Random walk in random environment

General definitions:

Nearest-neighbour walks on $\mathbb{Z}^d$

Today: $d=1$ but the following def. are for general $d$.

ENVIRONMENT

- $M^d$: The set of probability measures on $\{\mathbb{R}\}_{i=1}^d$ where $\mathbb{e}_i = (0,\ldots,1,0,\ldots)$ is the $i$th location.

- $\mathcal{W} = \{ w : \mathbb{Z}^d \to M^d \}$ where $\Sigma w(x, \cdot) = 1$ and $w(x, \cdot) \geq 0$.

- On $\mathcal{W}$ we have a prob. measure $P$ (the dist. of the environment), assume to be:

  1. Stationary and ergodic.

  for $z \in \mathbb{Z}^d$, the dist. of $w(\cdot, z)$ equals the dist. of $w(\cdot, z')$.

In fact we will mostly take $P$ to be i.i.d.

The $(w(x, \cdot))_{x \in \mathbb{Z}^d}$ all have the same dist on $M^d$ and they are indep.
Uniform ellipticity:
\[ \exists \varepsilon > 0 \text{ s.t. } P(\forall x \in \mathbb{Z}^d, \forall e \in \{\pm e_i\} : w(x,e) \leq \varepsilon) = 1 \]

**WALK**

Given \( w \), the walk \( (X_n)_{n=0}^{\infty} \) is the Markov chain with transition prob. \( P(X_{n+1} = x + e \mid X_n = x) = w(x,e) \)
\[ \forall x \in \mathbb{Z}^d, \forall e \in \{\pm e_i\} \]

The probe dist. \( P_w^x \) is the dist. over the walk in the environment \( w \) with \( P_w^x(X_0 = x) = 1 \)

We say that \( P_w^x \) is the quenched dist. of the walk.

**Joint law:** In addition, we give a name to the joint law of \( (w, (X_n)_{n=0}^{\infty}) \) when \( X_0 \sim P_w^x \) and call their dist. \( P^x \).

The marginal of \( P^x \) on \( (X_n)_{n=0}^{\infty} \), also denoted \( P^x \), is called the annealed (or averaged) dist. of the walk.

i.e., To sample from the quenched dist. we are given \( w \) and \( x \) and sample from \( P_w^x \).
To sample from the annealed measure, we are given \( x \), we sample from \( P \) and then we sample \( (X_n)_{n=0}^{\infty} \) from \( P_w^x \).
To illustrate the difference:

\[ P^x (X_0 = x, X_1 = x + e_1, X_2 = x, X_3 = x - e_2) = w(x, e_1) w(x, e_1) w(x - e_2) \]

\[ P^x (X_0 = x, X_1 = x + e_1, X_2 = x, X_3 = x - e_2) = \mathbb{E} [w(x, e_1) w(x, e_2) w(x - e_2)] \]

under \( P \)

The quenched law is a Markov chain. The annealed law is not, but is invariant in spirit to changing \( x \).

**Questions to ask**

1. **Recurrence/Transience**: does the walk return to its starting point inf. often
2. **Law of large numbers**: Does \( \frac{X_n}{n} \) have a limit? what is the limit?
3. **Central limit theorem.**

**One-dimensional case**

From now on, \( d = 1 \).

Instead of \( w(x, x+1) \) we write \( W_x \) (so that \( W_x \) is the prob. to go right from \( x \))

Set: \( f_x = \frac{1 - W_x}{W_x} \) \( e \left[ \frac{\epsilon}{1 - \epsilon}, \frac{1 - \epsilon}{\epsilon} \right] \)

**Recurrence/transience**

It is clear that \( P^0 (\limsup X_n < \infty) = 0 \)

\( \{\limsup X_n = k\} \) is the same as saying that \( X_n = k \) for infinitely many \( n \), but \( X_n \) only equals \( k+1 \) finitely many
times, so the claim follows from the strong Markov property and uniform ellipticity.

