(i) Let \((X_i), i \geq 1,\) be a sequence of independent, identically distributed random variables with \(E X_1 = 0\) and \(E X_1^2 = 1.\) Define \((S_n), n \geq 0,\) by \(S_0 := 0\) and

\[
S_n := \sum_{i=1}^{n} X_i, \quad n \geq 1.
\]

Denote \(M_n := \max(S_k: 0 \leq k \leq n)\) and \(T_n := \min(0 \leq k \leq n: S_k = M_n)\) (the maximum until time \(n\) and the first time at which it is attained). Similarly, for a standard Brownian motion \(B\) define \(M := \max(B(t): 0 \leq t \leq 1)\) and \(T := \min(0 \leq t \leq 1: B(t) = M)\). Prove that \(T_n/n\) converges in distribution to \(T\) as \(n \to \infty.\)

(ii) (a) Let \(S\) be a metric space with metric \(d.\) Suppose that \((X_n), (Y_n)\) are two sequences of random variables on the same probability space such that \(X_n\) converges in distribution to a random variable \(X\) and \(d(X_n, Y_n)\) converges to zero in probability. Prove that \(Y_n\) converges in distribution to \(X.\)

(b) Let \((X_i), i \geq 1,\) be a sequence of independent, identically distributed random variables with \(P(X_1 = 1) = P(X_1 = -1) = \frac{1}{2}.\) In a primitive model for a stock price one is given a volatility parameter \(\sigma > 0\) and a number \(n\) of epochs and defines the stock price process as the random \(P_{n,\sigma}: [0, 1] \to [0, \infty)\) given by \(P_{n,\sigma}(0) := 1,\)

\[
P_{n,\sigma}\left(\frac{k}{n}\right) := \prod_{i=1}^{k} \left(1 + \frac{\sigma}{\sqrt{n}} X_i\right), \quad 1 \leq k \leq n
\]

and \(P_{n,\sigma}(t)\) defined as the linear interpolation of these values. It is assumed that \(n > \sigma^2\) so that the process is indeed non-negative.

Let \(B\) be a standard Brownian motion. Prove that \((P_{n,\sigma}(t)), 0 \leq t \leq 1,\) converges in distribution (as a random function in the space \(C[0, 1]\)) to \(\left(\exp\left(\sigma B(t) - \frac{\sigma^2}{2} t\right)\right),\)

\(0 \leq t \leq 1,\) as \(n \to \infty\) (with \(\sigma\) fixed).

Hint: Show first that it suffices to prove convergence in distribution for the logarithms of the processes. Now use a Taylor expansion, Donsker’s invariance principle and the first part.

(iii) Solve exercise 5.2 from the Brownian motion book.