BROWNIAN MOTION HOMEWORK ASSIGNMENT 12

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(i) (a) Solve exercise 7.6 from the Brownian motion book.
(b) Solve exercise 7.7 from the Brownian motion book.

(ii) Let $(B(t))$, $t \geqslant 0$, be a standard Brownian motion. For every Borel set $S \subseteq \mathbb{R}$ define the *occupation time* of S up to time t as

$$\Gamma_t(S) := \int_0^t 1_S(B(s))ds,$$

i.e., the Lebesgue measure of the time which B spends at S up to time t. The *local time process* $(L_t(x))$, $(t, x) \in [0, \infty) \times \mathbb{R}$, is a random field, continuous in both variables together, which satisfies

$$\Gamma_t(S) = \int_S L_t(x)dx, \quad t \geqslant 0, \; S \text{ Borel}.$$

Assuming that a local time process as above exists, prove that the Brownian motion B is almost surely not differentiable at any point.

Remark: Of course, your proof should rely on the local time process, you cannot just cite the non-differentiability theorem we proved in class.

Hint: If B is differentiable at t then $|B(t+h) - B(t)| \leqslant Ch$ for small h and large C.

(iii) (Kolmogorov-Čentsov theorem). Let $(X(t))$, $0 \leqslant t \leqslant 1$, be a stochastic process (we assume no continuity or other regularity properties of X so that only the probability of events depending on countably many of the $(X(t))$ is defined). Suppose there exist $\alpha, \beta, C > 0$ such that

$$\mathbb{E}|X(t) - X(s)|^\alpha \leqslant C|t-s|^{1+\beta}, \quad 0 \leqslant s, t \leqslant 1. \tag{1}$$

Then there exists a continuous modification $(Y(t))$, $0 \leqslant t \leqslant 1$, of X. I.e., a continuous stochastic process Y satisfying $\mathbb{P}(X(t) = Y(t)) = 1$ for every $0 \leqslant t \leqslant 1$. Moreover, Y is almost surely locally γ-Hölder continuous for each $0 < \gamma < \frac{\beta}{\alpha}$. That is, for each such γ there exists a constant $\delta > 0$ and a random variable h with $\mathbb{P}(h > 0) = 1$ such that

$$\mathbb{P}(|Y(t) - Y(s)| \leqslant \delta|t-s|^\gamma \text{ for all } 0 \leqslant t, s \leqslant 1 \text{ satisfying } |t-s| \leqslant h) = 1.$$

In this exercise we establish the above theorem. Thus we assume X satisfies (1) and aim to show the existence of Y as above.

(a) Prove that X is continuous in probability. That is, for any $\varepsilon > 0$ and any $0 \leqslant t \leqslant 1$,

$$\mathbb{P}(|X_s - X_t| \geqslant \varepsilon) \to 0 \quad \text{as } s \text{ tends to } t.$$

(b) For the rest of the question let $D_n := \{\frac{k}{2^n} : 0 \leqslant k \leqslant 2^n\}$, $D := \cup_{n \geqslant 0} D_n$ and fix $0 < \gamma < \frac{\beta}{\alpha}$. Prove that there exists a random integer N such that almost surely,

$$\max_{1 \leqslant k \leqslant 2^n} \left| X\left(\frac{k}{2^n}\right) - X\left(\frac{k-1}{2^n}\right) \right| < 2^{-\gamma n} \quad \text{for all } n \geqslant N.$$

(c) Let $\delta := \frac{2}{1-2^{-\gamma}}$. Deduce that almost surely,

$$|X(t) - X(s)| \leqslant \delta |t-s|^\gamma \quad \text{for all } t, s \in D \text{ with } |t-s| < 2^{-N}.$$

Hint: It may help to show first that for every $m > n \geqslant N$ we have almost surely, $|X(t) - X(s)| \leqslant 2\sum_{j=n+1}^m 2^{-\gamma j}$ for all $t, s \in D_m$ satisfying $|t-s| < 2^{-n}$.

(d) Deduce that the required modification Y exists.

Remark: An analogous theorem holds when the index set of X is multi-dimensional.

The Brownian motion book is available at: http://research.microsoft.com/en-us/um/people/peres/brbook.pdf

Date: May 28, 2014.