Random Walks and Brownian Motion Tel Aviv University Spring 2011	Instructor: Ron Peled

Lecture 11

Lecture date: May 16, 2011 *Scribe: Jun Chen*

In this lecture, we will continue last lecture's topic: Markov property of Brownian Motion (writing 'BM' for short in this lecture). First we will talk about the application of Markov property to local maxima of 1D ('D' represents 'dimensional') BM. Then, starting with the definition of stopping time in continuous setting, we will move on to the Strong Markov property of BM. In the rest part of this lecture, we will show several applications of Strong Markov property of BM.

Tags for today's lecture: Markov property, stopping time, Strong Markov property, reflection principle, area of planar BM, Lévy's theorem, zero set of 1D BM

1 Markov Property of BM and its application

Denote

$$\mathcal{F}^0(t) := \sigma(B(s)|0 \leq s \leq t),$$
$$\mathcal{F}^+(t) := \bigcap_{\epsilon>0} F^0(t+\epsilon).$$

In last lecture, we have showed the Markov property of BM, including the following stronger version (allowing an additional infinitesimal glance into the future):

Theorem 1 *For a $d-$dimensional BM, and a fixed $s \geq 0$, $\{B(t+s) - B(s)|t \geq 0\}$ is a standard $d-$dimensional BM, independent of $\mathcal{F}^+(s)$.*

Application of Markov property (local maxima of 1D BM)

Definition 2 *t is a (strict) local maximum of f if $\exists \epsilon > 0$, s.t. $f(t) \geq (>) f(s)$ whenever $s \in [t-\epsilon, t+\epsilon]$.*

Last time we also showed:

Theorem 3

$$\mathbf{P}[\inf(t > 0|B(t) > 0) = 0] = 1 \qquad (\star)$$
$$\mathbf{P}[\inf(t > 0|B(t) < 0) = 0] = 1$$

Proposition 4 *For a 1D BM, a.s.*

1. *All local maxima are strict.*
2. *The set of times where local maxima are attained is dense and countable.*
3. *The global maximum, on, say* $[0, 1]$ *is attained only once.*

Proof First show that for any $0 \leq a < b < c < d$, the maxima on $I_1 = [a, b]$ and $I_2 = [c, d]$ are distinct a.s.

Denote

$$\begin{aligned} \hat{m}_1 &= \max_{t \in I_1} B(t) - B(b) \\ \hat{m}_2 &= \max_{t \in I_2} B(t) - B(c) \\ \text{and} \quad X &= B(c) - B(b). \end{aligned}$$

By the Markov property at b and c, all three RV's ('RV' means 'random variable' in this lecture) are jointly independent. Now note

$$\{\max_{t \in I_1} B(t) = \max_{t \in I_2} B(t)\} = \{X = \hat{m}_1 - \hat{m}_2\}.$$

This has probability 0 since X has an atomless distribution conditioning on $\hat{m}_1$ and $\hat{m}_2$.

1 follows since the maxima on all pairs of intervals with rational endpoints (rational interval) are different.

3 also follows.

To see 2, note that for any $a < b$, we have

$$\mathbf{P}[\max_{t \in [a,b]} B(t) \in \{B(a), B(b)\}] = 0.$$

This follows by $\star$ and the Markov property at a (and the same to the reversed BM). It follows that each rational interval contains a local maximum, showing denseness.

To show there is only countable many times of maxima, note each ne is the maximum of some rational interval, since local maxima are strict. $\square$

2 Stopping times and the strong Markov property

Definition 5 *A RV T taking values in* $[0, \infty]$ *is a stopping time w.r.t a filtration* $\{\mathcal{F}(t), t \geq 0\}$ *if*

$$\{T \leq t\} \in \mathcal{F}(t), \forall\ t.$$

Example 6 *1. Every deterministic time $t \geq 0$ is a stopping time.*

2. *Min $(min\{S,T\})$ of two stopping timeS is a stopping time.*

3. *Increasing limit: If T_n are stopping times and $T_n \uparrow T$ a.s. and $\mathcal{F}(t)$ contains all negligible sets (completeness) (Note it includes the event $\{T_n \nrightarrow T\}$), then T is a stopping time since*

$$\{T \leq t\} = \bigcap_{n=1}^{\infty} \{T_n \leq t\} \subset \mathcal{F}(t),$$

   where the last step is obtained by the fact that T_n is a stopping time.