Thus we are left with three options:

1. \( \lim_{n \to \infty} X_n = -\infty \) - transience to \(-\infty\)
2. \( \lim_{n \to \infty} X_n = +\infty \) - transience to \(+\infty\)
3. \( \limsup_{n \to \infty} X_n = -\infty, \liminf_{n \to \infty} X_n = +\infty \) - recurrence

Thom (Solomon 1975)

\( P \) is stationary and ergodic:

a) \( \mathbb{E}_P(\log s_n) < 0 \Rightarrow \lim_{n \to \infty} X_n = +\infty \) \( P \) a.s.

b) \( \mathbb{E}_P(\log s_n) > 0 \Rightarrow \lim_{n \to \infty} X_n = -\infty \) \( P \) a.s.

c) \( \mathbb{E}_P(\log s_n) = 0 \Rightarrow \limsup_{n \to \infty} X_n = +\infty \) \( P \) a.s.
\( \liminf_{n \to \infty} X_n = -\infty \) \( P \) a.s.

Proof:

For \( R, L \) positive integers we study \( V_{R, L}(x) = P_w^x (\text{the walk hits } +R \text{ before } -L) \)

depends also on \( w \)

We calculate \( V_{R, L} \) using recurrence relations.
\[ V_{L,L}(x) = w x V_{L,L}(x+1) + (1-w x) V_{L,L}(x-1) \quad \forall -L < x < R \]

and \( V_{L,L}(R) = 1 \), \( V_{L,L}(L) = 0 \)

we get:
\[
V_{L,L}(x) = \sum_{j=-L}^{x-1} \prod_{y=-L+1}^{j} g_y \quad \text{interpreting } \prod_{y=-L+1}^{j} g_y \text{ as } 1
\]

Derivation let \( g(x) = V_{L,L}(x+1) - V_{L,L}(x) \)

Then, \( \star \iff 0 = w x g(x) - (1-w x) g(x-1) \iff g(x) = x \cdot g(x-1) \)

\[
g(x) = x \cdot \frac{1}{x} \cdot \cdots \cdot \frac{1}{L+1} \cdot g(-L) = V_{L,L}(L+1)
\]

and \( \sum g(y) = 1 \quad -L \leq y \leq R-1 \)

Recurrence means that
\[
\begin{align*}
\lim_{R \to \infty} V_{L,L}(0) &= 0 = \lim_{R \to \infty} \frac{1}{\sum_{j=-L}^{R-1} \prod_{y=-L+1}^{j} g_y} \\
\lim_{L \to \infty} V_{L,L}(0) &= 1 = \lim_{R \to \infty} \frac{1}{\sum_{j=-L}^{R-1} \sum_{y=0}^{j} e^{s y} \prod_{y=-L+1}^{j} g_y}
\end{align*}
\]

For simplicity, focus on IID (get same results for ergodic \( \mathcal{P} \) by Birkhoff's ergodic thm and related arguments)

If \( \mathbb{E}(\log g_0) > 0 \) then \( \lim_{R \to \infty} V_{L,L}(0) = 0 \)

If \( \mathbb{E}(\log g_0) \leq 0 \) then \( \lim_{L \to \infty} V_{L,L}(0) = 1 \)
This already gives case (c) and using such ideas and the formula for $V_{R_L}(x)$ one gets also (a) and (b)

**Law of large numbers**

Why the behavior can differ from a homogeneous random walk? traps!

- **two sided trap:**
  - bias right
  - bias left

- **one sided trap:**
  - bias right
  - bias left

What determines a trap is the product of the $p_i$ by the formula for $V_{R_L}(x)$

**Theorem (Solomon 1975, Athre 1999, in ergodic case where statement slightly differs)**

a) $E_p(g_0) < 1 \Rightarrow \lim_{n \to \infty} \frac{X_n}{n} = \frac{1 - E_p(g_0)}{1 + E_p(g_0)} =: V_p \ p^0 - a.s.$

b) $E_p(g_0^+) < 1 \Rightarrow \lim_{n \to \infty} \frac{X_n}{n} = \frac{1 - E_p(g_0^+)}{1 + E_p(g_0^+)} \ p^0 - a.s.$

c) $E_p(g_0) = 1 \Rightarrow \lim_{n \to \infty} \frac{X_n}{n} = 0 \ p^0 - a.s.$

$E_p(g_0^+) = 1$

**Remark:** Comparing the two theorems we see that it is possible for a walk to be transient to $+\infty$ but non-ballistic.

I.e., $\lim_{n \to \infty} X_n = +\infty$, $\lim_{n \to \infty} \frac{X_n}{n} = 0$
Proof: prelude (Birkhoff ergodic thm - a generalization of the law of large numbers)

Suppose \((Y_1, Y_2, \ldots)\) is a stationary seq. taking values in some measurable space \((S, \mathcal{S})\), that is, \((Y_1, Y_2, \ldots) \equiv (Y_2, Y_3, \ldots)\).

