NOTES ON SIGMA ALGEBRAS FOR BROWNIAN MOTION COURSE

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These notes provide some basic facts regarding sigma algebras which arise in our Brownian motion course. We focus on sigma algebras for spaces of real-valued functions. For a deeper discussion see, for instance,
http://www.encyclopediaofmath.org/index.php/Measurable_space
and the related articles and books referenced there.

1. Sigma algebras on function spaces

The space $\mathbb{R}$ comes with its standard topology which we denote by $\mathcal{O}$ (that is, $\mathcal{O}$ is the collection of open sets in $\mathbb{R}$). It comes also with its Borel sigma algebra which we denote by $\mathcal{B}$. The Borel sigma algebra is the smallest one containing all open sets (i.e., the sigma algebra generated by $\mathcal{O}$).

Let I be an arbitrary non-empty set (finite, countable or uncountable). For $x \in I$, the coordinate function T_x is the function $T_x : \mathbb{R}^I \to \mathbb{R}$ defined by $T_x(f) := f(x)$. The set $\mathbb{R}^I$ is naturally endowed with both a topology and a sigma algebra named, naturally, the *product topology* and *product sigma algebra*. The product topology $\mathcal{O}^I$ is the smallest topology making all the (T_x), $x \in I$, continuous. The product sigma algebra $\mathcal{B}^I$ is the smallest sigma algebra making all the (T_x), $x \in I$, measurable (with respect to the Borel sigma algebra on $\mathbb{R}$). Slightly more explicitly, a base for the product topology is given by all *open cylinder sets* which are the sets of the form

$$\prod_{x \in I} U_x \quad \text{where } U_x \in \mathcal{O} \text{ and all but finitely many of the } U_x \text{ equal } \mathbb{R}. \tag{1}$$

Similarly, the product sigma algebra is the smallest sigma algebra containing all *measurable cylinder sets* which are sets of the form

$$\prod_{x \in I} A_x \quad \text{where } A_x \in \mathcal{B} \text{ and all but finitely many of the } A_x \text{ equal } \mathbb{R}. \tag{2}$$

The product sigma algebra may also be called the sigma algebra of *finite-dimensional distributions* as the cylinder sets (2) exactly correspond to the distributions of finitely many coordinates. A set in the product sigma algebra places restrictions on at most countably many coordinates of I, as the following claim shows.

Proposition 1.1. *Let us identify a set $A' \in \mathbb{R}^{I'}$, where $I' \subseteq I$, with the set $A \in \mathbb{R}^I$ given by $\{f \in \mathbb{R}^I : (f_x)_{x \in I'} \in A'\}$. With this identification we have*

$$\mathcal{B}^I \text{ is the union of all } \mathcal{B}^{I'} \text{ over all countable } I' \subseteq I.$$

Proof. Let us denote by $\widetilde{\mathcal{B}^I}$ the union of all $\mathcal{B}^{I'}$ over all countable $I' \subseteq I$. The fact that $\widetilde{\mathcal{B}^I}$ is contained in $\mathcal{B}^I$ is an immediate consequence of the definitions. It is also simple to check that $\widetilde{\mathcal{B}^I}$ is a sigma algebra. Finally, all coordinate functions (T_x), $x \in I$, are measurable with respect to $\widetilde{\mathcal{B}^I}$. Hence $\widetilde{\mathcal{B}^I} = \mathcal{B}^I$ by the definition of $\mathcal{B}^I$. □

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Since we are used to Borel sigma algebras, we may define an additional sigma algebra, $\sigma(\mathcal{O}^I)$, which is the smallest one making all sets in the product topology measurable. The question arises whether the two sigma algebras $\sigma(\mathcal{O}^I)$ and $\mathcal{B}^I$ are equal, and if not, which one is more useful. The answer depends on the cardinality of I.

