Random Walks and Brownian Motion Exercise 3

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The exercise needs to be handed in by July 12'th by email or by putting it in the Schreiber mailbox for Ron Peled.

In all of the following, unless otherwise indicated, $B(t)_{t=0}^{\infty}$ is a one-dimensional Brownian motion and we write $\mathbb{P}^x$ or $\mathbb{E}^x$ to indicate that $B(0) = x$ a.s. and $\mathbb{P}$ or $\mathbb{E}$ to indicate $B(0) = 0$ a.s..

1. Let $\{X(t)\}_{t=0}^{\infty}$ be a continuous martingale and T be a stopping time for which there exists an integrable random variable X satisfying $|X(t \wedge T)| \leq X$ a.s., for all $t > 0$.

   (a) Prove that $X(t \wedge T)_{t \geq 0}$ converges a.s. and in L_1 as $t \to \infty$.
   Hint: Use the corresponding theorem for discrete-time martingales.

   (b) Prove the optional stopping theorem: $\mathbb{E}(X(T)) = \mathbb{E}(X(0))$. Here, on the event $T = \infty$, $X(T)$ refers to the limit RV from part (a).

2. Prove that for each $\sigma \in \mathbb{R}$, the process $\left\{\exp(\sigma B(t) - \frac{\sigma^2 t}{2})\right\}_{t \geq 0}$ is a continuous martingale. Deduce that for $a, b > 0$, $\mathbb{P}(B(t) = a + bt \text{ for some t>0}) = \exp(-2ab)$.

3. For $t \geq 0$, let $M(t) := \max_{s \in [0,t]} B(s)$.

   (a) Show that for each $t > 0$, the joint density of $(B(t), M(t))$ exists (under $\mathbb{P}$) and equals $\frac{2(2m-b)}{\sqrt{2\pi t^3}} e^{-(2m-b)^2/2t}$ at the point (m, b) (for $m \geq 0$ and $-\infty < b \leq m$).
   Hint: Reflection principle.

   (b) Use the Donsker invariance principle to prove that for any RW $(S_n)_{n \geq 0}$ with $\mathbb{E}S_1 = 0$ and $\mathbb{E}S_1^2 = 1$, we have $(\frac{S_n}{\sqrt{n}}, \max_{0 \leq k \leq n} \frac{S_k}{\sqrt{n}}) \to (B(1), M(1))$ in distribution, as $n \to \infty$.
   Hint: Explain why it is sufficient to show convergence in distribution of any linear combination of the two RVs.

4. Show that for a tail event $A \in \mathcal{T}$ the probability $\mathbb{P}^x(A)$ is independent of x whereas for a germ event $A \in \mathcal{F}^+(0)$ the probability $\mathbb{P}^x(A)$ may depend on x (reminder: $\mathcal{T} = \cap_{t \geq 0} \sigma(B(s) \mid s \geq t)$).

5. Recall that a perfect set is a closed set with no isolated points. Prove that any non-empty perfect set in $\mathbb{R}$ is uncountable.
   Remark: Since we have shown that $\mathbb{P}$-a.s., the zero set of BM $\{t \geq 0 \mid B(t) = 0\}$ is perfect, the exercise implies that it is uncountable $\mathbb{P}$-a.s..

6. Complete the proof of the Azéma-Yor embedding theorem. Let X be a RV with $\mathbb{E}X = 0$ and $\mathbb{E}X^2 < \infty$. Let $\Psi(x) := \mathbb{E}(X \mid X \geq x)$ if $\mathbb{P}(X \geq x) > 0$ and $\Psi(x) = 0$ otherwise. For $t \geq 0$, let $M(t) := \max_{s\in[0,t]} B(s)$ and let $T := \inf\{t \geq 0 \mid M(t) \geq \Psi(B(t))\}$ be the Azéma-Yor stopping time.

