BROWNIAN MOTION HOMEWORK ASSIGNMENT 5

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(i) (a) (One-sided Chebyshev inequality) Let X be a random variable with $\mathbb{E}X^2 < \infty$. Denote $\sigma := \sqrt{\mathrm{Var}(X)}$. Prove that for any $t > 0$,

$$\mathbb{P}(X - \mathbb{E}(X) \geqslant t\sigma) \leqslant \frac{1}{t^2 + 1}.$$

Hint: If an event A and random variable Y satisfy $1_A \leqslant Y$ almost surely then $\mathbb{P}(A) \leqslant \mathbb{E}(Y)$.

(b) (Paley-Zygmund inequality) Deduce (or prove directly) that if X is a non-negative random variable with $\mathbb{E}(X^2) < \infty$ then for any $\lambda \in [0, 1]$,

$$\mathbb{P}(X > \lambda\mathbb{E}(X)) \geqslant (1 - \lambda)^2 \frac{(\mathbb{E}X)^2}{\mathbb{E}(X^2)}.$$

(c) (Kochen-Stone lemma): Deduce the following statement, of a similar flavor to the Borel-Cantelli lemma. If (A_n) is a sequence of events, not necessarily independent, satisfying

$$\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty \quad \text{and} \quad \liminf_{k\to\infty} \frac{\sum_{1\leqslant m,n\leqslant k} \mathbb{P}(A_m \cap A_n)}{(\sum_{n=1}^{k} \mathbb{P}(A_n))^2} < \infty$$

then

$$\mathbb{P}(\text{infinitely many of the } (A_n) \text{ occur}) > 0.$$

(ii) Solve exercise 3.3 from the Brownian motion book.

(iii) Solve exercise 3.9 from the Brownian motion book.

The Brownian motion book is available at: http://research.microsoft.com/en-us/um/people/peres/brbook.pdf

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