Lecture 5

Random walks in Random environment for $d \geq 2$

Walking on $\mathbb{Z}^d, d \geq 2$, nearest neighbour walk $P$, environment measure. Generally IID.

Uniformly ellipticity: $(\exists \varepsilon > 0, P(\omega(0) \geq \varepsilon) = 1 \quad \forall \omega \in \Omega)$

$P^x$ - quenched RW - (starting at $x$) and walking in $\omega$.
$P^x$ - annealed measure (also average) over $\omega$

**Question 1: Recurrence/Transience**

We study directional transience.

I.e., fix $l \in S^{d-1}$. Study $X_n \cdot l$

Claim: $P^0(\limsup X_n \cdot l \text{ is finite}) = 0$ for any unif. elliptic $\omega$

Saw this last time

So we have 3 options:

1. $\lim_{n \to \infty} X_n \cdot l = +\infty$

   Call this event $A_l$ - directional transience in direction $l$.

2. $\lim_{n \to \infty} X_n \cdot l = -\infty$

   Call this event $A_{-l}$

3. $\limsup_{n \to \infty} X_n \cdot l = +\infty \quad \liminf_{n \to \infty} X_n \cdot l = -\infty$

   Call this $O_l$ - projection to $l$ is neighbourhood recurrent.
By Claim: $P^x(A \cup U A \cup e \cup U O e) = 1$.

$\implies P^x(A \cup U A \cup e \cup U O e) = 1$

Important open question: $P^0(A e) \in \{0, 1\}$

What we know is only

1. Thm: (Kalikow 1981) : $P^0(O e) \in \{0, 1\}$.
   This is equivalent to $P^0(A e U U A \cup e U O e) \in \{0, 1\}$

2. In $d=2$,

   Thm (Merkl-Zerner 2001) : $P^0(A e) \in \{0, 1\}$

   They also showed that there exists a stationary and ergodic $P$ which is elliptic for which $P^0(A e) = \frac{1}{2}$

We proceed to prove Kalikow's 0-1 law:

Take $e = (1, 0, 0, \ldots)$ for notational simplicity.

Assume that $P^0(A e) > 0$ (or similarly $P^0(A \cdot e) > 0$)

Goal: $P^0(O e) = 0$

Claim: $P^0(\exists n \geq 1, X_n \cdot e < 0) < 1$

Proof: if the claim is not true, then $P^0(\exists n \geq 1, X_n \cdot e < 0) = 1$

This implies that for any $x \in \mathbb{Z}^d$, $P^x(\exists n \geq 1, X_n \cdot e < 0) = 1$ and this implies that $P^0(\liminf X_n \cdot e < 0) = 1$
and this contradicts the assumption that $P^0(Ae) > 0$.

To continue, denote $P = P^0(\exists n \geq 1, X_n \cdot l < 0) = 1$ for any $x \in \mathbb{Z}^d$ by trans-inv. = $P(\exists n \geq 1, X_n \cdot l < X \cdot l)$

Idea: Define "regeneration times" - times at which the walk first enters a half space it never visited before at each regeneration time, have probability $P$ to go back left of that point and these are "bounded above by independent".

Let $S_0 = 0$ and inductively, for each $k$,

$R_k := \min \{ n > S_k : X_n \cdot l < X_{S_k} \cdot l \} \quad k \geq 0$

$S_k := \min \{ n > R_{k-1} : X_n \cdot l > \max_{m \leq R_{k-1}} X_m \cdot l \} \quad k \geq 1$

By definition, $P^0(R_0 < \infty) = P$.

