PERCOLATION: HOMEWORK ASSIGNMENT 5

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This homework assignment needs to be submitted in class on April 23.

In the following questions, by C we always denote the connected component of the origin in the corresponding percolation.

(1) The graph $\vec{Z}^2$ is the directed graph with vertex set $\{(x, y) : x, y \in \{0, 1, 2, \ldots\}\}$ and directed edge set $\{(x, y), (x + 1, y)\} \cup \{(x, y), (x, y + 1)\}$ (i.e., edges of $\mathbb{Z}^2$ are oriented right and up). Define bond percolation with parameter p on $\vec{Z}^2$ by keeping each oriented edge with probability p independently and otherwise deleting it. Let

$$p_c(\vec{Z}^2) = \inf\big(p : \mathbb{P}_p(\text{there exists an infinite oriented path from } (0, 0)) > 0\big).$$

Prove that $p_c(\mathbb{Z}^2) < p_c(\vec{Z}^2)$ (prove the *strict* inequality).
Remark: The same result is true in higher dimensions as well. We give it in two dimensions to simplify the notation.

(2) Consider bond percolation on $\mathbb{Z}^2$ and assume that $\mathbb{E}\,|C| < \infty$ for all $p < p_c(\mathbb{Z}^2)$. Prove that $p_c(\mathbb{Z}^2) \leqslant \frac{1}{2}$.

(3) Recall the quantities $\theta(p, \gamma)$ and $\chi(p, \gamma)$ from class. Precisely, on top of bond percolation on $\mathbb{Z}^d$ with parameter p, color each vertex *green* with probability γ independently. Let G be the set of green vertices and define

$$\theta(p, \gamma) = \mathbb{P}(C \cap G \neq \emptyset), \quad \chi(p, \gamma) = \mathbb{E}[|C| \cdot 1_{(C\cap G=\emptyset)}]. \tag{1}$$

Define corresponding *finite volume* quantities as follows. Let $B(N)$ be the box $[-N, N]^d$ with periodic boundary conditions. I.e., corresponding vertices on opposite faces of the box are identified to one vertex (and parallel edges generated by this identification are replaced by a single edge). Consider bond percolation with parameter p on $B(N)$ and let each vertex be green with probability γ independently. Define the functions $\theta_N(p, \gamma)$ and $\chi_N(p, \gamma)$ with the same definitions (1), where the probability is now over the finite volume model $B(N)$ (and G stands for the green vertices in $B(N)$).

(a) Prove that $\theta(p, \gamma)$ is continuously differentiable in p, for each $\gamma > 0$.

(b) Prove that for each $0 < p, \gamma < 1$,

$$\theta_N(p, \gamma) \to \theta(p, \gamma), \tag{2}$$

$$\frac{\partial \theta_N}{\partial p} \to \frac{\partial \theta}{\partial p}, \tag{3}$$

$$\frac{\partial \theta_N}{\partial \gamma} \to \frac{\partial \theta}{\partial \gamma}, \tag{4}$$

as $N \to \infty$.

Date: April 14, 2013.