Lecture 2

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The following lecture will consider mainly the simple random walk in 1-dimension. The SRW will satisfy us for now as it can be generalized with a few simple assumptions to various other random walks. Our main topics are - the Wald identities along with the derivation of the moment generating function of the first passage time for 1-dimensional SRW and the SRW properties such as: the arcsine laws for the last zero and fraction of time above axis, time reversal and equidistribution of positive time for SRW bridge. We will see that there are several properties whose probability distribution is similar asymptotically to the arcsine function.

Tags for today's lecture: Wald identities, Moment generating function of the first passage time, Arcsine laws, Time reversal, Positive time for SRW bridge.

1 Wald Identities For General Random Walks

In the following statements, assume that $S_n = \sum_{i=1}^{n} X_i$ is a one-dimensional random walk with initial value $S_0 = 0$ (where, as before, X_i $\forall i \in \mathbb{N}$ are independent and identically distributed random variables ,i.i.d).

There are 3 Wald identities which need to be considered:

1. if $\mathbb{E}[X_1] = \mu < \infty$ then for any stopping time T with $\mathbb{E}[T] < \infty$,
$$\mathbb{E}[S_T] = \mu\mathbb{E}[T] \tag{1}$$
2. Assume that the random variable X_1 has finite second moment and let $\mathbb{E}[X_1] = \mu < \infty$ and $\text{var}[X_1] = \sigma^2 < \infty$, then for any stopping time T with $\mathbb{E}[T] < \infty$,
$$\mathbb{E}[S_T - \mu T]^2 = \sigma^2\mathbb{E}[T] \tag{2}$$
3. Assume that the moment generating function (mgf) $\varphi(\theta) = \mathbb{E}[e^{\theta \cdot X_1}]$ exists at θ then for every bounded stopping time T,
$$\mathbb{E}\left[\frac{e^{\theta \cdot S_T}}{\varphi(\theta)^T}\right] = 1 \tag{3}$$

We will prove only (1) and (3) as the proof of the 2nd wald identity is similar and is left as an exercise.

Proof

(1) Let us first consider the case where the stopping time T is bounded by some integer M, in this case S_T can be decomposed as a finite sum,

$$S_T = \sum_{n=0}^{M} S_n \cdot \mathbb{1}_{\{T=n\}} = \sum_{n=1}^{M} X_n \cdot \mathbb{1}_{\{T \geq n\}}$$

taking the expectation on both sides and using linearity we get,

$$\mathbb{E}[S_T] = \sum_{n=1}^{M} \mathbb{E}[X_n \cdot \mathbb{1}_{\{T \geq n\}}] = \sum_{n=1}^{M} \mathbb{E}[X_n] \cdot \mathbb{E}[\mathbb{1}_{\{T \geq n\}}]$$

where the last equality arises from the fact that the event $\{T \geq n\}$ depends only on the first $n-1$ increments and so it is independent of X_n. From probability theory we know that for any nonnegative integer-valued random variable, T ,the expectation of T equals

$$\mathbb{E}[T] = \sum_{n=1}^{\infty} \mathrm{Pr(T \geq n)}$$

and so,

$$\mathbb{E}[S_T] = \sum_{n=1}^{M} \mathbb{E}[X_n] \cdot \mathbb{E}[\mathbb{1}_{\{T \geq n\}}] = \sum_{n=1}^{M} \mathbb{E}[X_n] \cdot \mathrm{Pr(T \geq n)} = \mu \mathbb{E}[T].$$

