BROWNIAN MOTION HOMEWORK ASSIGNMENT 10

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(i) Solve exercise 7.1 from the Brownian motion book (you may use Lemma 1.41 as needed. See the beginning of Section 1.4 for the definition of the Dirichlet space $D[0,1]$).

(ii) Solve exercise 7.2 from the Brownian motion book.

(iii) Let $(\mathcal{F}(t))$, $t \geq 0$, be a complete filtration (i.e., $\mathcal{F}(t)$ contains all sets of measure 0 for each $t$). A local martingale $(M(t))$, $t \geq 0$, is an adapted (to $(\mathcal{F}(t))$) stochastic process for which there exists a sequence $(T_n)$ of stopping times satisfying

(a) $(T_n)$ are almost surely increasing to infinity: $\mathbb{P}(T_n \leq T_{n+1}) = 1$ and $\mathbb{P}(T_n \to \infty) = 1$.
(b) For each $n$, $(M(t \wedge T_n))$, $t \geq 0$, is a martingale.

In the next class we will see examples of local martingales which are not martingales. This exercise explores some basic properties of local martingales.

Let $(M(t))$, $t \geq 0$, be a continuous local martingale. That is, a local martingale whose sample paths are almost surely continuous.

(a) Prove that if $\mathbb{E}|M(0)| < \infty$ and $M$ is bounded from below in the sense that there exists some $C < \infty$ for which $\mathbb{P}(M(t) \geq -C) = 1$ for all $t$ then $M$ is a supermartingale.

Remark: in particular, if $M$ is bounded both from below and from above then $M$ is a martingale.

(b) Suppose there exists a sequence $(a_n)$ such that for any $n$,

$$\mathbb{P}\left(\sup_{0 \leq s < \infty} |M(s \wedge T_n)| \leq a_n\right) = 1. \quad (1)$$

Prove that for any fixed $t \geq 0$, the sequence $(M(t \wedge T_n))$ (indexed by $n$) is a discrete time martingale.

(c) Suppose that, in addition to (1),

$$\sup_n \mathbb{E}(M(t \wedge T_n)^2) < \infty \quad \text{for all } t. \quad (2)$$

Prove that $M$ is a martingale.