PERCOLATION: HOMEWORK ASSIGNMENT 6

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This homework assignment needs to be submitted in class on April 30.

(1) Consider percolation on $\mathbb{Z}^d$ with $0 < p < p_c(\mathbb{Z}^d)$. Let $C(x)$ be the connected component of the vertex x, and let $D(x)$ be the diameter of this component (that is, $D(x) = \max_{y,z\in C(x)} d(y,z)$, where $d(y,z)$ is the graph distance between y and z). Prove that there exist constants $0 < C, c < \infty$ (depending only on p and d) such that for all $n \geqslant 2$,

$$c \log n \leqslant \mathbb{E}_p \left[\max_{x\in[-n,n]^d} D(x) \right] \leqslant C \log n.$$

(2) Consider percolation on $\mathbb{Z}^d$ and let $e_1 := (1, 0, \ldots, 0) \in \mathbb{Z}^d$. Define $L(p)$, the *correlation length* of the percolation with parameter p, by

$$\frac{1}{L(p)} := -\lim_{n\to\infty} \frac{1}{n} \log(\mathbb{P}_p(0 \leftrightarrow ne_1)).$$

That is, $\mathbb{P}_p(0 \leftrightarrow ne_1)$ is approximately $\exp(-n/L(p))$ on a logarithmic scale. One observes the exponential decay only on scales larger than $L(p)$, which is the origin of the name correlation length.

(a) Prove that $L(p)$ is well-defined as an element of $(0,\infty]$ for all $p \in (0, 1]$, and that $\mathbb{P}_p(0 \leftrightarrow ne_1) \leqslant \exp(-n/L(p))$ for all n and p.
Hint: Recall the subadditive limit theorem from homework 2, exercise 4.

(b) Prove that $L(p)$ is non-decreasing on $(0, 1]$, that $L(p) = \infty$ for $p \geqslant p_c$ and that $L(p) < \infty$ for $0 < p < p_c$.

(c) Prove that $L(p) \to 0$ as $p \downarrow 0$.

(d) For the next three parts, fix $p \in (0, 1]$. Denote $\Lambda_n := [-n, n]^d$ and $\partial\Lambda_n := \Lambda_n \setminus \Lambda_{n-1}$. Prove that

$$\mathbb{P}_p(0 \leftrightarrow x)^2 \leqslant \mathbb{P}_p(0 \leftrightarrow 2ne_1) \quad \text{for every } x \in \partial\Lambda_n.$$

Deduce that for all n,

$$\mathbb{P}_p(0 \leftrightarrow \partial\Lambda_n) \leqslant |\partial\Lambda_n| \exp(-n/L(p)). \tag{1}$$

(e) Prove that

$$\frac{1}{L(p)} = -\lim_{n\to\infty} \frac{1}{n} \log(\mathbb{P}_p(0 \leftrightarrow \partial\Lambda_n)).$$

(f) Prove that for all $n, m \geqslant 1$,

$$\mathbb{P}_p(0 \leftrightarrow \partial\Lambda_{n+m}) \leqslant |\partial\Lambda_n| \mathbb{P}_p(0 \leftrightarrow \partial\Lambda_n) \mathbb{P}_p(0 \leftrightarrow \partial\Lambda_m).$$

Deduce that for some constants $0 < c, \alpha < \infty$ (depending only on d) and all $n \geqslant 1$,

$$\mathbb{P}_p(0 \leftrightarrow \partial\Lambda_n) \geqslant cn^{-\alpha} \exp(-n/L(p)). \tag{2}$$

Hint: The subadditive limit theorem may need to be used cleverly.

(g) Prove that $1/L(p)$ is continuous on $(0, 1]$.
Hint: It may help to use (1) and (2).

Date: April 25, 2013.