   (a) Show that there are finitely supported RVs $\{X_n\}_{n\geq 0}$ with $\mathbb{E}X_n = 0$ such that X_n converges to X in distribution and $T_n := \inf\{t \geq 0 \mid M(t) \geq \Psi_n(B(t))\}$ converge $\mathbb{P}$-a.s. to T. Here, $\Psi_n(x) := \mathbb{E}(X_n \mid X_n \geq x)$ if $\mathbb{P}(X_n \geq x) > 0$ and $\Psi_n(x) = 0$ otherwise.

   (b) Deduce that T solves the Skorohod embedding problem for X. That is, $B(T)$ has the same law as X and $\mathbb{E}T = \mathbb{E}X^2$.

7. In this exercise we use the law of the iterated logarithm for simple random walk to derive similar laws for the Brownian motion and other random walks. Let $\Psi(t) := \sqrt{2t \log\log t}$.

   (a) Let $f : [0,\infty) \to \mathbb{R}$ be a continuous function and $\{t_n\}_{n\geq 0} \subseteq \mathbb{R}$ satisfy $\frac{t_n}{n} \to 1$ as $n \to \infty$. Show that $\limsup_{t\to\infty} \frac{f(t)}{\Psi(t)} = \limsup_{n\to\infty} \frac{f(t_n)}{\Psi(t_n)}$ if the following condition holds:

$$\forall 2 > q > 1 \exists k_0 \text{ s.t } \max_{t\in[q^{k-1},q^k]} |f(t) - f(q^{k-1})| \leq 50\Psi(q^k)\sqrt{1 - \frac{1}{q}} \text{ for all } k > k_0. \tag{1}$$

   (b) Show that condition (1) holds $\mathbb{P}$-a.s. when f is a 1D BM.
   Remark: You may use that if $Z \sim N(0,1)$ then $\mathbb{P}(Z > x) \leq e^{-x^2/2}$ for $x > 1$.

   (c) Use the law of the iterated logarithm for simple random walk and Skorohod embedding to deduce that $\limsup_{t\to\infty} \frac{B(t)}{\Psi(t)} = 1$ $\mathbb{P}$-a.s..

   (d) Conclude that for any RW $(S_n)_{n\geq 0}$ with $\mathbb{E}S_1 = 0$ and $\mathbb{E}S_1^2 = 1$, we have $\limsup_{n\to\infty} \frac{S_n}{\Psi(n)} = 1$ a.s.. This is the Hartman-Wintner law of the iterated logarithm.

8. Let $\{H(s,\omega) \mid s \geq 0,\ \omega \in \Omega\}$ be a progressively measurable stochastic process and T a stopping time such that $\mathbb{E}\int_0^T H(s)^2 ds < \infty$. Prove that $\mathbb{E}\int_0^T H(s)dB(s) = 0$ and $\mathbb{E}\left(\int_0^T H(s)dB(s)\right)^2 = \mathbb{E}\int_0^T H(s)^2 ds$.

9. Let $f_1(x) := |x|$, $f_2(x) := \log|x|$ and $f_d(x) := |x|^{2-d}$ for $d \geq 3$ (where $|x|$ is the Euclidean norm of x). f_d is harmonic on $\mathbb{R}^d \setminus \{0\}$ (this is not part of the exercise).

   (a) Fix $0 < r < R < \infty$ and $x \in \mathbb{R}^d$ with $r < |x| < R$. Let $T := \inf\{t \mid |B(t)| \notin (r,R)\}$. Prove using Itô's formula that $\{f_d(B(t \wedge T))\}_{t\geq 0}$ is a martingale under $\mathbb{P}^x$ for a d-dimensional BM B.
   Hint: Apply Itô's formula to $(g \cdot f_d)(B(t))$ for a smooth function $g : \mathbb{R}^d \to \mathbb{R}$ satisfying $g(x) = 1$ whenever $r < |x| < R$ and $g(x) = 0$ outside some compact set $K \subseteq \mathbb{R}^d \setminus \{0\}$.

   (b) Use part (a) to find $\mathbb{P}^x(|B(T)| = r)$ for all $d \geq 1$.

   (c) Deduce that $\mathbb{P}(\lim_{t\to\infty} |B(t)| = \infty) = 1$ if and only if $d \geq 3$.