In order to complete the proof, we need to consider the case where T is not bounded, specifically, verifing that the interchange between summations and expectations still holds. This can be done by either the Fubini theorem or the dominated convergence theorem. Since $|X_n|$ and $\{T \geq n\}$ are still independent, the conditions for the theorems above stand,

$$\sum_{n=1}^{\infty} \mathbb{E}[|X_n| \cdot \mathbb{1}_{\{T \geq n\}}] = \sum_{n=1}^{\infty} \mathbb{E}[|X_n|] \mathbb{E}[\mathbb{1}_{\{T \geq n\}}] = \mathbb{E}[|X_1|] \sum_{n=1}^{\infty} \mathrm{Pr(T \geq n)} = \mathbb{E}[|X_1|] \cdot \mathbb{E}[T] < \infty$$

thus,

$$\mathbb{E}[S_T] = \mathbb{E}\left[\sum_{n=0}^{\infty} X_n \cdot \mathbb{1}_{\{T\geq n\}}\right] = \sum_{n=1}^{\infty} \mathbb{E}[X_n] \cdot \mathbb{E}[\mathbb{1}_{\{T\geq n\}}] = \sum_{n=1}^{\infty} \mathbb{E}[X_n] \cdot \Pr(\mathrm{T} \geq \mathrm{n}) = \mu\mathbb{E}[T].$$

(3) Since T is bounded by m,

$$\mathbb{E}\left[\frac{e^{\theta\cdot S_T}}{\varphi(\theta)^T}\right] = \sum_{n=0}^{m} \mathbb{E}\left[\frac{e^{\theta\cdot S_n}}{\varphi(\theta)^n}\mathbb{1}_{\{T=n\}}\right] = \sum_{n=0}^{m}\left(\mathbb{E}\left[\frac{e^{\theta\cdot S_n}}{\varphi(\theta)^n}\mathbb{1}_{\{T=n\}}\right]\right)\left(\mathbb{E}\left[\frac{e^{\theta\cdot S_m - S_n}}{\varphi(\theta)^{m-n}}\right]\right)$$

where the first term in the last sum is a function of $X_1, \ldots, X_n$ and the second term equals 1 and is a function of $X_{n+1}, \ldots, X_m$. By independence,

$$\mathbb{E}\left[\frac{e^{\theta\cdot S_T}}{\varphi(\theta)^T}\right] = \sum_{n=0}^{m} \mathbb{E}\left[\frac{e^{\theta\cdot S_m}}{\varphi(\theta)^m}\mathbb{1}_{\{T=n\}}\right] = \mathbb{E}\left[\frac{e^{\theta\cdot S_m}}{\varphi(\theta)^m}\sum_{n=0}^{m}\mathbb{1}_{\{T=n\}}\right] = \mathbb{E}\left[\frac{e^{\theta\cdot S_m}}{\varphi(\theta)^m}\right] = 1$$

□

That completes our proof of the Wald identities. Now let us discuss some of the uses for the Wald identities.

1.1 The Moment Generating Function Of The First Passage Time For 1-Dimensional SRW

Define the first passage time, τ, to the level 1 for SRW by $\tau = \min(n|S_n = 1)$. As we have previously seen $\Pr(\tau = \mathrm{n}) = \frac{1}{\mathrm{n}}\Pr(\mathrm{S_n} = 1)$ and we will now calculate $\mathbb{E}[S^\tau]$ (the moment generating function of τ) by using the 3rd wald identity.

If τ was bounded and we could use 3, then we would get,

$$\mathbb{E}\left[\frac{e^{\theta\cdot S_\tau}}{\varphi(\theta)^\tau}\right] = 1 = e^\theta\mathbb{E}\left[\frac{1}{\varphi(\theta)^\tau}\right] \tag{4}$$

where the last equality holds since $S_\tau \equiv 1$. Define $S := \frac{1}{\varphi(\theta)}$ and writing $e^{-\theta}$ as a function of S we get,

$$\mathbb{E}[S^\tau] = e^{-\theta} \tag{5}$$

By definition, $\frac{1}{S} = \varphi(\theta) = \mathbb{E}[e^{\theta X_1}] = \frac{1}{2}(e^{\theta} + e^{-\theta})$ combining with 5 and solving for $e^{-\theta}$ for $\theta > 0$,

$$\mathbb{E}[S^{\tau}] = e^{-\theta} = \frac{1 - \sqrt{1 - S^2}}{S} \tag{6}$$

which is exactly what we wanted ro find. To justify our assumption for τ, since it is unbounded, we will need the following lemma.