Proposition 1.2. *If I is finite or countable then $\sigma(\mathcal{O}^I) = \mathcal{B}^I$. If I is uncountable then $\mathcal{B}^I$ is a strict subset of $\sigma(\mathcal{O}^I)$.*

Proof. Let us first show that $\mathcal{B}^I$ is contained in $\sigma(\mathcal{O}^I)$. By the definition of $\mathcal{B}^I$ it suffices to show that all the coordinate functions (T_x), $x \in I$, are measurable with respect to $\sigma(\mathcal{O}^I)$. Indeed, these coordinate functions are continuous, by definition, with respect to $\mathcal{O}^I$ and hence also measurable with respect to $\sigma(\mathcal{O}^I)$.

We now show that if I is finite or countable then $\sigma(\mathcal{O}^I)$ is contained in $\mathcal{B}^I$. This hinges on the notion of separability. Specifically, the fact that the topology $\mathcal{O}^I$ is second countable for such I (that is, $\mathcal{O}^I$ has a base of *countable* cardinality). The countable base is obtained explicitly as all sets of the form (1) where all the U_x which are not $\mathbb{R}$ are open intervals with rational endpoints. The sets in the countable base are also of the form (2) and hence in $\mathcal{B}^I$. Since every set in $\mathcal{O}^I$ is the union of sets from the countable base (in particular, a countable union of such sets) we see that $\mathcal{O}^I$ is contained in $\mathcal{B}^I$, whence $\sigma(\mathcal{O}^I)$ is contained in $\mathcal{B}^I$.

Finally, let us show that if I is uncountable then $\sigma(\mathcal{O}^I)$ is strictly greater than $\mathcal{B}^I$. Indeed, the set

$$A = \{f \in \mathbb{R}^I : \exists x \in I, f(x) \in (-1, 1)\}$$

of all functions having at least one coordinate in $(-1, 1)$ is open, i.e., in $\mathcal{O}^I$ (and hence in $\sigma(\mathcal{O}^I)$) since it is the (uncountable) union of the open cylinder sets $(-1, 1) \times \prod_{x \in I \setminus \{y\}} \mathbb{R}$ over $y \in I$. However, it follows from Proposition 1.1 that the set A is not in $\mathcal{B}^I$ (since A does not belong to any of the $\mathcal{B}^{I'}$ for $I' \subseteq I$ countable). □

Regarding the usefulness question. For most applications the space $\mathbb{R}^I$ with I uncountable is itself not a useful space, being too large to work with. Thus, for such I, neither of the sigma algebras, $\mathcal{B}^I$ and $\sigma(\mathcal{O}^I)$, is too useful.

2. Sigma algebras on spaces of continuous functions

Let us now consider the space $C(I)$ of continuous functions $f : I \to \mathbb{R}$, where I is an arbitrary, positive length, interval in $\mathbb{R}$ (possibly unbounded). The space $C(I)$ comes with the supremum metric, $d(f, g) := \sup_{x\in I} |f(x) - g(x)|$ and its associated topology $\mathcal{O}(C(I))$. We again have the coordinate functions (T_x), $x \in I$, given by $T_x : C(I) \to \mathbb{R}$, $T_x(f) := f(x)$. We may again consider two sigma algebras, $\mathcal{B}(C(I))$, the analog of the product sigma algebra, which is the minimal sigma algebra making all coordinate functions measurable, and $\sigma(\mathcal{O}(C(I)))$, the sigma algebra generated from the open sets in $\mathcal{O}(C(I))$. Similarly to before, the sigma algebra $\mathcal{B}(C(I))$ is generated by the restriction to $C(I)$ of the measurable cylinder sets (2). Thus it may be thought of as the sigma algebra of finite-dimensional distributions on $C(I)$. However, it now has a more economic description.

Proposition 2.1. *For any countable dense set $Q \subseteq I$ the sigma algebra $\mathcal{B}(C(I))$ is generated by measurable cylinder sets which place restrictions only on the coordinates of Q. That is, by sets of the form*

$$C(I) \cap \left(\prod_{x\in Q} A_x \prod_{x\in I\setminus Q} \mathbb{R} \right) \quad \text{where } A_x \in \mathcal{B} \text{ for all } x \in Q. \tag{3}$$

