align*}
\]

(In dimensions \( d \geq 2 \)).

2) Percolation

Given \( 0 < p < 1 \), designate an edge open with prob. \( p \) and closed with prob. \( 1-p \), indep.

between edges.

Phase transition: \( \exists p_c \in (0, 1) \)

\[
\mathbb{P} \left( \left[ \begin{array}{c}
\text{origin has an open connection to } \infty \\
\text{in dimension } d \geq 2
\end{array} \right] \right) = \begin{cases} 
0 & p < p_c \\
\to & p > p_c
\end{cases}
\]
1) Random walk in random environment.

2) Random Field models or spin glasses.
   - Giorgio Parisi - Nobel Prize in Physics 2021

3) Noise sensitivity and dynamical percolation.

4) Localization in quantum systems.

Books:
- Ofer Zeitouni - Lectures on Probability Theory
- Christophe Garban - Noise Sensitivity and percolation
- Jeffrey Steif
- Random Operators - Disorder Effects
- Michael Aizenman and Simone Warzel

Titles available on course website.

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Random Walks in Random Environment

Brief discussion of random walks in a homogeneous environment:

- Walk on $\mathbb{Z}^d$.
- Nearest neighbor walk.

**Transition Kernel** = probability list on $\mathbb{Z}^d$, $\mathbb{Z}^{d-1}$

Write it as $w: \mathbb{Z}^d \times \mathbb{Z}^d \to [0,1]$, $\sum_i w(x_i) = 1$

elliptic: $\exists \varepsilon_0$ s.t. $w(x) \in [\varepsilon, 1 - \varepsilon]$.

Questions:
1) Recurrence/transience
2) Law of large numbers
3) Central limit theorem.
Recurrence/transience

Starting at 0 (more generally, \( \rho^x \) for the walk measure starting at \( x \)).

\[ P \left( \rho^0 \text{ (Walker returns to 0)} \right) = 1? \]

- \( = 1 \): recurrent
- \( < 1 \): transient

Investigate for simple random walk: \( \omega(\pm 1) = \frac{1}{2} \)

Polya's theorem: Recurrent when \( d = 1, 2 \), Transient when \( d > 2 \)

Idea of proof: Let \( P = P(C) \text{ (Walker returns to 0)} \)

\[ N = \# \text{ of visits to 0} \]. \( N \) is distributed geometric(1-\( P \))

\[ \Rightarrow \mathbb{E}[N] = \frac{1}{1-P} \text{ infinite } \iff P = 1 \]

However, \( N = \sum_{n=0}^{\infty} \mathbb{I}(X_n = 0) \)

\[ \mathbb{E}[N] = \sum_{n=0}^{\infty} \mathbb{P}(X_n = 0) \]

\( d = 1 \): \[ \mathbb{P}(X_n = 0) = \left\{ \begin{array}{ll}
0 & n \text{ odd} \\
\frac{1}{2^n} \cdot \binom{n}{n/2} & n \text{ even}
\end{array} \right. \Rightarrow \mathbb{E}[N] = \infty \]

\( d = 2 \): After a 45° rotation, the coords of \( X_n \) are independent simple random walk

\[ \Rightarrow \mathbb{P}(X_n = 0) = \left\{ \begin{array}{ll}
0 & n \text{ odd} \\
\frac{1}{2^n} \cdot \binom{n}{n/2} & n \text{ even}
\end{array} \right. \Rightarrow \mathbb{E}[N] = \infty \]
$d=3$: \[ P(\{X_n=0\}) = \begin{cases} 0 & n \text{ odd} \\ \frac{c}{n^{1/2}} & n \text{ even} \Rightarrow E|N| < \infty \end{cases} \]

requires further calculation

2) Law of large numbers: Understand $\lim_{n \to \infty} \frac{X_n}{n}$ for a general transition kernel $W$.

$X_n = \sum_{k=1}^{n} E_k$ where $(E_k)$ i.i.d. samples from $W$.

$\Rightarrow$ Since the $E_k$ are bounded in every coord., can apply the usual law of large numbers in every coord. to get:

$\lim_{n \to \infty} \frac{X_n}{n} = \mathbb{E}[E_1] - \rho - \text{a.s. and in } L^4$

If $\lim_{n \to \infty} \frac{X_n}{n} \to 0$ we say that the walk is ballistic.

In $d=1,2$ we have that either the walk is either recurrent or ballistic.

3) Central limit theorem: Thus:

$\frac{X_n - \mathbb{E}[E_1] \cdot n}{\sqrt{n^{1/2}}} \Rightarrow N(0, \Sigma)$

$\Sigma = $ Covariance matrix

$\Sigma_{ij} = \text{Cov}(E_i, E_j)$

If $Z \sim N(0, \Sigma)$ then $\mathbb{E}[Z_i Z_j] = \Sigma_{ij}$

$\Sigma_{ij} = \text{Cov}(Z_i, Z_j)$

coordinates of the vector $d=2$ $d=1$
Setup for central limit theorem:

\( (E_j) \) IID random vectors in \( \mathbb{R}^d \).

For simplicity, assume \( M = \max_{1 \leq i \leq d} I\mathbb{E}(1E_i, 1E_j) < \infty \)

\[ S_n := \frac{1}{n} \sum_{j=1}^{n} E_j, \quad \bar{S}_n = \mathbb{E}[E_j] \]

We want \( S_n \Rightarrow \mathbb{E}(E_i) \) i.i.d. \( \mathbb{N}(0, \Sigma) \)

\[ \Sigma_{ij} = \text{Cov}(E_i, E_j) \]

This means that for every continuous and bounded \( F: \mathbb{R}^d \rightarrow \mathbb{R} \) it holds:

\[ \mathbb{E} \left[ F \left( \frac{S_n - n \bar{S}_n}{\sqrt{n}} \right) \right] \xrightarrow{n \to \infty} \mathbb{E}(F(z)) \]

where \( Z \sim \mathbb{N}(0, \Sigma) \)

In this, it actually suffices to consider only \( F: \mathbb{R}^d \rightarrow \mathbb{R} \) smooth and compactly supported.