4. *Upper approximation: For a stopping time T, one can define*

$$T_n := (m+1)2^{-n}$$

   if

$$m2^{-n} \leq T < (m+1)2^{-n}$$

   and ∞ if $T = \infty$, then T_n is a stopping time.

For BM, which filtration to use, $\mathcal{F}^0(t)$ or $\mathcal{F}^+(t)$? We will take $\mathcal{F}^+(t)$ and even complete it.

Points to note:

1. $\mathcal{F}^0(t) \subset \mathcal{F}^+(t)$, so we only gain more stopping times.
2. First hitting time of a closed set is a stopping time w.r.t $\mathcal{F}^0(t)$.
3. First hitting time of an open set

$$T = \inf(t \geq 0 | B(t) \in G(\text{ open set }))$$

   is not a stopping time w.r.t $\mathcal{F}^0(t)$, but is a stopping time w.r.t $\mathcal{F}^+(t)$.

Definition 7 *A filtration $\{\mathcal{F}(t)\}_{t \geq 0}$ is right continuous if*

$$\mathcal{F}(t) = \bigcap_{\epsilon > 0} F^0(t+\epsilon) \,\forall\, t.$$

$\mathcal{F}^0(t)$ is not right continuous, but $\mathcal{F}^+(t)$ is right continuous.

Proposition 8 *If T is a RV taking values in $[0,\infty]$ s.t. $\{T < t\} \in \mathcal{F}(t)$ for all t and a right continuous filtration $\mathcal{F}(t)$, then T is a stopping time.*

Proof

$$\{T \leq t\} = \bigcap_n \{T < t + \frac{1}{n}\} \in \bigcap_n \mathcal{F}(t + \frac{1}{n}) = \mathcal{F}(t),$$

where the last step used right continuity. $\square$

From now on, we always complete $\mathcal{F}^+(t)$ and we work with this larger filtration and still call it $\mathcal{F}^+(t)$.

Definition 9 *For a stopping time T, w.r.t $\mathcal{F}^+(t)$, let*

$$\mathcal{F}^+(T) = \{\text{ event } A | A \cap \{T \leq t\} \in \mathcal{F}^+(t)\}.$$

This is the collection of events known by time T. In particular, every event depending on $\{B(s)|0 \leq s \leq T\}$.

Theorem 10 *Strong Markov property For a $d-$dimensional BM B and any a.s. finite time T w.r.t $\mathcal{F}^+(t)$, $\{B(T+t)-B(T)|t \geq 0\}$ is a standard $d-$dimensional BM independent of $\mathcal{F}^+(t)$.*

Remark 11 *If $S = max\{t \leq 1|B(t) = 0\}$, note $\{X(t) := B(S+t) - B(S)|t \geq 0\}$ does satisfy*

$$\mathbf{P}[\inf(t > 0|X(t) = 0) = 0] = 0,$$

Hence X is not a standard BM.

Proof of theorem 10

Suppose first that T takes any countably many values. For the k^{th} value a_k, let $\{B_k(t)(t \geq 0)\}$ be defined by $B_k(t) = B(a_k + t) - B(a_k)$ (a standard BM by Markov property), let also $\{B^\star(t)|t \geq 0\}$ be defined by $\{B^\star(t) = B(T+t) - B(T)\}$. Finally fix $E \in \mathcal{F}^+(T)$, we need to show that for any event $\{B^\star \in A\}$, we have

$$\mathbf{P}(\{B^\star \in A\} \cap E) = \mathbf{P}(\{B \in A\})\mathbf{P}(E).$$

In fact,

$$\begin{aligned}
\mathbf{P}(\{B^\star \in A\} \cap E) &= \sum_k \mathbf{P}(\{B^\star \in A\} \cap E \cap \{T = a_k\}) \\
&= \sum_k \mathbf{P}(\{B_k \in A\} \underbrace{\cap E \cap \{T = a_k\}}_{\text{measurable w.r.t. } \mathcal{F}^+(a_k)}) \\
&= \sum_k \underbrace{\mathbf{P}(\{B_k \in A\})}_{\text{Markov property at } a_k} \mathbf{P}(E \cap \{T = a_k\}) \\
&= \mathbf{P}(\{B \in A\}) \sum_k \mathbf{P}(\cap E \cap \{T = a_k\}) \\
&= \mathbf{P}(E),
\end{aligned}$$

For a general T, let T_n be an upper approximation, $T_n = (m+1)2^{-n}$ on the event $m2^{-n} \leq T < (m+1)2^{-n}$. Notice $T_n \downarrow T$ a.s.. We know $\{B(T_n + t) - B(T_n) | t \geq 0\}$ is a standard BM independent of $\mathcal{F}^+(T_n) \supset \mathcal{F}^+(T)$ (since $T_n \geq T$). Finally,

$$B(T+t) - B(T) = \lim_{n\to\infty} B(T_n + t) - B(T_n),$$

so the process $B(T+t) - B(T)$ has the same finite dimensional distribution as a standard BM and continuous paths, hence it is a standard BM.

