PERCOLATION: HOMEWORK ASSIGNMENT 4

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This homework assignment needs to be submitted in class on April 9.

(1) Prove that if the events $A, B \subseteq \{0,1\}^E$ satisfy that A is increasing and B is decreasing then $A \Box B = A \cap B$.

(2) Let A be the event that, after bond percolation on $\mathbb{Z}^d$, the origin is connected to distance n (in the graph metric). Prove that there exists a constant $c = c(d) > 0$ such that
$$\mathbb{P}_{p_c(d)}(A) \geqslant cn^{-(d-1)/2}. \tag{1}$$
Hint: The BK inequality.
Remark: We may interpret (1) as saying that the "one-arm exponent" for critical percolation on $\mathbb{Z}^d$ is at most $\frac{d-1}{2}$.

(3) (a) Prove that percolation on $\mathbb{Z}^d$ is *ergodic*. Precisely, let τ be a non-zero translate of $\mathbb{Z}^d$. Suppose that A is an event which is invariant under τ, that is a configuration ω is in A if and only if the configuration translated by τ is in A. Prove that $\mathbb{P}_p(A) \in \{0,1\}$ for all $p \in [0,1]$.
Hint: The space of configurations of percolation is $\Omega := \{0,1\}^{E(\mathbb{Z}^d)}$ where $E(\mathbb{Z}^d)$ is the edge set of $\mathbb{Z}^d$. Recall that an event is a measurable subset $A \subseteq \Omega$. What does measurability imply?
(b) Give an example of an event to which the 0-1 law of the previous part may be applied, but to which Kolmogorov's 0-1 law does not apply (that is, a translation-invariant event which is not a tail event). Also give an example of the other direction (a tail event which is not translation-invariant).

(4) Prove Russo's formula. Let $G = (V, E)$ be a finite graph. Consider bond percolation on G and let X be a bounded random variable for this percolation process. Prove that
$$\frac{d}{dp}\left(\mathbb{E}_p X\right) = \sum_{e \in E} \mathbb{E}_p \left[X(\omega^e) - X(\omega_e)\right] \quad \text{for } p \in (0,1),$$
where by ω we denote the random subgraph of G sampled by the percolation process, and by ω^e (ω_e) we denote the subgraph obtained from ω by adding the edge e (removing the edge e, respectively). In other words, if we identify ω as an element of $\{0,1\}^E$ then
$$\omega^e(f) := \begin{cases} \omega(f) & f \neq e \\ 1 & f = e \end{cases}, \quad \omega_e(f) := \begin{cases} \omega(f) & f \neq e \\ 0 & f = e \end{cases}.$$

(5) (a) Let G be a connected graph and x, y, z three vertices of it. Prove that there is a vertex u having edge-disjoint paths from it to x, y and z (where u may belong to $\{x, y, z\}$).
(b) Consider percolation on $\mathbb{Z}^d$ with $p < p_c(d)$. Let $C(0)$ be the connected component of the origin after percolation. Write $|C(0)|$ for the number of vertices in $C(0)$. Prove that
$$\mathbb{E}\left[|C(0)|^2\right] \leqslant \left(\mathbb{E}\left[|C(0)|\right]\right)^3.$$

Date: March 22, 2013.