We prove using Lindeberg's method (1922a)

**Step 1:** Let \( (Z_j) \) be IID \( \mathbb{N}(0, \Sigma) \)

Then \( S_n := \frac{1}{n} \sum_{j=1}^{n} Z_j \)

\[ S_n \sim \mathbb{N}(0, m\Sigma) \]

since a sum of independent Gaussian is a Gaussian

\[ S_n \xrightarrow{\mathcal{D}} \mathbb{N}(0, \Sigma) \]

**Step 2:** Idea: In \( S_n \), replace the \( (E_j) \) by the \( (Z_j) \) one by one.

Let \( S_n^k := \frac{1}{k} \sum_{j=1}^{k} Z_j + \frac{n-k}{n} \sum_{j=k+1}^{n} E_j \) for \( 0 < k < n \)

\[ S_n^0 = S_n, \quad S_n^n = S_n^1 \]
Let $F : \mathbb{R}^d \to \mathbb{R}$ be smooth and compactly supported.

Write:

$$E \left[ F \left( \frac{S_{n_i} - n \cdot \mu_i}{\sqrt{n}} \right) \right] - E \left[ F \left( N(0, \Sigma) \right) \right]$$

$$= \sum_{i=0}^{n_i-1} E \left[ F \left( \frac{S_{n_i} - n \cdot \mu_i}{\sqrt{n}} \right) \right] - E \left[ F \left( \frac{k \cdot \mu_i + \epsilon_{n_i}}{\sqrt{n}} \right) \right]$$

Call this $d_k$.

Goal: $\sum_{i=0}^{n_i-1} d_k \to 0$ as $n \to \infty$.

Proof: Develop $F$ in a Taylor expansion:

$$F(X + \delta) = F(X) + \frac{\delta}{1!} \nabla F(X) \cdot \delta + \frac{\delta^2}{2!} J_{ij} \nabla X_i \cdot \delta_j + \frac{\delta^3}{3!} C_{ijk} \nabla^3 X_i \cdot \delta_j \cdot \delta_k + \mathcal{O}(\max_{i,j,k} |\delta_i^3|)$$

$F$ is smooth and compactly supported.

$$F(X + \delta) = F(X) + \frac{\delta}{1!} \nabla F(X) \cdot \delta$$

Note that:

$$F \left( \frac{S_{n_i} - n \cdot \mu_i}{\sqrt{n}} \right) = F \left( \frac{1}{\sqrt{n}} \left( \sum_{j=1}^{k_i} (Z_j - \mu_i) \right) + \frac{1}{\sqrt{n}} \left( \sum_{j=k_i+1}^{n_i} (\sum_{j=k_i+1}^{n_i} (Z_{k_i} - \mu_i)) \right) \right)$$

$$= F \left( \frac{1}{\sqrt{n}} \left( \sum_{j=1}^{k_i} (Z_j - \mu_i) \right) \right)$$

Note also, by the Taylor development:

$$E(F(X + \delta, y)) = E[F(y)] + \sum_{i=1}^{k_i} E[J_i^2 F(X_i + \delta_i, y)]$$

$$+ \sum_{i,j} C_{ij} E[J_i^2 J_j^2 F(X_i, X_j)]$$

$$+ \mathcal{O}(\max_i |\delta_i^3|)$$

$$E[J_i^2 F(X_i)]. E[J_i^2, J_j^2]$$
Since $E_{i1}$ and $Z_{i1}$ have the same first and second moments we get

$$d_k = O\left( \max_i \left| \delta_{i1} \right|^3 + \max_i \left| \delta_{kj} \right|^3 \right) = O\left( \left( \frac{1}{n} \right)^{3k} \right)$$

by def. of $\delta_{i1}$ and $\delta_{i1}$ and the assumption $\max_i |E_{i1}|^3 < \infty$.

$$\Rightarrow \sum_{n=0}^{k-1} d_n = O\left( \frac{1}{n^3} \right) \quad \xrightarrow{n \to \infty} 0$$

Finishing the proof. \( \Box \)

Remarks:

1) The method is sometimes useful when Fourier methods are not applicable.

It has been extended to showing that:

$\mathbb{E}F(E_1, \ldots, E_n) - \mathbb{E}F(Z_1, \ldots, Z_n)$ is small when the $E_i$ share many moments with $Z_j$ and $F$ has a good Taylor expansion.

Tao-Vu use this to show higher matrix universality, when first 4-moments match.

2) Also some extensions to dependent random variables.

Random Walk in Environment

Setup: at each vertex of $\mathbb{Z}^d$ we put a prob. distance over $\{e_{i1} \}_{i=1}^d$.

Then the walker walks in this environment.

There are two prob. meas:

1) $P$ - the distribution of the environment $w$. $w : \mathbb{Z}^d \times \{e_{i1} \}_{i=1}^d \to [0,1]$

2) Uniform ellipticity: $w : \mathbb{Z}^d \times \{e_{i1} \}_{i=1}^d \to [0,1]$ for some $\varepsilon > 0$
Many times we will take $P$ to be a product measure, 
$(\omega(\cdot, \cdot))_m$ IID.

People also study the case that $P$ is stationary and ergodic.

2) Given $\omega$, $P^x_\omega$ is the prob. dist. of the walk
started at $x$ and walking in the environment $\omega$. 