Lemma 1. *if τ is a stopping time, then for every nonnegative integer n the random variable $\tau \wedge n$ is also a stopping time.*

The proof of Lemma 1 is left as an exercise. Since $\tau \wedge n$ is also a stopping time, the 3rd wald identity along with the lemma gives

$$\mathbb{E}\left[\frac{e^{\theta \cdot S_{\tau \wedge n}}}{\varphi(\theta)^{\tau \wedge n}}\right] = 1$$

Considering $\Pr(\tau < \infty) = 1$ which was proved earlier, we conclude that

$$\lim_{n \to \infty} \frac{e^{\theta \cdot S_{\tau \wedge n}}}{\varphi(\theta)^{\tau \wedge n}} = \frac{e^{\theta \cdot S_{\tau}}}{\varphi(\theta)^{\tau}}.$$

Since $S_{\tau \wedge n} \leq 1$ we derive that $e^{\theta \cdot S_{\tau \wedge n}} \leq e^{\theta}$ for $\theta > 0$. As mentioned above, $\varphi(\theta) = \cosh(\theta) \geq 1$ for any θ, so $\frac{e^{\theta \cdot S_{\tau \wedge n}}}{\varphi(\theta)^{\tau \wedge n}} \leq e^{\theta}$ we may apply again the dominated convergence theorem and obtain,

$$\mathbb{E}\left[\frac{e^{\theta S_{\tau}}}{\varphi(\theta)^{\tau}}\right] = 1 \quad for\ \theta > 0.$$

2 Sample Path Properties Of SRW

2.1 Arcsine distributions - Main lemma

How much time does the SRW spends above level 0? Our intuition maybe a bit misleading when pondering upon this question and as we are about to see the answer is a bit surprising since the probability that no return to the origin occurs up to and including epoch $2n$ is the same as the probability that a return occurs at epoch $2n$.

Lemma 2.

$$\mathrm{Pr}(\mathrm{S_1} \neq 0, \ldots, \mathrm{S_{2n}} \neq 0) = \mathrm{Pr}(\mathrm{S_{2n}} = 0) \tag{7}$$

Proof Where the event on the left occurs either all the S_j are positive, or all are negative. The two contingencies being equally probable we can restate lemma 2,

$$\mathrm{Pr}(\mathrm{S_1} > 0, \ldots, \mathrm{S_{2n}} > 0) = \frac{1}{2}\mathrm{Pr}(\mathrm{S_{2n}} = 0) \tag{8}$$

From the Ballot theorem we know that,

$$\mathrm{Pr}(\mathrm{S_1} > 0, \ldots, \mathrm{S_{2n-1}} > 0, \mathrm{S_{2n}} = 2\mathrm{r}) = \frac{\mathrm{r}}{\mathrm{n}}\mathrm{Pr}(\mathrm{S_{2n}} = 2\mathrm{r})$$

and by a bit of algebra (or a part of the ballot theorem) we derive that,

$$\mathrm{Pr}(\mathrm{S_1} > 0, \ldots, \mathrm{S_{2n-1}} > 0, \mathrm{S_{2n}} = 2\mathrm{r}) = \frac{1}{2}[\mathrm{Pr}(\mathrm{S_{2n-1}} = 2\mathrm{r} - 1) - \mathrm{Pr}(\mathrm{S_{2n+1}} = 2\mathrm{r} + 1)]$$

Let us now consider the right hand side of equation 8 with specifically using the fact that the epoch is even ($2n$), we can rewrite the term as the following sum,

$$\mathrm{Pr}(\mathrm{S_1} > 0, \ldots, \mathrm{S_{2n}} > 0) = \sum_{\mathrm{r}=1}^{\infty} \mathrm{Pr}(\mathrm{S_1} > 0, \ldots, \mathrm{S_{2n-1}} > 0, \mathrm{S_{2n}} = 2\mathrm{r}) = \frac{1}{2}\mathrm{Pr}(\mathrm{S_{2n-1}} = 1)$$

where the last equality holds from the fact that all the terms in the sum with $r > n$ equal 0 and the rest of the terms cancel each other (a telescopic series). And so we are done since,

$$\mathrm{Pr}(\mathrm{S_{2n}} = 0) = \frac{1}{2}[\mathrm{Pr}(\mathrm{S_{2n-1}} = 1) + \mathrm{Pr}(\mathrm{S_{2n-1}} = -1)] = \mathrm{Pr}(\mathrm{S_{2n-1}} = 1)$$

