This homework assignment needs to be submitted in class on March 5.

(1) Let $G = (V, E)$ be an infinite connected graph. Perform a percolation on $G$, retaining each edge with probability $0 \leq p \leq 1$ independently. Let $G_p$ be the subgraph of $G$ obtained after the percolation. Let $E$ be the event that there exists an infinite connected component in $G_p$.

(a) Prove that $E$ is an event, i.e., that it is measurable. Here, we identify a subgraph $G_p$ as an element of $\{0, 1\}^E$, and take our probability space to be $\{0, 1\}^E$ with the Borel sigma-algebra induced by the product topology. By saying that $G$ is a graph we implicitly mean that $V$ and $E$ are countable (or finite) sets.

(b) Prove that $P_p(E) \in \{0, 1\}$ for every $p$ (hint: the Kolmogorov 0-1 law).

(c) For a vertex $v \in V$, let $C_v$ be the connected component of $v$ in $G_p$. Prove that the following are equivalent:

$P_p(E) = 1$, $P_p(|C_v| = \infty) > 0$ for every $v \in V$, $P_p(|C_v| = \infty) > 0$ for some $v \in V$.

(2) Complete the proof of the theorem that a Galton-Watson tree has positive probability to be infinite if and only if $E(X) > 1$ (or $P(X = 1) = 1$), where $X$ is a random variable with the offspring distribution of the tree. As in class, assume (without loss of generality) that $m := E(X) < \infty$, $P(X = 1) < 1$ and $0 < P(X = 0) < 1$. Let $(Z_n)$ be the generation sizes of the Galton-Watson tree ($Z_n$ is the size of the $n$th generation), with $Z_0 := 1$. Use the following steps:

(a) Define the moment-generating function $f(s) := E(s^X)$. Prove that

(i) $f$ is continuous and non-decreasing on $[0, 1]$.

(ii) $f$ is strictly convex on $[0, 1]$ if $P(X \geq 2) > 0$.

(iii) $f(1) = 1$ and $0 < f(0) < 1$.

(iv) $f'(s)$ exists on $[0, 1]$ and satisfies $f'(s) = E(Xs^{X-1})$.

(b) Define $f_n(s) := E(s^{Z_n})$. Recall from class (no need to prove again) that $f_{n+1}(s) = f(f_n(s))$. Let $q_n := P(Z_n = 0) = f_n(0)$. Prove that $q_n \to q = P(\text{tree is finite})$ and $q$ satisfies $f(q) = q$.

(c) Use the previous parts to conclude that $q < 1$ if and only if $m > 1$, proving the theorem (hint: by the first part, $m = f'(1)$).