Proof. Write $\widetilde{\mathcal{B}(C(I))}$ for the sigma algebra generated by the sets of the form (3). It is simple to see that $\widetilde{\mathcal{B}(C(I))} \subseteq \mathcal{B}(C(I))$. To see the converse inclusion, it suffices to check that the coordinate functions are measurable with respect to $\widetilde{\mathcal{B}(C(I))}$. This follows since the function T_x, $x \in I$, is the pointwise limit (using continuity of $f \in C(I)$) of the functions (T_{x_n}), $x_n \in Q$, $x_n \to x$, which are measurable with respect to $\widetilde{\mathcal{B}(C(I))}$. □

Having two sigma algebras on $C(I)$, namely $\mathcal{B}(C(I))$ and $\sigma(\mathcal{O}(C(I)))$, we again ask whether these two sigma algebras are equal, and if not, which is more useful. The answer now depends on the topological properties of I.

Proposition 2.2. *If I is a compact interval then $\sigma(\mathcal{O}(C(I))) = \mathcal{B}(C(I))$. Otherwise, $\sigma(\mathcal{O}(C(I)))$ is strictly larger than $\mathcal{B}(C(I))$.*

Proof. The proof is similar to that of Proposition 1.2. Let us first show that $\mathcal{B}(C(I))$ is contained in $\sigma(\mathcal{O}(C(I)))$. Again, it suffices to show that all coordinate functions are measurable with respect to $\sigma(\mathcal{O}(C(I)))$ and this follows since the coordinate functions are continuous in the supremum metric.

Now assume that I is a compact interval. In this case the topology $\mathcal{O}(C(I))$ is second countable. This follows from the well-known fact that $C(I)$ is a separable metric space in the supremum norm, by noting that we may take as a base of the topology the set of all open metric balls of rational radius around the countable dense set of functions. Thus to show that $\sigma(\mathcal{O}(C(I))) = \mathcal{B}(C(I))$ when I is compact it suffices to show that for every $f \in C(I)$ and every $0 < r < \infty$ the ball $B(f,r) := \{g \in C(I) \colon d(f,g) < r\}$ is in $\mathcal{B}(C(I))$. This ball is the union of the closed balls $\bar{B}(f,r') := \{g \in C(I) \colon d(f,g) \leqslant r'\}$, $r' < r$ (and this may be written as a countable union). Now $\bar{B}(f,r')$ equals the set $\{g \in C(I) \colon |g(x) - f(x)| \leqslant r' \text{ for all } x \in \mathbb{Q} \cap I\}$ which is in $\mathcal{B}(C(I))$ by Proposition 2.1.

Finally, suppose I is not a compact interval. For concreteness, say $I = [0,\infty)$ (other cases are analogous). We sketch a proof that $\sigma(\mathcal{O}(C(I)))$ is strictly larger than $\mathcal{B}(C(I))$ by a cardinality argument. It is known that the cardinality of $\mathcal{B}$ (the Borel sigma algebra of $\mathbb{R}$) is that of the continuum (we will not prove this here). One may then deduce from Proposition 2.1 that the cardinality of $\mathcal{B}(C(I))$ is also that of the continuum. However, when I is not compact, the cardinality of $\sigma(\mathcal{O}(C(I)))$ is larger than that of the continuum. Indeed, this is the case already for $\mathcal{O}(C(I))$. To see this, in the case $I = [0,\infty)$, for each binary sequence $b \in \{0,1\}^{\{0,1,2,\ldots\}}$ define a function $f_b \in C(I)$ as follows: on the interval $[k, k+1]$, according to the bit b_k, the function f_b is either the zero function or a piecewise linear function equalling 0 at k and $k+1$ and equalling 1 at $k + \frac{1}{2}$. Thus the set of such functions (f_b) forms a set of continuum cardinality in $C(I)$ with the property that the supremum distance of every two functions in the set is 1. Consider the open balls $B_b := B(f_b, 1/2)$. Then these are a continuum of disjoint open sets. Since an arbitrary union of them is in $\mathcal{O}(C(I))$ we deduce that $\mathcal{O}(C(I))$ has cardinality larger than the continuum. □

We proceed to the question of which sigma algebra is more useful. Here the answer is clear-cut, the sigma algebra $\mathcal{B}(C(I))$ is more useful and is the only one we will use in our study of Brownian motion. In particular, we consider Brownian motion as a random function in $C[0,\infty)$ with respect to the sigma algebra $\mathcal{B}(C([0,\infty)))$. We mention three additional characterizations of $\mathcal{B}(C(I))$:

(i) $\mathcal{B}(C(I))$ is the sigma algebra generated by the topology of uniform convergence on compact sub-intervals of I. That is, the minimal sigma algebra making every open set in this topology measurable.