$P(R_0 < \infty, S_1 < \infty, R_1 < \infty) = \sum_{x \in \mathbb{Z}^d} P^0(R_0 < \infty, S_1 < \infty, R_1 < \infty, X_{S_1} = x)$
\[ \mathbb{E}_p \left[ P^0_w(R_0 < \infty, S_1 < \infty, R_1 < \infty, X_{s_1} = z) \right] = \text{Strong Markov property} \]

\[ = \mathbb{E}_p \left[ P^0_w(R_0 < \infty, S_1 < \infty, X_{s_1} = z) \right] \mathbb{P}^2 \left( \exists n \geq 1, X_{n+1} \not\in \mathbb{Z}^d \right) \]

A fcn. only of \((Wx)_{(x \cdot l \leq z \cdot l)}\)

A fcn. only of \((Wx)_{x \cdot l \geq z \cdot l}\)

\[ = \mathbb{E}_p \left[ P^0_w(R_0 < \infty, S_1 < \infty, X_{s_1} = z) \right] \mathbb{E}_p \left[ \mathbb{P}^2 \left( \exists n \geq 1, X_{n+1} \not\in \mathbb{Z}^d \right) \right] = \mathbb{P}^0_w(R_0 < \infty, S_1 < \infty, X_{s_1} = z) \mathbb{P}^2 \left( \exists n \geq 1, X_{n+1} \not\in \mathbb{Z}^d \right) = p \]

In summary,

\[ \mathbb{P}^0(R_0 < \infty, S_1 < \infty, R_1 < \infty) = p \cdot \sum_{z \in \mathbb{Z}^d} \mathbb{P}^0(R_0 < \infty, S_1 < \infty, X_{s_1} = z) = \]

\[ = p \cdot \mathbb{P}^0(R_0 < \infty, S_1 < \infty) \leq p \cdot \mathbb{P}^0(R_0 < \infty) = p^2 \]

By the same argument,

\[ \mathbb{P}^0(R_0 < \infty, S_1 < \infty, \ldots, R_k < \infty) \leq p^{k+1} \]

\[ \Rightarrow \mathbb{P}^0(\text{all } R_k \text{ and } S_k \text{ are finite}) = 0 \]

On \(O_e\), all \(R_k\) and \(S_k\) are finite so that \(\mathbb{P}(O_e) = 0\), as we wanted to prove.

We now show Merkl-Zerner thm:

Planarity will be used to create intersections between random walk trajectories.
Recall: Lévy's upward thm:

if \((F_n)_{n \geq 0}\) is a filtration and \(A\) an event then,
\[
P(A | F_n) \xrightarrow{n \to \infty} P(A | \sigma(\bigcup_n F_n)) \quad \text{a.s.}
\]

We use this for the RW and the event \(A_e\). (we assume for simplicity that \(L^\circ = (1,0)\))

\[F_n = \sigma(X_1, \ldots, X_n)\]. Then, for each \(\omega\) and any \(x \in \mathbb{Z}^d\)
\[
P_{X_0}^\omega(A_e) = P_{X_0}^\omega(A_e | F_n) \xrightarrow{n \to \infty} P_{X_0}^\omega(A_e | \sigma(X_1, X_2, \ldots)) = 1_{A_e}
\]

We are assuming that \(P^\circ(A_e | \sigma(L_e \cup A_e)) = 1\)

Idea of the proof

Start one random walker at \((-L,0)\) and another one at \((L, L)\)

After the first \(L\) steps, both walkers are nearly certain if \(A_e\) happens for them or not.

But since they're still in independent environments then these decisions are independent and it is possible (if \(P^\circ(A_e) > 0, P^\circ(A \setminus e) > 0\)) that the one started at \((L, L)\) will satisfy \(A_e\).

On this event, it is very unlikely that they intersect

since: \(P_{X_0}^{L_0} (A_e) \approx 1\) \(P_{X_0}^{L_0} (A_e) \approx 0\)
Lastly, using planarity, it is shown that they intersect with a uniformly positive prob. when \((y_L)\) is well chosen

We sketch the proof (not in every detail)
First, let us obtain an version of \(\otimes\) with a uniform rate of convergence.

**Claim 1:** \(\forall \varepsilon > 0 \exists M_\varepsilon > 0 \text{ s.t. } \forall x \in \mathbb{Z}^d,\)
\[ P^x(\lim_{n \to \infty} P^x_n(A_\varepsilon) - 1_{A_\varepsilon} < \varepsilon \; \forall n > M_\varepsilon) = 1 - \varepsilon \]

**Proof:**

The proof is the same for all \(x\), so we prove for \(x = 0\).