Furthermore, since $B(T_n + t) - B(T_n)$ is independent of $\mathcal{F}^+(T_n)$, the limit $\{B(T+t) - B(T)\}_{t\geq 0}$ is independent of $\bigcap_n \mathcal{F}^+(T_n) = \mathcal{F}^+(T)$. □

3 Application of Strong Markov Property

3.1 Reflection Principle

Given a 1D BM, if T is a stopping time w.r.t $\mathcal{F}^+$, then the process

$$B^\star(t) = \begin{cases} B(t), & \text{if } t \leq T; \\ 2B(T) - B(t), & \text{if } t > T. \end{cases}$$

($B^\star(t) = B(t)$ for all t, if $T = \infty$)is also a standard BM.

Proof First assume T is finite a.s.. By the strong Markov property, $\{B(T+t) - B(T) | t \geq 0\}$ and $\{-(B(T+t) - B(T)) | t \geq 0\}$ are both standard BM independent of $\mathcal{F}^+(T)$. Gluing the first process to $\{B(t) | 0 \leq t \leq T\}$, We get back $B(t)$. Gluing the second process, giving $B^\star(t)$, gives a process distributed (having the same distribution) as B.

For general T, we can first apply the principle to $T_n = min(T, n)$, obtaining $B^\star_n$, $B^\star_n \to B^\star$ stabilizing each initial segment. □

Corollary 12 *Denote $M(t) = \max_{0 \le s \le t} B(s)$. If $a > 0$, then*

$$\mathbf{P}(M(t) \ge a) = 2\mathbf{P}(B(t) \ge a).$$

Proof Apply reflection principle to $T = \inf\{t > 0 | B(t) = a\}$,

$$\{M(t) \ge a\} = \{B(t) \ge a\} \uplus \{B(t) < a, M(t) \ge a\} = \{B^\star(t) > a\},$$

where $\biguplus$ means disjoint union. $\square$

3.2 Area of planar BM

Theorem 13 *(Lévy 1940) A.s. the area of points visited by a planar BM is 0.*

Remark 14 *However, this set has Hausdorff dimension 2. Stronger statement is true:*

$$dim(B(A)) = 2dim(A) \wedge d,$$

where $A \subset [0, \infty)$ is closed set, $\wedge$ means minimum, and d is the dim of the BM, only important for $d = 1$. See BM book, Eaufman's doubling (theorem 9.28) or earlier result of M-Kean (theorem 4.33).

Denote $\mathcal{L}_d$ the Lebesgue measure in R^d, and $f \star g$ the convolution of the given functions f and g, whenever well-defined, by

$$(f \star g)(x) = \int f(y)g(x - y)dy.$$

Lemma 15 *If A_1, A_2 are Borel sets in R^d with positive measure, then*

$$\mathcal{L}_d(\{x \in R^d | \mathcal{L}_d(A_1 \cap (A_2 + x)) > 0\}) > 0.$$

Proof Assume WLOG, A_1, A_2 are bounded sets. Note $1_{A_1} \star 1_{-A_2}(x) = \mathcal{L}_d(A_1 \cap (A_2 + x))$. By Fubini's theorem,

$$\begin{aligned}\int_{R^d} 1_{A_1} \star 1_{-A_2}(x)dx &= \int_{R^d}\int_{R^d} 1_{A_1}(y)1_{-A_2}(x-y)dydx \\ &= \int_{R^d} 1_{A_1}(y)\int_{R^d} 1_{-A_2}(x-y)dxdy \\ &= \mathcal{L}_d(A_1)\mathcal{L}_d(A_2) > 0.\end{aligned}$$