□

The lemma can be restated in several ways; for example,

Lemma 3.

$$\Pr(S_1 \geq 0, \ldots, S_{2n} \geq 0) = \Pr(S_{2n} = 0) \tag{9}$$

Proof Continuing with equation 8, indeed, a path of length $2n$ with all vertices strictly above the x-axis passes through the point $(1,1)$. Taking this point as a new origin we obtain a path of length $2n-1$ with all vertices above or on the new x-axis. It follows that,

$$\frac{1}{2}\Pr(S_{2n} = 0) = \Pr(S_1 > 0, \ldots, S_{2n} > 0) = \frac{1}{2}\Pr(S_1 \geq 0, \ldots, S_{2n-1} \geq 0)$$

by parity consideration,

$$\frac{1}{2}\Pr(S_{2n} = 0) = \frac{1}{2}\Pr(S_1 \geq 0, \ldots, S_{2n} \geq 0)$$

□

Note that,

$$\Pr[\text{first retuen to 0 is at 2n}] = \Pr(S_1 > 0, \ldots, S_{2n-2} > 0) - \Pr(S_1 > 0, \ldots, S_{2n} > 0) \tag{10}$$

and by using equality 8 we get that equation 10 equals,

$$\frac{1}{2}[\Pr(S_{2n-2} = 0) - \Pr(S_{2n} = 0)] = \frac{1}{2n-1}\Pr(S_{2n} = 0)$$

same as in the ballot theorem.

2.2 Last Zero

Our next goal is to determine the probability for the last visit to the origin in a SRW up to a certain time. Why is that important to us?

Well, consider a long coin-tossing game with 2 players. Our intuition might tell us that each player will be on the winning side for about half the time, and that the lead will pass not infrequently from one player to the other. However (!), the next theorem will show that with probability $\frac{1}{2}$ no equalization occurred in the second half of the game, regardless of the length of the game. furthermore, the most probable last equalization is at the extremes!

Theorem 4. *Define $T = T_{2n} = \max(k \leq n | S_{2n} = 0)$ (not a stopping time), then*

$$\Pr(T = 2k) = \Pr(S_{2k} = 0) \cdot \Pr(S_{2n-2k} = 0) \tag{11}$$

Remark 5. *The result is symmetric around n. By the Sterling formula,*

$$\Pr(S_{2k} = 0) \sim \frac{1}{\sqrt{\pi k}} \quad \text{as } k \to \infty$$

and so,

$$\Pr(T = 2k) \approx \frac{1}{\pi\sqrt{k(n-k)}} = \frac{1}{\pi\sqrt{x(1-x)}} \quad \text{for } x = \frac{k}{n}$$

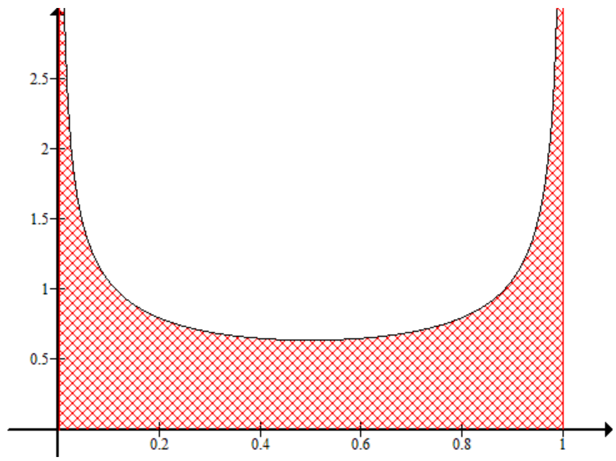

Figure 1: Graph of $\frac{1}{\pi\sqrt{x(1-x)}}$, obviously areas near the edges are more significant than the area around 0.5

Corollary 6.

$$\lim_{\substack{n\to\infty \\ \frac{k}{n}\to x}} \Pr(T \le 2k) = \int_0^X \frac{1}{\pi\sqrt{y(1-y)}} dy = \frac{2}{\pi}\arcsin(\sqrt{x}).$$

We will later see that the arcsine law also holds for brownian motion.

