PERCOLATION: HOMEWORK ASSIGNMENT 3

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This homework assignment needs to be submitted in class on March 19.

(1) Show that the critical percolation probability, $p_c(2)$, for the two-dimensional lattice $\mathbb{Z}^2$ satisfies $p_c(2) \leqslant 1 - \frac{1}{\lambda(2)}$, where $\lambda(2)$ is the connective constant of $\mathbb{Z}^2$.

(2) Show that for any $d \geqslant 2$, $p_c(d+1) < p_c(d)$ (that is, show the *strict* inequality).

(3) (a) Prove Chebyshev's other inequality. For any real-valued random variable X and any two bounded non-decreasing functions $f, g : \mathbb{R} \to \mathbb{R}$,

$$\mathbb{E}(f(X)g(X)) \geqslant \mathbb{E}\, f(X)\, \mathbb{E}\, g(X).$$

(b) In the next two exercises we finish the proof of Harris' inequality. Let $\Omega := \{0,1\}^m$ be endowed with the probability measure making the coordinates independent, with each coordinate having probability p to be 1. Put a partial order $\succeq$ on Ω by $\omega^1 \preceq \omega^2$ iff $\omega_i^1 \leqslant \omega_i^2$ for all i. A function $X : \Omega \to \mathbb{R}$ is called *non-decreasing* if $X(\omega^1) \leqslant X(\omega^2)$ whenever $\omega^1 \preceq \omega^2$. Let $X, Y : \Omega \to \mathbb{R}$ be bounded and non-decreasing. Prove that if $m < \infty$ then

$$\mathbb{E}(XY) \geqslant \mathbb{E}\, X\, \mathbb{E}\, Y. \tag{1}$$

Hint: For $m = 1$, use the first part of the question. For $m > 1$, use induction on m.

(c) Prove that (1) continues to hold also when $m = \infty$ (that is, when Ω is the set of all infinite binary sequences).
Hint: Recall that if $(\mathcal{F}_n)$ is a filtration then the sequence (M_n) defined by $M_n := \mathbb{E}(X \mid \mathcal{F}_n)$ is a martingale. Use the L^2-martingale convergence theorem.

Date: March 13, 2013.