This homework assignment needs to be submitted in class on June 18.

(1) Consider bond percolation on the edges of $[-n, n]_d$. Recall that a crossing cluster of $[-n, n]_d$ is a connected set $A \subseteq [-n, n]_d$ satisfying that for every $1 \leq i \leq d$ there exist $x, y \in A$ with $x_i = -n$ and $y_i = n$. Recall that for $B \subseteq \mathbb{Z}^d$ we denote $\text{diam}_\infty(B) = \max\{\|x - y\|_\infty : x, y \in B\}$.

Let $T_{m,n}$ denote the event that after bond percolation on the edges in $[-n, n]_d$ there are (at least) two distinct connected components $A, B \subseteq [-n, n]_d$ such that $A$ is a crossing cluster and $\text{diam}_\infty(B) \geq m$. Prove that for every $d \geq 3$, $p > p_c(\mathbb{Z}^d)$ and integers $n, m \geq 1$, $m \leq 2n + 1$, there exist some $C, c > 0$, depending only on $d$ and $p$, such that

$$\Pr_p(T_{m,n}) \leq Cn^{2d}\exp(-cm).$$

Hint: Similar to a lemma from class.

(2) (continuing an exercise from last homework) Consider bond percolation on $\mathbb{Z}^d$. Prove that for every $d \geq 3$ and $p > p_c(\mathbb{Z}^d)$ there exists some constant $C = C(d, p)$ such that for every $x, y \in \mathbb{Z}^d$,

$$\mathbb{E}_p(d_G \text{open graph}(x, y) \cdot 1_{(x, y \text{ in infinite component})}) \leq C \cdot d_Z(x, y),$$

where $d_G$ is the graph distance in $G$ and by ‘open graph’ we mean the subgraph of $\mathbb{Z}^d$ of open edges.

Hint: Static renormalization.