PERCOLATION: HOMEWORK ASSIGNMENT 10

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This homework assignment needs to be submitted in class on June 18.

(1) Consider bond percolation on the edges of $[-n,n]^d$. Recall that a *crossing cluster* of $[-n,n]^d$ is a connected set $A \subseteq [-n,n]^d$ satisfying that for every $1 \leqslant i \leqslant d$ there exist $x, y \in A$ with $x_i = -n$ and $y_i = n$. Recall that for $B \subseteq \mathbb{Z}^d$ we denote $\mathrm{diam}_\infty(B) = \max\{\|x-y\|_\infty : x, y \in B\}$.
Let $T_{m,n}$ denote the event that after bond percolation on the edges in $[-n,n]^d$ there are (at least) two distinct connected components $A, B \subseteq [-n,n]^d$ such that A is a crossing cluster and $\mathrm{diam}_\infty(B) \geqslant m$. Prove that for every $d \geqslant 3$, $p > p_c(\mathbb{Z}^d)$ and integers $n, m \geqslant 1$, $m \leqslant 2n+1$, there exist some $C, c > 0$, depending only on d and p, such that

$$\mathbb{P}_p(T_{m,n}) \leqslant C n^{2d} \exp(-cm).$$

Hint: Similar to a lemma from class.

(2) (continuing an exercise from last homework) Consider bond percolation on $\mathbb{Z}^d$. Prove that for every $d \geqslant 3$ and $p > p_c(\mathbb{Z}^d)$ there exists some constant $C = C(d,p)$ such that for every $x, y \in \mathbb{Z}^d$,

$$\mathbb{E}_p(d_{\text{open graph}}(x,y) \cdot 1_{(x,\, y \text{ in infinite component})}) \leqslant C \cdot d_{\mathbb{Z}^d}(x,y),$$

where d_G is the graph distance in G and by 'open graph' we mean the subgraph of $\mathbb{Z}^d$ of open edges.
Hint: Static renormalization.

Date: June 13, 2013.