□

Proof of Lévy's theorem

It is sufficient to show a.s. $\mathcal{L}_2(B[0,1]) = 0$. Let $X = \mathcal{L}_2(B[0,1])$. First, we show $EX < \infty$. Let W be 1D BM. Note

$$\begin{aligned}\mathbf{P}(X > a) &\le 2\mathbf{P}(\max_{0\le t\le 1}|W(t)| > \frac{\sqrt{a}}{2})\\ &\le 4\mathbf{P}(\max_{0\le t\le 1}W(t) \ge \frac{\sqrt{a}}{2})\\ &= 8\mathbf{P}(W(1) \ge \frac{\sqrt{a}}{2})\end{aligned}$$

decays exponentially in a. Second, by scaling invariance, $\{B(t)\}$ and $\{\sqrt{3}B(\frac{t}{3})|t \ge 0\}$ are standard BM. Hence

$$E\mathcal{L}_2(B[0,3]) = 3E\mathcal{L}_2(B[0,1]) = 3EX$$

Next, note that

$$\mathcal{L}_2(B[0,3]) \le \sum_{j=0}^{2} \mathcal{L}_2(B[j,j+1])\ \forall\ 0 \le i < j \le 2$$

with equality if and only if $\mathcal{L}_2(B[i,i+1] \cap B[j,j+1]) = 0\ \forall\ 0 \le i < j \le 2$. We need only $i = 0, j = 2$. Now notice

$$3EX = E\mathcal{L}_2(B[0,3]) \le E\sum_{j=0}^{2} \mathcal{L}_2 B[j,j+1] = 3EX,$$

So $\mathcal{L}_2(B[0,3]) = \sum_{j=0}^{2} \mathcal{L}_2 B[j,j+1]$ a.s.. In particular, a.s. $\mathcal{L}_2(B[0,1] \cap B[2,3]) = 0$. Without writing all details, this implies $\mathcal{L}_2(B[0,1]) = 0$ a.s.. In fact, $\mathcal{L}_2(B[0,1])$ is identically distributed and independent of $\mathcal{L}_2(B[2,3])$ and the overlap of these two sets is determined by $B(2) - B(1)$ which is an independent Gaussian, and we can use the lemma. □

Corollary 16 *For $d \ge 2$, $\forall\ x, y \in R^d$, $\mathbf{P}^x(y \in B(0,\infty)) = 0$, i.e. BM does not hit points.*

Proof WLOG, we may assume $d = 2$. Note by Fubini's theorem,

$$\int_{R^2} \mathbf{P}_y(x \in B[0,\infty))dx = E\mathcal{L}_2(B[0,\infty)) = 0.$$

Hence for every y and a.e. x(Lebesgue measure)

$$\mathbf{P}^y(x \in B[0,\infty]) = 0.$$

To get rid of a.e., note

$$\begin{aligned}\mathbf{P}^y(x \in B(0,\infty)) &= \mathbf{P}^0(x - y \in B(0,\infty)) \\ &= \underbrace{\mathbf{P}^0(y - x \in B(0,\infty))}_{\text{by symmetry of BM}} \\ &= \mathbf{P}^x(y \in B(0,\infty))\end{aligned}$$

for every y and a.e. x. Now we finish since

$$\begin{aligned}\mathbf{P}^x(y \in B(0,\infty)) &= \lim_{\epsilon \downarrow 0} \mathbf{P}^x(y \in B[\epsilon,\infty)) \\ &= \lim_{\epsilon \downarrow 0} \underbrace{E_x \mathbf{P}^{B(\epsilon)}(y \in B[\epsilon,\infty))}_{\text{by Markov property}} = 0\end{aligned}$$

□

3.3 Zero Set of 1D BM

Denote

$$Zero = \{t \geq 0 | B(t) = 0\}.$$

Theorem 17 *For a 1D BM, a.s. Zero is a perfect set(i.e. closed with no isolated points).*

Proof Zeros is closed since BM is a.s. continuous. For $a \geq 0$, let

$$\tau_a = \min\{t \geq a | B(t) = 0\} (\text{i.e. first zero after } a).$$

This is a stopping time. It follows from distribution of maximum in previous corollary that τ_a is a.s. finite. By the strong Markov property at τ_a and theorem 3, we have

$$\mathbf{P}(\inf(t > \tau_a | B(t) = 0) = \tau_a) = 1.$$

Hence the zero at τ_a not isolated from right. This holds simultaneously for τ_q for all rational q a.s..

This remains to show any $z \in Zero$ not equal to τ_q for a rational q is not isolated from left. This follows by definition, since if $q_n \uparrow z$, then $\tau_q \uparrow z$. □

References

[1] *P.Mörters* and *Y.Peres*. Browniam Motion. Chapter 2. 46-58.