Exercise 1

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The exercise needs to be handed in by May 2’nd in class.

In all of the following, unless otherwise indicated, we assume \((S_n)_{n=0}^{\infty}\) is a random walk in \(\mathbb{R}^d\) with \(S_n := \sum_{i=0}^{n} X_i\) (the walk may or may not be on \(Z^d\)).

1. Prove the second Wald identity (directly, without use of martingale theory)

2. Biased 1D RWs: The following exercise discusses the 1D random walk with
\[ P(X_1 = 1) = 1 - P(X_1 = -1) = p \text{ for } p \neq \frac{1}{2}. \]
Let \(T := \min \{ n \mid S_n \in \{0, M\} \}\) be the hitting time of 0 or \(M\).

(a) Calculate \(P_x(S_T = 0)\) for \(0 < x < M\) and use it to find \(P_x(\text{no return to 0})\) for \(x > 0\). Compare the result with the Kesten-Spitzer-Whitman theorem.

(b) Calculate \(E_x T\) for \(0 < x < M\).

3. Do Exercises 5 and 12 from the notes of Steve Lalley:
http://galton.uchicago.edu/~lalley/Courses/312/RW.pdf.

4. Prove that the minimum of two stopping times is also a stopping time. That is, if \(T\) and \(S\) are stopping times with respect to a filtration \(\mathcal{F}_n\) \(n \geq 0\) then \(\min(T, S)\) is also such a stopping time.

5. Suppose \(X_1\) is uniform on the interval (0,1) and let \(T := \min \{ n \mid S_n > 1 \}\).
Show that \(P(T > n) = 1/n!\), so \(E T = e\) and \(E S_T = e/2\).

6. Prove that each of the following conditions is sufficient to deduce that \(\lim \inf_{n \to \infty} S_n = -\infty\) and \(\lim \sup_{n \to \infty} S_n = \infty\).

(a) \(X_1\) has a symmetric distribution and \(P(X_1 = 0) < 1\).
(b) \(E(X_1) = 0\) and \(0 < E(X_1^2) < \infty\) (prove this without using the Chung-Fuchs theorem).

7. Let \(P := \{ x \mid \exists n \ P(S_n = x) > 0 \}\) be the set of possible values for \(S_n\). Prove that if \(S_n\) is point-recurrent then \(P(\forall x \in P, S_n = x \text{ infinitely often}) = 1\) (note that the walk is not necessarily on \(Z^d\)).
8. Prove that if $P(|S_n| < 1 \text{ infinitely often}) = 1$ then for every $\varepsilon > 0$, $P(|S_n| < \varepsilon \text{ infinitely often}) = 1$. This justifies defining neighbourhood-recurrence using one particular value of $\varepsilon$.

9. Let $P := \{x \mid \forall \varepsilon > 0 \exists n \ P(|S_n - x| < \varepsilon) > 0\}$ be the set of neighbourhood possible values for $S_n$. Prove that if $S_n$ is neighbourhood-recurrent then $P$ is a group under addition in $\mathbb{R}^d$.

10. Give an example of a point-recurrent 1D RW whose set of possible values $P := \{x \mid \exists n \ P(S_n = x) > 0\}$ is dense in $\mathbb{R}$.

11. Give an example of a 1D RW which is neighbourhood-recurrent but not point-recurrent.

12. Prove that if $S_n$ is recurrent on $\mathbb{Z}^d$ then so is its symmetrized version $\tilde{S}_n$, the walk whose increments are distributed as $X_1 - X'_1$ where $X_1, X'_1$ are independent copies of $X_1$. 
   Hint: Use the Fourier-analytic criterion for recurrence and compare $S_{2n}$ with $\tilde{S}_n$.  
   Remark: The same result is true also for RW in $\mathbb{R}^d$ and neighbourhood-recurrence.

13. Prove that if $S_n$ is a RW on $\mathbb{Z}$ satisfying the weak law of large numbers, i.e., for every $\varepsilon > 0$, $P(S_n > \varepsilon) \to 0$ as $n \to \infty$, then $S_n$ is recurrent.  
   Hint: Similar to the 2D recurrence theorem.

14. Let $p$ be the transition kernel of an irreducible Markov chain on a countable state space $S$. That is, for every $x \in S$, $\sum_{y \in S} p(x, y) = 1$ ($p(x, y)$ is the probability to go from $x$ to $y$ in one step) and for every $x, y \in S$, $\exists n$ such that the probability to go from $x$ to $y$ in $n$ steps is non-zero. A function $h : S \to \mathbb{R}$ is called superharmonic with respect to the Markov chain if $h(x) \geq \sum_{y \in S} p(x, y) h(y)$. Show that the Markov chain is recurrent (that is, every $x \in S$ is recurrent) if and only if all non-negative superharmonic functions with respect to the Markov chain are constants.

15. (* Optional exercise) Construct arbitrary heavy tail recurrent 1D distributions. More precisely, show that for any $\varepsilon(x) \downarrow 0$ as $x \to \infty$ there exists a recurrent 1D RW on $\mathbb{Z}$ such that $P(|X_1| \geq x) \geq \varepsilon(x)$ for all large $x$. 

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