(ii) $\mathcal{B}(C(I))$ is the minimal sigma algebra containing all $\mathcal{B}(C(J))$, where J is a compact subinterval of I. Here we identify a set A in $C(J)$ with the set $\{f \in C(I)\colon (f(x))_{x\in J} \in A\}$.

(iii) For any countable dense set $Q \subseteq I$, $B(C(I))$ may be written as the collection of sets of the form $\{f \in C(I)\colon (f(x))_{x\in Q} \in A\}$ where $A \in \mathcal{B}^Q$. In particular, $\mathcal{B}(C(I))$ is separable, that is, there is a countable collection of sets such that $\mathcal{B}(C(I))$ is the minimal sigma algebra containing this collection. In fact, $\mathcal{B}(C(I))$ is a so-called *standard Borel space*.

3. Uniqueness of measures

Let $(\Omega, \mathcal{F})$ be a measurable space. The question arises from time to time, given two measures on $(\Omega, \mathcal{F})$, how to show that they are equal? For instance, we have defined Brownian motion as the measure on $(C([0,\infty)), \mathcal{B}(C([0,\infty))))$ supported on functions f with $f(0) = 0$ and having stationary independent increments with the increment from t to s distributed $N(0, t-s)$. Do these properties uniquely define the measure? To tackle these questions define a *π-system* to be a collection $\mathcal{P}$ of subsets of Ω which is closed under finite intersections. That is, $A \cap B \in \mathcal{P}$ whenever $A, B \in \mathcal{P}$. The following result is quite useful.

Proposition 3.1. *Let $(\Omega, \mathcal{F})$ be a measurable space such that $\mathcal{F} = \sigma(\mathcal{P})$ for some π-system $\mathcal{P}$. If two probability measures on $(\Omega, \mathcal{F})$ coincide on $\mathcal{P}$ then they are equal.*

This is a consequence of Dynkin's $\pi - \lambda$ theorem which we state next. A collection $\mathcal{L}$ of subsets of Ω is called a *λ-system* if (i) $\Omega \in \mathcal{L}$, (ii) $B \setminus A \in \mathcal{L}$ whenever $A, B \in \mathcal{L}$ and $A \subseteq B$ and (iii) $A_n \in \mathcal{L}$ and A_n increases to A implies that $A \in \mathcal{L}$.

Theorem 3.2. *(Dynkin's $\pi - \lambda$ theorem) If $\mathcal{P}$ is a π-system and $\mathcal{L}$ is a λ-system containing $\mathcal{P}$ then $\sigma(\mathcal{P}) \subseteq \mathcal{L}$.*

The $\pi - \lambda$ theorem (which we do not prove here) directly implies Proposition 3.1 as follows. Suppose $\mathbb{P}_1, \mathbb{P}_2$ are two probability measures on $(\Omega, \mathcal{F})$ which coincide on the given π-system $\mathcal{P}$. Define $\mathcal{L} \subseteq \mathcal{F}$ to be the collection of sets in $\mathcal{F}$ on which $\mathbb{P}_1$ and $\mathbb{P}_2$ coincide. It is not difficult to check that $\mathcal{L}$ is a λ-system, whence $\mathcal{L} = \mathcal{F}$ by the $\pi - \lambda$ theorem since $\mathcal{P} \subseteq \mathcal{L}$ and $\mathcal{F} = \sigma(P)$.

In the case of Brownian motion we conclude that the properties in its definition uniquely determine the Brownian motion measure on $(C([0,\infty)), \mathcal{B}(C([0,\infty))))$. Indeed, the sigma algebra $\mathcal{B}(C([0,\infty)))$ is generated by the π-system of measurable cylinder sets (the restriction to $C([0,\infty))$ of the sets in (2)) and the distribution on these measurable cylinder sets is given by the specified finite-dimensional distributions.