PERCOLATION: HOMEWORK ASSIGNMENT 9

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This homework assignment needs to be submitted in class on June 11.

(1) Consider bond percolation on $\mathbb{Z}^d$ and assume $p > p_c$. Let A be an infinite subset of vertices of $\mathbb{Z}^d$. Prove that

$$\mathbb{P}_p(\text{the intersection of } A \text{ with the infinite component is infinite}) = 1.$$

Hint: The Kolmogorov 0-1 law.

(2) Let G be a graph with maximal degree Δ and v a vertex of G. Show that there exists some C, depending only on the maximal degree Δ, such that for every $L \geqslant 1$,

$$|\{A : A \text{ is a connected set in } G,\ |A| = L \text{ and } v \in A\}| \leqslant C^L.$$

Remark: $C = \frac{\Delta^\Delta}{(\Delta-1)^{\Delta-1}} \leqslant e\Delta$ suffices (for $\Delta \geqslant 2$).

(3) Consider bond percolation on $\mathbb{Z}^d$. Prove that for every $d \geqslant 2$ there exists some $p(d) < 1$, with $p(2) = p_c(\mathbb{Z}^2)$, such that for every $p > p(d)$ and every $x, y \in \mathbb{Z}^d$, there exists a constant $C = C(d, p)$ satisfying

$$\mathbb{E}_p(d_{\text{open graph}}(x,y) \cdot 1_{(x,\, y \text{ in infinite component})}) \leqslant C \cdot d_{\mathbb{Z}^d}(x,y),$$

where d_G is the graph distance in G and by 'open graph' we mean the subgraph of $\mathbb{Z}^d$ of open edges.

(4) Consider a site percolation on $\mathbb{Z}^d$, written as $(Y_x)_{x\in\mathbb{Z}^d}$ with $Y_x \in \{0,1\}$, which has possibly different probabilities per site and which is possibly not independent. For $k \geqslant 1$, we say that (Y_x) is *k-dependent* if for every two sets $A, B \subseteq \mathbb{Z}^d$ such that

$$\min_{\substack{x\in A \\ y\in B}} d_{\mathbb{Z}^d}(x,y) > k$$

we have that $(Y_x)_{x\in A}$ and $(Y_x)_{x\in B}$ are independent. Prove that there exist some $p, C, c > 0$, depending only on k and d, such that for every k-dependent site percolation satisfying $\mathbb{P}(Y_x = 1) < p$ for every $x \in \mathbb{Z}^d$ we have

$$\mathbb{P}(\text{the origin is connected to } \partial[-n,n]^d \text{ by a path with } Y_x = 1) \leqslant C\exp(-cn).$$

Date: July 10, 2013.