By \(\otimes\) we know that \(\lim_{n \to \infty} P^x_n(A_\varepsilon) - 1_{A_\varepsilon} < \varepsilon\) occurs for all \(n > N_\varepsilon(w, x_0)\) where \(N_\varepsilon(w, x_0) < \infty\)

Now, take \(M_\varepsilon\) so large s.t. \(P^0(N_\varepsilon > M_\varepsilon) < \varepsilon\)

We now consider a RW \((X^1_n)\) starting at \((-L, 0)\) and another RW \((X^2_n)\) starting at \((L, y_L)\) for some \(y_L\) to be chosen later.

Call the annealed measure of both RWs by \(P_L\)

**Claim 2:** for any \((y_L)_L \geq 1,\)
\[ P_L(\lim_{n \to \infty} X^1_n : l = +\infty, \lim_{n \to \infty} X^2_n : l = -\infty) \xrightarrow{L \to \infty} P^0(A_\varepsilon) \cdot P^0(A_{-\varepsilon}) \]
**Proof:** for a small \( \varepsilon > 0 \), take \( L > M \).

Then, \( P_{\mathbb{A}}(\lim X_n^1 : l = -\infty, \lim X_n^2 : l = -\infty) \) is almost equal to:

\[
\mathbb{E}_{\mathbb{A}}[P_{\mathbb{A}}(A_1 \cdot P_{\mathbb{A}}(A_2)]
\]

depends on \( \varepsilon \)

The distribution \( X_n^1 \) only depends on \( (w_x)x \cdot t < 0 \) and \( X_n^2 \) only depends on \( (w_x)x \cdot t > 0 \).

So they are independent under \( P \). This allows us to get the claim.

We assume that \( P^0(A_1), P^0(A_2) > 0 \) in order to get a contradiction.

**Claim 3:** for any \( (y_d) \)

\[
\lim_{l \to \infty} P_{\mathbb{A}}(\lim X_n^1 : l = -\infty, \lim X_n^2 : l = -\infty, \text{the walk trajectories intersect}) = 0
\]

**Proof:** Breaking the event according to where the intersect at some \( X \) with \( x \cdot l \leq 0 \).

The intersection necessarily happens after more than \( L \) steps of \( X_n^2 \). Again with \( L > M \), we have that \( P^{X_n^2}_{\mathbb{A}}(A_1 \cdot l < \varepsilon \text{ for } n > L \text{ with high prob}) \)

In particular, at the intersection point.

Define a stopping time \( T = \min \{ n \geq 0 : P_{\mathbb{A}}(X_n^1(\text{intersection}) \leq \varepsilon \} \)

on the event of intersection and when (***) happens.
and \( \lim X_n^2 l = -\infty \) then \( T < \infty \).

Finally, for the first random walk, for any \( k < \infty \),
\[
P_{\infty}^+ (\lim X_n \cdot l = \infty) \leq 3 + P_{\infty}^+ (T > k)
\]

**strong markov property for \( \min(T, k) \)**

The claim follows from these considerations.

It remains to "make the walk intersect" to get a contradiction.

**Claim 4:** For a specific \( (y_l) \)
\[
P_{\infty} (\lim X_n \cdot l = +\infty, \lim X_n \cdot l = -\infty, \text{ no intersection}) \leq \frac{1}{2} P_{\infty} (\lim X_n \cdot l = +\infty, \lim X_n \cdot l = -\infty)
\]

**Proof:**

Choose \( y_l \) as the \( P_{\infty} \)-median of the \( y \)-coord for the last visit of \( X_n \) to the line \( X = L \), conditioned on \( \lim X_n \cdot l = +\infty \).

By def, we have prob. \( \geq \frac{1}{2} \) to have the last visit with \( y \)-coord \( \geq y_l \)

and prob. \( \leq \frac{1}{2} \) for \( y \)-coord \( \leq y_l \).

This is enough if \( X_n^2 \) doesn't go to positions with \( X \)-coord \( \geq L \)