This homework assignment needs to be submitted in class on March 19.

(1) Show that the critical percolation probability, \( p_c(2) \), for the two-dimensional lattice \( \mathbb{Z}^2 \) satisfies
\[
p_c(2) \leq 1 - \frac{1}{\lambda(2)},
\]
where \( \lambda(2) \) is the connective constant of \( \mathbb{Z}^2 \).

(2) Show that for any \( d \geq 2 \), \( p_c(d+1) < p_c(d) \) (that is, show the strict inequality).

(3) (a) Prove Chebyshev’s other inequality. For any real-valued random variable \( X \) and any two bounded non-decreasing functions \( f, g : \mathbb{R} \to \mathbb{R} \),
\[
\mathbb{E}(f(X)g(X)) \geq \mathbb{E}f(X)\mathbb{E}g(X).
\]
(b) In the next two exercises we finish the proof of Harris’ inequality. Let \( \Omega := \{0, 1\}^m \) be endowed with the probability measure making the coordinates independent, with each coordinate having probability \( p \) to be 1. Put a partial order \( \succeq \) on \( \Omega \) by \( \omega^1 \succeq \omega^2 \) iff \( \omega^1_i \leq \omega^2_i \) for all \( i \). A function \( X : \Omega \to \mathbb{R} \) is called non-decreasing if \( X(\omega^1) \leq X(\omega^2) \) whenever \( \omega^1 \preceq \omega^2 \). Let \( X, Y : \Omega \to \mathbb{R} \) be bounded and non-decreasing. Prove that if \( m < \infty \) then
\[
\mathbb{E}(XY) \geq \mathbb{E}X \mathbb{E}Y.
\]
Hint: For \( m = 1 \), use the first part of the question. For \( m > 1 \), use induction on \( m \).

(c) Prove that (1) continues to hold also when \( m = \infty \) (that is, when \( \Omega \) is the set of all infinite binary sequences). Hint: Recall that if \( (\mathcal{F}_n) \) is a filtration then the sequence \( (M_n) \) defined by \( M_n := \mathbb{E}(X | \mathcal{F}_n) \) is a martingale. Use the \( L^2 \)-martingale convergence theorem.