This homework assignment needs to be submitted in class on March 12.

In the first three exercises we use the notation of Galton-Watson trees: $X$ is the offspring distribution, $Z_n$ the number of descendants at level $n$ (with $Z_0 := 1$), $f(s) := \mathbb{E} s^X$ and $f_n(s) := \mathbb{E} s^{Z_n}$. We always assume that 

$$\mathbb{E} X < \infty \text{ and } \mathbb{P}(X = 1) < 1.$$  

(1) Assume that $\mathbb{E} X = 1$ and $\mathbb{E} X^2 < \infty$.

(a) We proved in class the Kesten-Ney-Spitzer theorem,

$$\lim_{n \to \infty} \frac{1}{n} \left( \frac{1}{1 - f_n(s)} - \frac{1}{1 - s} \right) = \frac{\text{Var}(X)}{2} \text{ uniformly for } 0 \leq s < 1.$$  

A tool in the proof was defining a function $\varepsilon$ by

$$f(s) = s + \left( \frac{\text{Var}(X)}{2} - \varepsilon(s) \right) (1 - s)^2.$$  

Prove that 

$$\varepsilon(s) = \sum_{k=3}^{\infty} \mathbb{P}(X = k) \sum_{j=2}^{k-1} \sum_{v=1}^{j-1} (1 - s^v) \quad 0 \leq s \leq 1.$$  

(b) Prove that

$$\lim_{n \to \infty} n^2 \mathbb{P}(Z_n > 0, Z_{n+1} = 0) = \frac{2}{\text{Var}(X)}.$$  

Informally, the chance that a critical Galton-Watson tree survives for exactly $n$ generations is about $\frac{2}{\text{Var}(X)n^2}$.

Hint: Relate this to the facts in the previous part.

(2) For a Galton-Watson tree (satisfying (1)) and integer $i \geq 1$, let $N_i$ be the number of $n$ such that $Z_n = i$. Prove that $\mathbb{E} N_i < \infty$ for all $i \geq 1$.

Remark: In sophisticated terminology, one might say that the Green’s function of the Markov chain $(Z_n)$ is finite.

*(3) Let $T_n$ denote the set of all (unrooted) trees on $n$ labeled vertices. In other words, $T_n$ is the set of spanning trees of the complete graph on $n$ labeled vertices. Let $S_n$ be a random tree with $n$ vertices obtained by conditioning a Galton-Watson tree with offspring distribution $X \sim \text{Poisson}(1)$ to have exactly $n$ vertices. Let $T_n$ be the tree on $n$ labeled vertices obtained by assigning the labels $\{1, \ldots, n\}$ to the vertices of $S_n$ in a uniformly random way and then forgetting the location of the root of $S_n$. Prove that $T_n$ is uniformly distributed on $T_n$.

(4) Let $(x_n)$, $n \geq 1$, be a sequence of real numbers satisfying $x_{m+n} \leq x_m + x_n$ for all $m, n \geq 1$. Prove that $\lim_{n \to \infty} \frac{x_n}{n}$ exists in $[\infty, \infty)$ and equals $\inf_n \frac{x_n}{n}$.

Remark: we used this fact, with $x_n = \log(\sigma(n))$, to show the existence of the connective constant $\lambda(d)$ of $\mathbb{Z}^d$.  

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Date: March 6, 2013.