Proof Of theorem 4. By the Markov property we know that,

$$\Pr(S_{2k+1} \neq 0, \ldots, S_{2n} \neq 0) = \Pr(S_1 \neq 0, \ldots, S_{2n-2k} \neq 0)$$

and combing this with Lemma 2 we get,

$$\Pr(T = 2k) = \Pr(S_{2k} = 0)\Pr(S_1 \neq 0, \ldots, S_{2n-2k} \neq 0) = \Pr(S_{2k} = 0) \cdot \Pr(S_{2n-2k} = 0)$$

□

2.3 Positive Time

The next theorem will deal with the amount of time that a SRW spends above or below the axis. Again, we will see an amazing result - The probability that in the time interval from 0 to $2n$ the random walk spends $2k$ time units on the positive side and $2n-2k$ time units on the negative side equals $\Pr(S_{2k}=0)\cdot\Pr(S_{2n-2k}=0)=\Pr(T=2k)$, just as the previous result.

For a SRW of length $2n$ let S be the amount of time above the axis, and formally,

$$S = |\{k|S_k, S_{k+1} \geq 0\}|$$

Theorem 7.

$$\Pr(S=2k) = \Pr(S_{2k}=0)\cdot\Pr(S_{2n-2k}=0)$$

Proof Denote $b_{2k,2n} = \Pr(S=2k)$ and let us first consider the cases where $k \in \{0, n\}$,

- if <u>*k=n:*</u> $b_{2n,2n} = \Pr(S=2n) = \Pr(S_1 \geq 0, \ldots, S_{2n} \geq 0) = \Pr(S_{2n}=0)$ (by lemma 9, and considering $\Pr(S_0=0)=1$).
- if <u>*k=0:*</u> Similarly $b_{0,2n} = \Pr(S_{2n}=0)$.

Suppose that $1 \leq k \leq n-1$, denote $f_{2r} = \Pr(\text{first return to 0 is at 2r})$ then,

$$\begin{aligned} b_{2k,2n} &= \Pr(S=2k) = \sum_{r=0}^{n} \Pr(S=2k,\ \text{first return } = 2r) \\ &= \frac{1}{2}\left[\sum_{r=0}^{k} f_{2r} b_{2k-2r,2n-2r} + \sum_{r=0}^{n-k} f_{2r} b_{2k,2n-2r}\right] \end{aligned}$$

Since there are 2 possible ways for the walk to occur,

(I) First $2r$ steps until the first return to 0 are above the axis and the last $2n-2r$ have $2k-2r$ time above the axis ($r \leq k \leq n-1$).

(II) First $2r$ steps until the first return to 0 are under the axis, then $2k$ time above the axis ($2n-2r \geq 2k$).

By induction on n, $b_{2m,2n-2r} = \Pr(S_{2m}=0)\Pr(S_{2n-2r-2m}=0)$, therefore $b_{2k,2n}$ can be rewritten as,

