PERCOLATION: HOMEWORK ASSIGNMENT 2

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This homework assignment needs to be submitted in class on March 12.

In the first three exercises we use the notation of Galton-Watson trees: X is the offspring distribution, Z_n the number of descendants at level n (with $Z_0 := 1$), $f(s) := \mathbb{E}\, s^X$ and $f_n(s) := \mathbb{E}\, s^{Z_n}$. We always assume that

$$\mathbb{E}\, X < \infty \text{ and } \mathbb{P}(X = 1) < 1. \tag{1}$$

(1) Assume that $\mathbb{E}\, X = 1$ and $\mathbb{E}\, X^2 < \infty$.

(a) We proved in class the Kesten-Ney-Spitzer theorem,

$$\lim_{n\to\infty} \frac{1}{n}\left(\frac{1}{1-f_n(s)} - \frac{1}{1-s}\right) = \frac{\mathrm{Var}(X)}{2} \quad \text{uniformly for } 0 \leqslant s < 1.$$

A tool in the proof was defining a function ε by

$$f(s) = s + \left(\frac{\mathrm{Var}(X)}{2} - \varepsilon(s)\right)(1-s)^2.$$

Prove that

$$\varepsilon(s) = \sum_{k=3}^{\infty} \mathbb{P}(X = k) \sum_{j=2}^{k-1}\sum_{v=1}^{j-1}(1 - s^v) \qquad 0 \leqslant s \leqslant 1.$$

(b) Prove that

$$\lim_{n\to\infty} n^2 \mathbb{P}(Z_n > 0, Z_{n+1} = 0) = \frac{2}{\mathrm{Var}(X)}.$$

Informally, the chance that a critical Galton-Watson tree survives for exactly n generations is about $\frac{2}{\mathrm{Var}(X)n^2}$.

Hint: Relate this to the facts in the previous part.

(2) For a Galton-Watson tree (satisfying (1)) and integer $i \geqslant 1$, let N_i be the number of n such that $Z_n = i$. Prove that $\mathbb{E}\, N_i < \infty$ for all $i \geqslant 1$.

Remark: In sophisticated terminology, one might say that the *Green's function* of the Markov chain (Z_n) is finite.

*(3) Let $\mathcal{T}_n$ denote the set of all (unrooted) trees on n labeled vertices. In other words, $\mathcal{T}_n$ is the set of spanning trees of the complete graph on n labeled vertices. Let S_n be a random tree with n vertices obtained by conditioning a Galton-Watson tree with offspring distribution $X \sim \text{Poisson}(1)$ to have exactly n vertices. Let T_n be the tree on n labeled vertices obtained by assigning the labels $\{1, \ldots, n\}$ to the vertices of S_n in a uniformly random way and then forgetting the location of the root of S_n. Prove that T_n is uniformly distributed on $\mathcal{T}_n$.

(4) Let (x_n), $n \geqslant 1$, be a sequence of real numbers satisfying $x_{m+n} \leqslant x_m + x_n$ for all $m, n \geqslant 1$. Prove that $\lim_{n\to\infty} \frac{x_n}{n}$ exists in $[-\infty, \infty)$ and equals $\inf_n \frac{x_n}{n}$.

Remark: we used this fact, with $x_n = \log(\sigma(n))$, to show the existence of the connective constant $\lambda(d)$ of $\mathbb{Z}^d$.

Date: March 6, 2013.