# BROWNIAN MOTION HOMEWORK ASSIGNMENT 1

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Some reminders: The characteristic function (or Fourier transform) of a random vector $X$ taking values in $\mathbb{R}^n$ is the continuous function $\hat{X} : \mathbb{R}^n \to \mathbb{C}$ defined by

$$\hat{X}(\theta) := \mathbb{E}e^{i<\theta,X>},$$

where $< \theta, X >:= \sum_{i=1}^n \theta_i X_i$. If two random vectors have the same characteristic function then they have the same distribution. A sequence $X_1, X_2, \ldots,$ of random vectors converges in distribution to a random vector $X$ if and only if $\hat{X}_j$ converges pointwise to $\hat{X}$. Furthermore, if $\hat{X}_j$ converges pointwise to a continuous function then $X_j$ converges in distribution to some random vector.

A random variable $X$ is distributed $N(\mu, \sigma^2)$, $\mu \in \mathbb{R}, \sigma \geqslant 0$, if it has the density

$$\frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \quad x \in \mathbb{R}.$$

or, in the case $\sigma = 0$, if it is equal to $\mu$ with probability 1. Its characteristic function is given by

$$\hat{X}(\theta) := \exp\left(i\mu\theta - \frac{1}{2}\sigma^2\theta^2\right).$$

We say that a random vector $X$ in $\mathbb{R}^n$ is a Gaussian vector if there exists a vector $\mu \in \mathbb{R}^n$ and an $n \times n$ real matrix $A$ such that

$$X = AY + \mu, \tag{1}$$

where $Y$ is a random vector in $\mathbb{R}^n$ all of whose coordinates are independent and have the distribution $N(0, 1)$.

(i) (a) Prove that if $X$ is given by (1) then

$$\hat{X}(\theta) = \exp\left(i(\mu, \theta) - \frac{1}{2}(\theta, \Sigma\theta)\right), \tag{2}$$

where $\Sigma := AA^t$ and $A^t$ is the transpose of $A$.

(b) Deduce that the distribution of a Gaussian vector is uniquely determined by its mean vector and covariance matrix.

(c) Deduce that if $X$ is a random vector in $\mathbb{R}^n$ whose characteristic function is given by (2) for some $\mu \in \mathbb{R}^n$ and some non-negative definite matrix $\Sigma$ then $X$ is a Gaussian vector.

(d) Suppose that $X_1, X_2, \ldots,$ are a sequence of Gaussian vectors in $\mathbb{R}^n$ which converges in distribution to a random vector $X$. Prove that $X$ is a Gaussian vector having mean vector and covariance matrix which are the limits (coordinate-wise) of the mean vectors and covariance matrices of $X_j$ (in particular, these limits exist).

(ii) Solve exercise 1.1 from the Brownian motion book.

(iii) Solve exercise 1.8 from the Brownian motion book.

The Brownian motion book is available at: http://research.microsoft.com/en-us/um/people/peres/brbook.pdf

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