$$
\begin{aligned}
b_{2k,2n} &= \frac{1}{2}\left[\sum_{r=0}^{k} f_{2r} b_{2k-2r,2n-2r} + \sum_{r=0}^{n-k} f_{2r} b_{2k,2n-2r}\right] = \\
&= \frac{1}{2}\mathrm{Pr}(\mathrm{S}_{2n-2k}=0)\sum_{r=0}^{k}\mathrm{f}_{2r}\mathrm{Pr}(\mathrm{S}_{2k-2r}=0) + \frac{1}{2}\mathrm{Pr}(\mathrm{S}_{2k}=0)\sum_{r=0}^{n-k}\mathrm{f}_{2r}\mathrm{Pr}(\mathrm{S}_{2n-2r-2k}=0) = \\
&= \frac{1}{2}\mathrm{Pr}(\mathrm{S}_{2n-2k}=0)\mathrm{Pr}(\mathrm{S}_{2k}=0) + \frac{1}{2}\mathrm{Pr}(\mathrm{S}_{2k}=0)\mathrm{Pr}(\mathrm{S}_{2n-2k}=0) = \\
&= \mathrm{Pr}(\mathrm{S}_{2n-2k}=0)\mathrm{Pr}(\mathrm{S}_{2k}=0).
\end{aligned}
$$

the last equalities follow from the fact that the sums above are actually conditioning on the first return to zero along with the Markov property. □

3 Duality (Time Reversal)

Every path we have seen up to this point consists of increments $X_1, \ldots, X_n$ in a certain order determined a priori from 1 to n. We now may ask ourselves - what happens if we reverse the order of the increments? Geometrically speaking, this means rotating the given path through 180 degrees about its right endpoint, taking the latter as origin of a new coordinate system and and reflecting around the new x-axis. Obviously, the probability of a path with increments $X_1, \ldots, X_n$ equals the probability of a path with increments $X_n, \ldots, X_1$.

Denote a segment of the original path by $S_j = X_1 + \cdots + X_j$ and the reversed path with respect to length n by

$$S^*_{j,n} = X_n + X_{n-1} + \cdots + X_{n-j+1} = S_n - S_{n-j}. \tag{12}$$

Let's review a few examples and interesting results:

1 *Strict ladder time (record time):* by 12 we conclude,

$$\{S^*_{j,n} > 0 \ \forall \ 1 \leq j \leq n\} = \{S_n > S_j \ \forall \ 0 \leq j \leq n-1\}$$

thus combining with 8 we get,

$$\begin{aligned} \mathrm{Pr}(\mathrm{S_{2n} > S_j \ \forall \ 0 \leq j \leq 2n-1}) &= \mathrm{Pr}(\mathrm{S^*_{j,2n} > 0 \ \forall \ 1 \leq j \leq 2n}) = \\ &= \frac{1}{2}\mathrm{Pr}(\mathrm{S^*_{2n,2n} = 0}) = \frac{1}{2}\mathrm{Pr}(\mathrm{S_{2n} = 0}) \end{aligned}$$

and we get a very interesting result about the probability to finish a walk in a record level.

1.1 *Weak ladder time:* with a similar calculation and using 9,

$$\mathrm{Pr}(\mathrm{S_{2n} \geq S_j \ \forall \ 0 \leq j \leq 2n-1}) = \mathrm{Pr}(\mathrm{S_{2n} = 0}).$$

2 *Arcsine law for the 1st visit to terminal point:*

$$\mathrm{Pr}(\mathrm{min(j|S_{2j} = S_{2n}) = k}) = \mathrm{Pr(last\ visit\ to\ 0\ for\ the\ rev.\ walk\ is\ at\ 2n-2k)}.$$

3 *Arcsine law for position of the first\last maxima:*

$$\begin{aligned} \mathrm{Pr(1st\ global\ max = 2k)} &= \mathrm{Pr}(\mathrm{S_0 < S_{2k}, \ldots, S_{2k-1} < S_{2k}, S_{2k+1} \leq S_{2k}, \ldots, S_{2n} \leq S_{2k}}) = \\ &= \mathrm{Pr}(\mathrm{S_0 < S_{2k}, \ldots, S_{2k-1} < S_{2k}})\mathrm{Pr}(\mathrm{S_1 \leq 0, \ldots, S_{2n-2k} \leq 0}) = \\ &= \mathrm{Pr}(\mathrm{S_1 > 0, \ldots, S_{2k} > 0})\mathrm{Pr}(\mathrm{S_1 \geq 0, \ldots, S_{2n-2k} \geq 0}) = \\ &= \frac{1}{2}\mathrm{Pr}(\mathrm{S_{2k} = 0})\mathrm{Pr}(\mathrm{S_{2n-2k} = 0}) \end{aligned}$$

where we used the Markov property along with the other results previously mentioned. Note that the same formula holds for the last maxima as well.