Example: E.g., \((Y_n)_{n=1}^{\infty}\) are sampled from a (time-homogeneous) Markov chain with \(Y_1\) dist. as the stationary dist.

Thm: (Birkhoff)

For every meas. \(f: S \rightarrow \mathbb{R}\) s.t. \(E|f(Y)| < \infty\)
\[
\frac{1}{n} \sum_{k=1}^{n} f(Y_k) \xrightarrow{a.s.} E[f(Y_1) | \mathcal{I}] \text{ as } n \rightarrow \infty
\]

\(\mathcal{I}\) is the sigma algebra of invariant events that is, \(\mathcal{I}\) contains all the measurable \(E \subseteq S^*\) such that \((Y_1, Y_2, \ldots) \in E \iff (Y_2, Y_3, \ldots) \in E\).

The seq. \((Y_n)_{n=1}^{\infty}\) is called **ergodic** if: \(P(E) \in \{0, 1\}, \forall \mathcal{E} \in \mathcal{I}\) under the dist of \((Y_n)_{n=1}^{\infty}\).

In this case the right-hand side of Birkhoff’s thm is just \(E[f(Y_1)]\).

Moving to the proof of Solomon’s thm with Birkhoff’s thm in mind, we can write \(\frac{1}{n} (X_n - X_0) = \frac{1}{n} \sum_{k=1}^{n} (X_k - X_{k-1})\)
but unfortunately \((X_k - X_{k-1})_{k=1}^{\infty}\) is not stationary, neither under \(P^x\) nor under \(P^z\). Instead proceed as follows:
Without loss of generality, assume \( \limsup_{n \to \infty} X_n = +\infty \) \( p^0 \)-a.s.

\( \mathbb{E}[\log p] < 0 \)

Define \( T_n := \min \{ k \geq 0 : X_k = n \} \) for \( n \geq 0 \)

\( \mathcal{Z}_0 = \emptyset, \quad \mathcal{Z}_n = T_n - T_{n-1} \quad \text{for} \quad n > 1 \)

Claim: \( (\mathcal{Z}_n)_{n=1}^\infty \) is a stationary and ergodic seq. under \( p^0 \)

taking the claim for granted let’s proceed.

Birkhoff: \( \frac{T_n}{n} \xrightarrow{n \to \infty} \mathbb{E}^0[\mathcal{Z}] \) \( p^0 \)-a.s. (and in \( l^1 \) if \( \mathbb{E}[\mathcal{Z}] < \infty \))

Indeed \( \frac{1}{n} T_n = \frac{1}{n} \sum_{k=1}^{n} \mathcal{Z}_k \) and if \( \mathbb{E}[\mathcal{Z}] = \infty \) we can use

that \( \frac{1}{n} T_n \geq \frac{1}{n} \sum_{k=1}^{n} \mathcal{Z}_k \cdot 1_{\{\mathcal{Z}_k \leq M\}} \xrightarrow{n \to \infty} \mathbb{E}[\mathcal{Z} \cdot 1_{\{\mathcal{Z} \leq M\}}] \xrightarrow{n \to \infty} \mathbb{E}[\mathcal{Z}] \)

Lemma

\( \lim_{n \to \infty} \frac{X_n}{n} = \frac{1}{\mathbb{E}^0[\mathcal{Z}]} = \lim_{n \to \infty} \frac{1}{\mathbb{E}^0[\mathcal{Z}] \frac{T_n}{n}} \)

Proof:

let \( k_n \) be such that \( T_{k_n} \leq n \leq T_{k_n+1} \)

since \( \frac{T_n}{n} \xrightarrow{n \to \infty} \alpha = \mathbb{E}^0(\mathcal{Z}) \)

We also have \( \frac{k_n}{n} - \frac{1}{n} (n - T_{k_n}) \leq \frac{X_n}{n} \leq \frac{k_n}{n} \xrightarrow{n \to \infty} \frac{1}{\alpha} \)

\( \xrightarrow{\alpha} \frac{1}{\alpha} = 1 - \frac{T_{k_n}}{n} \xrightarrow{n \to \infty} 0 \) exercise

exercise