4 Equidistribution Of Positive Time For SRW Bridge

Definition 8. *A simple random walk bridge is a simple random walk conditioned to return to the starting point level after 2n time steps.*

A question one might ask is how much time a SRW bridge will be above the axis? The amazing fact that the next theorem will show is that the probability for the time spent above the axis to be 2k is independent of k!

Theorem 9. $\mathrm{Pr(time\ above\ axis = 2k|S_{2n} = 0) = \frac{1}{n+1}}$

Proof

Denote

$$\begin{aligned} C_{2k,2n} &= \mathrm{Pr(time\ above\ the\ axis = 2k|S_{2n} = 0)} = \\ &= \frac{\mathrm{Pr(time\ above\ the\ axis = 2k, S_{2n} = 0)}}{\mathrm{Pr(S_{2n} = 0)}} = \frac{d_{2k,2n}}{\mathrm{Pr(S_{2n} = 0)}} \end{aligned}$$

we will prove that $d_{2k,2n}$ is independent of k.

For $k = n$,

$$\begin{aligned} d_{2n,2n} &= \mathrm{Pr(S_1 \geq 0, \ldots, S_{2n-1} \geq 0, S_{2n} = 0) = 2^{-2n}C_n} = \\ &= 2^{-2n}\frac{1}{n+1}\binom{2n}{n} = \frac{1}{n+1}\mathrm{Pr(S_{2n} = 0)} \end{aligned}$$

and by symmetry the same holds for $k = 0$.[1]

For $1 \leq k \leq n-1$, using the earlier notations, $f_{2r} = \frac{1}{2r-1}\mathrm{Pr(S_{2r} = 0)} = \frac{1}{\mathrm{2r}}\mathrm{Pr(S_{2r-2} = 0)}$ and similar calculation gives,

$$\begin{aligned} d_{2k,2n} &= \sum_{r=1}^{n} \mathrm{Pr(time\ above\ the\ axis = 2k, S_{2n} = 0, first\ return\ to\ 0\ is\ at\ 2r)} = \\ &= \frac{1}{2}\left[\sum_{r=1}^{k} f_{2r}d_{2k-2r,2n-2r} + \sum_{r=1}^{n-k} f_{2r}d_{2k,2n-2r}\right] \end{aligned}$$

By induction, $d_{2m,2n} = \frac{1}{n+1}\mathrm{Pr(S_{2n} = 0)}$,

[1] C_n - the Catelan number from the ballot theorem which was used for the last equality

$$
\begin{aligned}
d_{2k,2n} &= \frac{1}{2}\sum_{r=1}^{k} \frac{1}{r(n-r+1)} \Pr(\mathrm{S_{2r-2}} = 0) \Pr(\mathrm{S_{2n-2r}} = 0) + \\
&+ \frac{1}{2}\sum_{r=1}^{n-k} \frac{1}{r(n-r+1)} \Pr(\mathrm{S_{2r-2}} = 0) \Pr(\mathrm{S_{2n-2r}} = 0)
\end{aligned}
$$

the last two sums are symmetrical under the exchange of r with $n - r + 1$, and so we get,

$$
d_{2k,2n} = \frac{1}{2}\sum_{r=1}^{n} \frac{1}{r(n-r+1)} \Pr(\mathrm{S_{2r-2}} = 0) \Pr(\mathrm{S_{2n-2r}} = 0)
$$

which is independent of k. □

References

[1] Steve Lalley. Lecture notes - One-Dimensional Random Walks. *The University of Chicago*, 2010,8-12.

[2] William Feller. An introduction to probability theory and its applications \volume 1, 3rd edition. *John Wiley & Sons, Inc.*, 1967,67-95.